STUDIA MATHEMATICA, T. XCV (1989)

Some remarks on Orlicz's interpolation theorem

by

LECH MALIGRANDA (Poznań and Caracas)

Dedicated to Professor Władysław Orlicz on the occasion of his 85th birthday

Abstract. A simple proof of a more general version of the Orlicz interpolation theorem for Lipschitz operators is given along with some applications.

0. Introduction. The first interpolation theorem concerning Orlicz spaces (not only L_p spaces) as intermediate spaces is due to Orlicz. He proved in [17] that any separable Orlicz space $L_\varphi(a, b)$ is an interpolation space between $L_1(a, b)$ and $L_\infty(a, b)$ for linear operators. Next, in [18] he extended his theorem to arbitrary Orlicz spaces and even to Lipschitz operators. In this paper we give a generalization of Orlicz's nonlinear interpolation theorem with a simple proof and with some applications to inequalities involving nonincreasing rearrangements. Orlicz's theorem is generalized to the case of arbitrary measures both in the domain and in the image of operators. We did not succeed in proving the theorem for the Orlicz space $L_\varphi(\mu)$ but only for $L_\varphi(\mu) \cap L_1(\mu)$. Of course, if $\mu\Omega < \infty$ or if φ satisfies condition δ_2 for small u then we have the Orlicz theorem on the whole of $L_\varphi(\mu)$, because $L_\varphi(\mu) \subset L_1(\mu)$ in the first case and $L_\varphi(\mu) \cap L_1(\mu)$ is dense in $L_\varphi(\mu)$ in the second. Simplicity of our proof of the main theorem stems from the application of a representation (2) of Orlicz functions and a special property (7) of the operators considered. Moreover, our proof gives an exact estimate of the norm of the operator. In Orlicz's paper there was an additional constant connected with his technique of proof.

The paper is divided into five sections. Section 1 contains necessary definitions and basic facts. In Section 2 we give two proofs of the main theorem which are based on a special representation of Orlicz functions and a special property of operators. In Section 3 we show a generalization of Orlicz's nonlinear interpolation theorem to arbitrary measures. The derivation of Theorem 2 from the main Theorem 1 has a general character and can be formulated in the language of abstract interpolation (cf. [19] and [16]). For a study of interpolation of nonlinear operators, with particular reference to

Lipschitz operators, we refer to [16]. Pointwise nonlinear operators are considered in Section 4. This was partly motivated by results in [5] and [11]. Applications of the main Theorem 1 and the Orlicz type Theorem 2 yielding a classical interpolation result for linear operators, the Hardy–Littlewood–Pólya theorem, the Lorentz–Shimogaki inequality and a generalization of Brudnyi's result on the modulus of continuity of x and x^* are considered in Section 5.

1. Preliminaries. In this section Orlicz spaces and rearrangements are defined and some of their properties are given. Also a result on truncation is included for use later on.

Let (Ω, Σ, μ) be a σ-finite measure space and let φ: $[0, \infty) \to [0, \infty)$ be an *Orlicz function*, i.e., a nondecreasing continuous convex function such that $\varphi(0) = 0$.

Let $S(\mu)$ be the space of equivalence classes of real-valued measurable functions on Ω.

Then the functional I_φ: $S(\mu) \to [0, \infty]$ defined by

$$I_\varphi(x) = \int_\Omega \varphi(|x(t)|)\,d\mu$$

is a modular on $S(\mu)$. The *Orlicz space* $L_\varphi = L_\varphi(\mu) = L_\varphi(\Omega)$ is the space of all $x \in S(\mu)$ for which $I_\varphi(rx) < \infty$ for some $r > 0$, depending on x. The functional

$$\|x\|_\varphi = \inf\{r > 0\colon\ I_\varphi(x/r) \leqslant 1\}$$

defined on the whole of $S(\mu)$ is a norm on L_φ. The *Orlicz class* $L_\varphi^0 = L_\varphi^0(\mu)$ is the set of all $x \in S(\mu)$ for which $I_\varphi(x) < \infty$. This set is, in general, only convex. $L_\varphi^a = L_\varphi^a(\mu)$ is the space of all $x \in S(\mu)$ for which $I_\varphi(rx) < \infty$ for any $r > 0$. This is a closed subspace in L_φ and the norm $\|\cdot\|_\varphi$ on L_φ^a is continuous, i.e., if $x \in L_\varphi^a$ and $|x| \geqslant x_n \downarrow 0$ then $\|x_n\|_\varphi \to 0$. In the case when $\varphi(u) = u^p$ we write the letter p instead of φ.

The *nonincreasing* left-continuous *rearrangement* of $x \in S(\mu)$ is the function $x^* = x_\mu^*$: $(0, \infty) \to [0, \infty]$ defined by

$$x_\mu^*(t) = \inf\{\lambda > 0\colon\ d_x(\lambda) < t\},$$

where $d_x(\lambda) = \mu(\{t \in \Omega\colon\ |x(t)| > \lambda\})$ and $\inf\emptyset = \infty$.

The collection of all $x \in S(\mu)$ for which $d_x(\lambda) \not\equiv \infty$ will be denoted by $S_0(\mu)$. For every $x \in S_0(\mu)$ we have $d_x(\lambda) \to 0$ as $\lambda \to \infty$ and so $x^*(t)$ is finite for any $t > 0$. The rearrangement has the following properties (cf. [9]):

(i) $0 \leqslant x_n \nearrow x$ implies $x_n^*(t) \nearrow x^*(t)$ for all $t > 0$.

(ii) $0 \leqslant x \leqslant y$ implies $x^*(t) \leqslant y^*(t)$ for all $t > 0$.

(iii) If m denotes the Lebesgue measure then

$$\mu(\{t \in \Omega\colon\ |x(t)| > \lambda\}) = m(\{s > 0\colon\ x^*(s) > \lambda\}) \quad \text{for all } \lambda > 0,$$

and we say that x and x^* are equimeasurable even though they are defined on different measure spaces. Moreover,

$$\int_\Omega |x|\,d\mu = \int_0^\infty x^*\,dm.$$

(iv) For x and y in $S(\mu)$ we have

$$\int_0^t (x+y)^*\,dm \leqslant \int_0^t x^*\,dm + \int_0^t y^*\,dm \quad \text{for all } t > 0.$$

Suppose now that x: $\Omega \to \boldsymbol{R}$, and define the *$\alpha$-truncation* $(\alpha > 0)$ $x^{(\alpha)}$ by

$$x^{(\alpha)}(t) = \min(|x(t)|, \alpha)\operatorname{sgn} x(t).$$

One can show that for functions x and y defined on Ω we have

$$|x^{(\alpha)}(t) - y^{(\alpha)}(t)| \leqslant |x(t) - y(t)| \quad \text{for all } t \in \Omega. \tag{1}$$

Finally, recall that an operator T in a Banach space X is called *nonexpansive* if $\|Tx - Ty\|_X \leqslant \|x-y\|_X$ for all $x, y \in X$.

2. Main theorem. In the proof of the main theorem we will need the following lemma about representation of Orlicz functions.

LEMMA 1. *Every Orlicz function φ has a representation*

$$\varphi(u) = au + \int_0^\infty (u-s)_+\,dp(s), \tag{2}$$

where p is a nondecreasing nonnegative right-continuous function on $[0, \infty)$ and $a = p(0^+)$ (here c_+ means $\max(c, 0)$).

Proof. It is well known that every Orlicz function φ can be represented in the form $\varphi(u) = \int_0^u p(s)\,ds$, where p is the right-derivative of φ. By integration by parts we get

$$\int_0^u p(s)\,ds = up(u) - \int_0^u s\,dp(s) = up(0^+) + \int_0^u u\,dp(s) - \int_0^u s\,dp(s)$$

$$= au + \int_0^u (u-s)\,dp(s) = au + \int_0^\infty (u-s)_+\,dp(s).$$

Let us now explain with some examples how to interpret (2) with $dp(s)$ as a measure:

1° If $p = \varphi'$ is absolutely continuous then according to Lemma 1 we have $\varphi(u) = au + \int_0^\infty (u-s)_+\,\varphi''(s)\,ds$. Note that in [8] the existence and integrability of φ'' is assumed.

2° If $\varphi(u) = 0$ for $0 \leqslant u \leqslant 1$ and $u-1$ for $u > 1$, then $p(s) = 0$ for $0 \leqslant s < 1$ and 1 for $s \geqslant 1$, and for $u > 1$ we have

$$\int_0^u (u-s)\,dp(s) = u[p(0^+)-p(0)] + (u-1)[p(1^+)-p(1^-)] = u-1.$$

3° If $\varphi(u) = u$ for $0 \leqslant u \leqslant 1$ and u^2 for $u > 1$, then $p(s) = 1$ for $0 \leqslant s < 1$ and $2s$ for $s \geqslant 1$, and for $u > 1$ we have

$$\int_0^u (u-s)\,dp(s) = u[p(0^+)-p(0)] + (u-1)[p(1^+)-p(1^-)] + \int_1^u (u-s)2\,ds$$
$$= u-1+2u^2-u^2-2u+1 = u^2-u.$$

In the sequel (Ω', Σ', ν) will be a σ-finite measure space. The basic result is

THEOREM 1 (Main Theorem). *Let $T\colon L_1(\mu)+L_\infty(\mu) \to L_1(\nu)+L_\infty(\nu)$ be an operator such that T maps $L_1(\mu)$ into $L_1(\nu)$ and $L_\infty(\mu)$ into $L_\infty(\nu)$, and*

(3) $$\|Tx-Ty\|_1 \leqslant M\|x-y\|_1 \quad \forall\, x, y \in L_1(\mu),$$

(4) $$\|Tx\|_\infty \leqslant M\|x\|_\infty \quad \forall\, x \in L_\infty(\mu).$$

Then T maps $L_\varphi(\mu) \cap L_1(\mu)$ into $L_\varphi(\nu)$ and

(5) $$I_\varphi(Tx/M) \leqslant I_\varphi(x) \quad \forall\, x \in L^0_\varphi(\mu) \cap L_1(\mu),$$

(6) $$\|Tx\|_\varphi \leqslant M\|x\|_\varphi \quad \forall\, x \in L_\varphi(\mu) \cap L_1(\mu).$$

First proof. By taking T/M instead of T if necessary, we may assume that $M = 1$. First we prove that if (4) holds and M is 1 then for each $x \in L_1(\mu) + L_\infty(\mu)$

(7) $$|Tx(t)-(Tx)^{(\alpha)}(t)| \leqslant |Tx(t)-T(x^{(\alpha)})(t)| \quad \nu\text{-a.e.}$$

Indeed, if $|Tx(t)| \leqslant \alpha$ then (7) is obvious. On the other hand, if $|Tx(t)| > \alpha$ then since $\|T(x^{(\alpha)})\|_\infty \leqslant \|x^{(\alpha)}\|_\infty \leqslant \alpha$, it follows that $|T(x^{(\alpha)})(t)| \leqslant \alpha$ ν-a.e. Hence

$$|Tx(t)-(Tx)^{(\alpha)}(t)| = |Tx(t)-\alpha\,\mathrm{sgn}\,Tx(t)| = |Tx(t)|-\alpha$$
$$\leqslant |Tx(t)|-|(Tx)^{(\alpha)}(t)| \leqslant |Tx(t)-T(x^{(\alpha)})(t)| \quad \nu\text{-a.e.}$$

Now, if $x \in L^0_\varphi(\mu) \cap L_1(\mu)$ then from the representation (2) of φ and the Fubini theorem

$$I_\varphi(Tx) = \int_{\Omega'} \varphi(|Tx(t)|)\,d\nu = \int_{\Omega'} \Big[a|Tx(t)| + \int_0^\infty (|Tx(t)|-s)_+\,dp(s)\Big]\,d\nu$$
$$= a\|Tx\|_1 + \int_0^\infty \int_{\Omega'} (|Tx(t)|-s)_+\,d\nu\,dp(s)$$
$$= a\|Tx\|_1 + \int_0^\infty \int_{\Omega'} |Tx(t)-(Tx)^{(s)}(t)|\,d\nu\,dp(s).$$

Using (7) and (3) with $M = 1$ we have

$$I_\varphi(Tx) \leqslant a\|Tx\|_1 + \int_0^\infty \|Tx-T(x^{(s)})\|_1\,dp(s) \leqslant a\|x\|_1 + \int_0^\infty \|x-x^{(s)}\|_1\,dp(s)$$
$$= a\|x\|_1 + \int_0^\infty \int_\Omega (|x(t)|-s)_+\,d\mu\,dp(s).$$

Again, from the Fubini theorem and (2)

$$I_\varphi(Tx) \leqslant \int_\Omega \Big[a|x(t)| + \int_0^\infty (|x(t)|-s)_+\,dp(s)\Big]\,d\mu = \int_\Omega \varphi(|x(t)|)\,d\mu = I_\varphi(x).$$

Hence $Tx \in L^0_\varphi(\nu)$ and $I_\varphi(Tx) \leqslant I_\varphi(x)$.

The remaining assertions follow immediately from the above and the definitions of L_φ space and $\|\cdot\|_\varphi$-norm.

Second proof (when $\nu(\Omega') \leqslant \mu\Omega$). As in the first proof, let $M = 1$. Given $u > 0$ and $x \in L_1(\mu)$, let $\alpha = x^*(u)$ and $x_1 = x^{(\alpha)}$. Then by properties (iv) and (iii) of rearrangement

$$\int_0^u (Tx)^*_\nu(t)\,dt \leqslant \int_0^u (Tx-Tx_1)^*_\nu(t)\,dt + \int_0^u (Tx_1)^*_\nu(t)\,dt$$
$$\leqslant \int_0^\infty (Tx-Tx_1)^*_\nu(t)\,dt + u\|Tx_1\|_\infty = \int_{\Omega'} |Tx(t)-Tx_1(t)|\,d\nu + u\|Tx_1\|_\infty.$$

Using (3), (4), (iii) and the fact that $d_x(\alpha) \leqslant u$ we get

$$\int_0^u (Tx)^*_\nu(t)\,dt \leqslant \int_\Omega |x(t)-x_1(t)|\,d\mu + u\|x_1\|_\infty = \int_\Omega (|x(t)|-\alpha)_+\,d\mu + u\alpha$$
$$\leqslant \int_0^{d_x(\alpha)} (x^*_\mu(t)-\alpha)\,dt + u\alpha \leqslant \int_0^u (x^*_\mu(t)-\alpha)\,dt + u\alpha = \int_0^u x^*_\mu(t)\,dt.$$

Hence, if $x \in L_1(\mu)$ then

(8) $$\int_0^u (Tx)^*_\nu(t)\,dt \leqslant \int_0^u x^*_\mu(t)\,dt \quad \text{for all } 0 \leqslant u \leqslant \mu\Omega.$$

Now, we prove that if (8) holds and $\nu(\Omega') \leqslant \mu\Omega$ then for $x \in L^0_\varphi(\mu) \cap L_1(\mu)$ we have

$$\int_{\Omega'} \varphi(|Tx(t)|)\,d\nu \leqslant \int_\Omega \varphi(|x(t)|)\,d\mu.$$

Indeed, let $A_s = \{t > 0\colon x^*_\mu(t) > s\}$, $B_s = \{t > 0\colon (Tx)^*_\nu(t) > s\}$ and $a_s = mA_s$, $b_s = mB_s$. Then from property (iii) of rearrangement, representation (2) of φ and the Fubini theorem we have

$$\begin{aligned} I_\varphi(Tx) &= \int_{\Omega'} \varphi(|Tx(t)|)\,dv = \int_0^{\nu(\Omega')} \varphi((Tx)^*_\nu(t))\,dt \\ &= a \int_0^{\nu(\Omega')} (Tx)^*_\nu(t)\,dt + \int_0^\infty \int_0^{\nu(\Omega')} [(Tx)^*_\nu(t)-s]_+\,dt\,dp(s) \\ &\leqslant a \int_0^{\nu(\Omega')} (Tx)^*_\nu(t)\,dt + \int_0^\infty \int_0^{b_s} [(Tx)^*_\nu(t)-s]_+\,dt\,dp(s) \end{aligned}$$

and from the assumption (8), this is

$$\leqslant a \int_0^{\mu\Omega} x^*_\mu(t)\,dt + \int_0^\infty \Big(\int_0^{b_s} x^*_\mu(t)\,dt - sb_s\Big)\,dp(s).$$

But now, if $b_s \leqslant a_s$ then

$$\begin{aligned} \int_0^{b_s} x^*(t)\,dt - sb_s &= \int_0^{a_s} x^*(t)\,dt - \int_{b_s}^{a_s} x^*(t)\,dt - sb_s \\ &\leqslant \int_0^{a_s} x^*(t)\,dt - x^*(a_s)(a_s-b_s) - sb_s \\ &\leqslant \int_0^{a_s} x^*(t)\,dt - s(a_s-b_s) - sb_s \\ &= \int_0^{a_s} x^*(t)\,dt - sa_s, \end{aligned}$$

and if $b_s > a_s$ then

$$\int_0^{b_s} x^*(t)\,dt - sb_s = \int_0^{a_s} x^*(t)\,dt + \int_{a_s}^{b_s} x^*(t)\,dt - sb_s,$$

and $t > a_s$ implies $x^*(t) \leqslant s$, so the last expression is

$$\leqslant \int_0^{a_s} x^*(t)\,dt + s(b_s-a_s) - sb_s = \int_0^{a_s} x^*(t)\,dt - sa_s.$$

Hence, from the Fubini theorem, (2) and (iii) we get

$$\begin{aligned} I_\varphi(Tx) &\leqslant a \int_0^{\mu\Omega} x^*(t)\,dt + \int_0^\infty \Big(\int_0^{a_s} x^*(t)\,dt - sa_s\Big)\,dp(s) \\ &= a \int_0^{\mu\Omega} x^*(t)\,dt + \int_0^\infty \int_0^{\mu\Omega} [x^*(t)-s]_+\,dt\,dp(s) \\ &= \int_0^{\mu\Omega} \varphi(x^*(t))\,dt = \int_\Omega \varphi(|x(t)|)\,d\mu = I_\varphi(x). \end{aligned}$$

In the case when T is a linear operator and $\varphi(u) = u^p$ we have a simple proof of a particular case of the M. Riesz interpolation theorem, i.e., for $p_0 = 1$, $p_1 = \infty$ and $1 < p < \infty$ (cf. also [8], where it is assumed that T is also positive). In the nonlinear case with $\varphi(u) = u^p$ the above theorem follows also from Lions [12] and Peetre [19] (see also Krengel and Lin [11], where additionally T is order preserving). Brezis and Strauss [1] proved Theorem 1 under the additional assumption of T being positive. Moreover, their theorem is for convex lower semicontinuous functions φ on $\boldsymbol{R}$ such that $\min\varphi = \varphi(0) = 0$. In [14] it is proved that if both measure spaces are the same and either nonatomic or counting then (3) and (4) with $M = 1$ imply (8). Then the Calderón–Mityagin theorem (see [9], p. 105) and the fact that the space $L_\varphi(\mu) \cap L_1(\mu)$ has the Fatou property imply (6). However, in this paper we wanted to prove the theorem without references to other theorems.

3. Interpolation of Lipschitz operators. Using Theorem 1 and considerations from Orlicz's paper [18] we prove a more general version of Orlicz's theorem about interpolation of Lipschitz operators in L_1 and L_∞.

THEOREM 2. *Let $T\colon L_1(\mu)+L_\infty(\mu) \to L_1(\nu)+L_\infty(\nu)$ be an operator such that T maps $L_1(\mu)$ into $L_1(\nu)$ and $L_\infty(\mu)$ into $L_\infty(\nu)$, and*

(3) $$\|Tx-Ty\|_1 \leqslant M\|x-y\|_1 \quad \forall\, x, y \in L_1(\mu),$$

(4′) $$\|Tx-Ty\|_\infty \leqslant M\|x-y\|_\infty \quad \forall\, x, y \in L_\infty(\mu).$$

Then T maps $L_\varphi(\mu) \cap L_1(\mu)$ into $L_\varphi(\nu)$ and

(6′) $$\|Tx-Ty\|_\varphi \leqslant M\|x-y\|_\varphi \quad \forall\, x, y \in L_\varphi(\mu) \cap L_1(\mu).$$

Proof. For any fixed $x_0 \in L_1(\mu) \cap L_\infty(\mu)$ and for $x \in L_1(\mu)+L_\infty(\mu)$ let

$$T_1 x := T(x+x_0) - Tx_0.$$

Then

$$\|T_1x - T_1y\|_1 = \|T(x+x_0)-T(y+x_0)\|_1 \leqslant M\|x-y\|_1 \quad \forall\, x, y \in L_1(\mu),$$

$$\|T_1x\|_\infty = \|T(x+x_0)-Tx_0\|_\infty \leqslant M\|x\|_\infty \quad \forall\, x \in L_\infty(\mu).$$

From Theorem 1 we get

$$\|T_1x\|_\varphi \leqslant M\|x\|_\varphi \quad \forall\, x \in L_\varphi(\mu) \cap L_1(\mu).$$

This means that

$$\|T(x+x_0)-Tx_0\|_\varphi \leqslant M\|x\|_\varphi \quad \forall\, x \in L_\varphi(\mu) \cap L_1(\mu), \quad \text{or}$$

(9) $$\|Tx-Tx_0\|_\varphi \leqslant M\|x-x_0\|_\varphi \quad \forall\, x \in L_\varphi(\mu) \cap L_1(\mu),\ x_0 \in L_1(\mu) \cap L_\infty(\mu).$$

For arbitrary $x, y \in L_\varphi(\mu) \cap L_1(\mu)$ we consider the truncations $x^{(k)}$, $y^{(k)}$. Then $z_k := T(x^{(k)}) - T(y^{(k)})$ converges to $Tx-Ty$ in the $L_1(\nu)$-norm, by (3) and the continuity of the $\|\cdot\|_1$-norm. Consequently, the same convergence holds in the measure ν. Therefore, for a properly chosen sequence k_n, the sequence z_{k_n} converges ν-a.e. to $Tx-Ty$. Then $x_n := x^{(k_n)}$ and $y_n := y^{(k_n)}$ have the following property:

$$(10)\quad \begin{aligned} &x_n,\, y_n \in L_1(\mu)\cap L_\infty(\mu), \quad |x_n-y_n| \leqslant |x-y| \quad \mu\text{-a.e.},\\ &Tx_n - Ty_n \to Tx - Ty \quad \nu\text{-a.e.} \end{aligned}$$

Now, by (10), the Fatou property of the norm and (9) we get

$$\|Tx-Ty\|_\varphi \leqslant \liminf_{n\to\infty} \|Tx_n - Ty_n\|_\varphi \leqslant M \liminf_{n\to\infty} \|x_n-y_n\|_\varphi \leqslant M\|x-y\|_\varphi .$$

COROLLARY 1 (Orlicz's Theorem, 1954). *If T: $L_1(a,b)\to L_1(a,b)$ is a Lipschitz operator in $L_1(a,b)$ and in $L_\infty(a,b)$ then T is also Lipschitz in $L_\varphi(a,b)$.*

COROLLARY 2. *If the operator T: $L_1(\mu)\cap L_\infty(\mu)\to L_1(\nu)\cap L_\infty(\nu)$ satisfies* (3) *and* (4) *for $x, y\in L_1(\mu)\cap L_\infty(\mu)$ then* (6) *holds for $x, y \in L_1(\mu)\cap L_\infty(\mu)$.*

Corollary 2 with the additional assumption that $L_1(\mu)\cap L_\infty(\mu)$ is dense in both $L_\varphi(\mu)$ and $L_\infty(\mu)$ (this means that $\mu\Omega<\infty$ and $L_\varphi = L_\varphi^a$) is a particular case of a general theorem of Browder [2].

Remark 1. If $\mu\Omega = \infty$ and for arbitrary $x, y\in L_\varphi(\mu)$ it is possible to construct sequences x_n, y_n with the property (10) then it easy to see that (6) holds even for $x, y\in L_\varphi(\mu)$. On the other hand, if $\mu\Omega=\infty$ and $L_\varphi(\mu)\cap L_1(\mu)$ is dense in $L_\varphi(\mu)$ then (6) also holds for $x, y\in L_\varphi(\mu)$, by continuity.

We prove now that the density of $L_\varphi\cap L_1$ in L_φ is equivalent to condition δ_2 for φ for small u.

PROPOSITION 1. *Let μ be a σ-finite measure and $\mu\Omega=\infty$. Assume that Ω contains a nonatomic part of infinite measure or that there are atoms $\{e_n\}_{n=1}^\infty$ such that $0<\inf_n \mu e_n \leqslant \sup_n \mu e_n<\infty$. Then the following conditions are equivalent:*

(a) *$L_\varphi(\mu)\cap L_1(\mu)$ is dense in $L_\varphi(\mu)$.*
(b) *$L_\varphi(\mu)\cap L_\infty(\mu) = L_\varphi^a(\mu)\cap L_\infty(\mu)$.*
(c) *φ satisfies condition δ_2: $\limsup_{u\to 0^+} \varphi(2u)/\varphi(u)<\infty$.*

Proof. (a)$\Rightarrow$(b). If $0\leqslant x\in L_\varphi\cap L_\infty$ then by assumption there is a sequence $0\leqslant x_n \in L_\varphi\cap L_1$ such that $\|x-x_n\|_\varphi\to 0$. Let $y_n=\min(x_n, \|x\|_\infty)$. Then $y_n \in L_1\cap L_\infty$ and

$$|x-y_n| = |\min(x, \|x\|_\infty) - \min(x_n, \|x\|_\infty)| \leqslant |x-x_n| \quad \mu\text{-a.e.},$$

i.e., $\|x-y_n\|_\varphi \to 0$. If we prove that $y_n\in L_\varphi^a$ then $x\in L_\varphi^a$ (because L_φ^a is closed in L_φ) and so $x\in L_\varphi^a\cap L_\infty$. For any fixed $r>0$ let

$$c=\varphi(r\|y_n\|_\infty)/(r\|y_n\|_\infty) \leqslant \varphi(r\|x\|_\infty)/(r\|x\|_\infty) < \infty .$$

Then $I_\varphi(ry_n)\leqslant cr\int_\Omega y_n(t)\,d\mu<\infty$ and so $y_n\in L_\varphi^a$.

(b)$\Rightarrow$(a). Let $x\in L_\varphi$ and $r>0$ be such that $I_\varphi(rx)<\infty$. If $A_n=\{t\in\Omega:\ r|x(t)| > n^{-1}\}$ then

$$\varphi(n^{-1})\mu A_n = \int_{A_n} \varphi(n^{-1})\,d\mu \leqslant \int_{A_n} \varphi(r|x(t)|)\,d\mu \leqslant I_\varphi(rx)<\infty$$

for any natural number n. Hence $\mu A_n<\infty$ and so $x1_{A_n}\in L_\varphi\cap L_1$.

On the other hand, there is a sequence (s_n) of simple functions such that $s_n\nearrow x$ μ-a.e. Of course, $s_n\in L_\varphi\cap L_1$. Then putting

$$x_n = s_n 1_{\Omega\setminus A_n} + x1_{A_n}$$

we have $x_n\in L_\varphi\cap L_1$. Moreover, $(x-s_n)1_{\Omega\setminus A_n}\downarrow 0$, $x-x1_{A_n} = x1_{\Omega\setminus A_n}\downarrow 0$ and $x1_{\Omega\setminus A_n}\in L_\varphi\cap L_\infty = L_\varphi^a\cap L_\infty$. Therefore

$$\|x-x_n\|_\varphi \leqslant \|(x-s_n)1_{\Omega\setminus A_n}\|_\varphi + \|x-x1_{A_n}\|_\varphi \to 0 \quad \text{as } n\to\infty .$$

(b)$\Rightarrow$(c). Assume that $\limsup_{u\to 0^+}\varphi(2u)/\varphi(u)=\infty$. Let $(x_n)\in l_\varphi\setminus l_\varphi^a$. If μ is nonatomic then there are pairwise disjoint sets B_n such that $\mu B_n=1$ for each natural number n. Let

$$x=\sum_{n=1}^\infty x_n 1_{B_n} \quad (\text{convergence in } \mu).$$

Then $x\in(L_\varphi\cap L_\infty)\setminus(L_\varphi^a\cap L_\infty)$. In the second case, let

$$x=\sum_{n=1}^\infty x_n 1_{e_n} \quad (\text{convergence in } \mu);$$

then also $x\in(L_\varphi\cap L_\infty)\setminus(L_\varphi^a\cap L_\infty)$.

(c)$\Rightarrow$(b). Let $x\in L_\varphi\cap L_\infty$ and $I_\varphi(rx)<\infty$ for some $r>0$. Put

$$c_n = \sup\{\varphi(2u)/\varphi(u):\ 0<u\leqslant 2^{n-1}r\|x\|_\infty\}.$$

Then $c_n<\infty$ (because φ satisfies condition δ_2) and

$$I_\varphi(2^n rx)\leqslant c_n I_\varphi(2^{n-1}rx)\leqslant \ldots \leqslant \prod_{k=1}^n c_k I_\varphi(rx)<\infty,$$

i.e., $rx\in L_\varphi^a$ and so $x\in L_\varphi^a\cap L_\infty$.

It may not be a simple matter to check directly whether an operator is nonexpansive in L_φ. Then the following result may be useful:

COROLLARY 3. *If T: $L_1(\mu)+L_\infty(\mu)\to L_1(\mu)+L_\infty(\mu)$ is nonexpansive in $L_1(\mu)$ and in $L_\infty(\mu)$, then T is nonexpansive in $L_\varphi(\mu)$ provided $\mu\Omega<\infty$ or μ is such as in Proposition 1 and φ satisfies condition δ_2.*

Remark 2. If an operator T is such as in Theorem 2, then (with the same proof) $I_\varphi((Tx-Ty)/M)\leqslant I_\varphi(x-y)$ for $x, y\in L_1(\mu)$ such that $x-y\in L_\varphi^0(\mu)$. We note that in [1], this is proved under the additional assumption that T is positive. Moreover, if for a space X the inclusions $L_1(\mu)+L_\infty(\mu) \subset X \subset S(\mu)$ hold, and for any $x, y \in X$ it is possible to construct sequences x_n, y_n with the property (10) then

$$I_\varphi\left(\frac{Tx-Ty}{M}\right) \leqslant I_\varphi(x-y) \quad \text{for } x, y \in X. \tag{11}$$

4. Interpolation of pointwise nonlinear operators. In many problems of analysis, an operator T as considered above is not linear, but order preserving and C-sublinear ($C \geqslant 1$) in X, i.e., it satisfies

$$\begin{cases} T \text{ maps } X \subset S(\mu) \text{ into } S(\nu) \text{ and for any } x, y \in X, \lambda \in \mathbf{R}: \\ 0 \leqslant x \leqslant y \text{ implies } Tx \leqslant Ty, \\ |T(\lambda x)| = |\lambda|\,|Tx|, \\ |T(x+y)| \leqslant C(|Tx|+|Ty|). \end{cases} \tag{12}$$

For such type of operators the proof of the interpolation theorem is simpler than the proof of Theorem 1.

THEOREM 3. *Let T be an order preserving and C-sublinear operator in $X \supset L_\infty(\mu)$ and let*

$$\|Tx\|_\infty \leqslant M\|x\|_\infty \quad \forall\, x \in L_\infty(\mu). \tag{4}$$

Then for $x \in X$

$$\varphi(T(|x|)) \leqslant M^{-1}T(\varphi(CM|x|)) \quad \nu\text{-a.e.} \tag{13}$$

Moreover, if we assume that $X \supset L_1(\mu)+L_\infty(\mu)$ and

$$\|Tx\|_1 \leqslant M\|x\|_1 \quad \forall\, x \in L_1(\mu), \tag{3'}$$

then

$$I_\varphi(Tx/(CM)) \leqslant I_\varphi(x) \quad \forall\, x \in L^0_\varphi(\mu), \tag{5'}$$

$$\|Tx\|_\varphi \leqslant CM\|x\|_\varphi \quad \forall\, x \in L_\varphi(\mu). \tag{6'}$$

Proof. Let $\varphi^*(v) = \sup_{u \geqslant 0}[uv-\varphi(u)]$ and $\varphi^{-1}(v) = \inf\{u > 0:\ \varphi(u) > v\}$. Then $\varphi^{**} = \varphi$, $\varphi(\varphi^{-1}(u)) \leqslant u$ and by the Young inequality

$$u\varphi^{*-1}(v) \leqslant \varphi(u)+\varphi^*(\varphi^{*-1}(v)) \leqslant \varphi(u)+v.$$

Therefore, for any $v > 0$,

$$|x| \leqslant \frac{\varphi(|x|)+v}{\varphi^{*-1}(v)} \quad \mu\text{-a.e.}$$

and from (12)

$$T(|x|) \leqslant C\frac{T\varphi(|x|)+vT1}{\varphi^{*-1}(v)} \quad \nu\text{-a.e.}$$

Then using (4) we get

$$T(|x|) \leqslant C\frac{T\varphi(|x|)+Mv}{\varphi^{*-1}(v)} \quad \nu\text{-a.e.}$$

Note that

$$\varphi^{*-1}(u) = \inf_{v>0}\frac{u+v}{\varphi^{-1}(v)};$$

if φ' is an increasing function then this follows from the fact that for $v = \varphi'(u)$ we have $u+v = \varphi^{-1}(u)\varphi^{*-1}(v)$—equality in the Young theorem. For the general case, see, for example, [15], Lemma 2. Thus

$$T(|x|) \leqslant CM\varphi^{**-1}(T\varphi(|x|)/M) = CM\varphi^{-1}(T\varphi(|x|)/M)$$

and (13) holds. The proof of the next part follows immediately from (13) and (3′).

COROLLARY 4. *Order preserving and sublinear operators in L_1+L_∞ which are nonexpansive in L_1 and in L_∞ are also nonexpansive in Orlicz spaces L_φ.*

Of course, the above corollary holds also for positive nonexpansive linear operators. We wish to point out that this result generalizes the corresponding result in [10] for linear positive operators on L_p spaces.

5. Applications. We now consider some applications of the results of Sections 2 and 3.

a) *Orlicz's interpolation theorem for linear operators.* Looking at the first proof of Theorem 1 and using the proposition given below, which is of independent interest, we will have a simple proof of the well known fact that any Orlicz space is an interpolation space between L_1 and L_∞ for linear operators.

PROPOSITION 2. *For any Orlicz space L_φ either $L_\varphi \subset (L_1+L_\infty)^a$ or $L_\varphi = L_{\varphi_0}+L_\infty$ for some φ_0 with $L_{\varphi_0} \subset (L_1+L_\infty)^a$.*

Proof. First, we note that $(L_1+L_\infty)^a = \overline{L_1 \cap L_\infty}$, where $\overline{L_1 \cap L_\infty}$ denotes the closure of $L_1 \cap L_\infty$ in L_1+L_∞.

If $\mu\Omega < \infty$ then there is nothing to prove, because $L_\infty \subset L_\varphi \subset L_1$ and L_∞ is dense in L_1. Therefore, let $\mu\Omega = \infty$.

We divide the remaining proof into two parts (always $\mu\Omega = \infty$):

I. If $\lim_{u\to 0^+}\varphi^{-1}(u) = 0$, then $L_\varphi \subset \overline{L_1 \cap L_\infty}$. Indeed, we have by Hölder's inequality, for $x \in L_\varphi$

$$\begin{aligned} x^*(t) &\leqslant t^{-1}\int_0^t x^*(s)\,ds = t^{-1}\|x^*1_{[0,t]}\|_1 \leqslant 2t^{-1}\|x^*\|_\varphi\|1_{[0,t]}\|_{\varphi^*} \\ &= 2t^{-1}\|x^*\|_\varphi/\varphi^{*-1}(1/t) \leqslant 2\varphi^{-1}(1/t)\|x\|_\varphi, \end{aligned}$$

and so $\lim_{t\to\infty} x^*(t) = 0$. Hence, $d_x(\lambda) < \infty$ for any $\lambda > 0$ and writing $A_k = \{t\in\Omega\colon |x(t)| > 1/k\}$ $(k = 1, 2, \ldots)$ we have $\mu A_k < \infty$ for any k. Thus $x1_{A_k}\in L_1$ and

$$\|x-x1_{A_k}\|_{L_1+L_\infty} \leqslant \|x-x1_{A_k}\|_\infty \leqslant 1/k,$$

and so $x\in\overline{L_1}$. But $\overline{L_1} = \overline{L_1} \cap \overline{L_\infty} = \overline{L_1 \cap L_\infty}$ and the result follows.

II. If $\lim_{u\to 0^+} \varphi^{-1}(u) = b > 0$, then φ_0 is an Orlicz function, $\lim_{u\to 0^+} \varphi_0^{-1}(u) = 0$ and $L_\varphi = L_{\varphi_0}+L_\infty$, where $\varphi_0(u) = \varphi(u+b)$. Indeed, if $x\in L_\varphi$ and $I_\varphi(rx) < \infty$, and $B_r = \{t\in\Omega\colon r|x(t)| < 2b\}$, then $x1_{B_r}\in L_\infty$ and

$$I_{\varphi_0}(\tfrac{1}{2}rx1_{\Omega\setminus B_r}) = \int_{\Omega\setminus B_r} \varphi(\tfrac{1}{2}r|x(t)|+b)\,d\mu$$
$$\leqslant \int_{\Omega\setminus B_r} \varphi(r|x(t)|)\,d\mu \leqslant I_\varphi(rx) < \infty,$$

i.e., $x1_{\Omega\setminus B_r}\in L_{\varphi_0}$, which proves the inclusion $L_\varphi \subset L_{\varphi_0}+L_\infty$.

Conversely, let $x\in L_{\varphi_0}+L_\infty$, i.e., $x = x_0+x_1$, where $x_0\in L_{\varphi_0}$ and $x_1\in L_\infty$. Choose $r > 0$ such that $r|x_1(t)| < b$ μ-a.e. and $I_{\varphi_0}(rx_0) < \infty$. Then putting $A = \{t\in\Omega\colon r|x_0(t)| > b\}$ we have

$$I_\varphi(\tfrac{1}{2}rx) = I_\varphi(\tfrac{1}{2}r(x_0+x_1)) \leqslant I_\varphi(rx_0)+I_\varphi(rx_1) = I_\varphi(rx_0)$$
$$= \int_A \varphi(r|x_0(t)|)\,d\mu = \int_A \varphi_0(r|x_0(t)|-b)\,d\mu$$
$$\leqslant \int_A \varphi_0(r|x_0(t)|)\,d\mu \leqslant I_{\varphi_0}(rx_0) < \infty.$$

Hence $x\in L_\varphi$ and so $L_{\varphi_0}+L_\infty \subset L_\varphi$.

THEOREM 1'. *Let $T\colon L_1(\mu)+L_\infty(\mu)\to L_1(\nu)+L_\infty(\nu)$ be a linear operator which is bounded from $L_1(\mu)$ into $L_1(\nu)$ with norm M_1 and from $L_\infty(\mu)$ into $L_\infty(\nu)$ with norm M_2. Then T is bounded from $L_\varphi(\mu)$ into $L_\varphi(\nu)$ with norm $M \leqslant C\max(M_1, M_2)$, where the constant C depends on φ and μ.*

Proof. For any $x\in L_\varphi(\mu)$ the sequence $x_n = x1_{A_n}$ is in $L_\varphi(\mu)\cap L_1(\mu)$, where $A_n \nearrow \Omega$ and $\mu A_n < \infty$ (such a sequence of sets exists because μ is σ-finite).

If $L_\varphi \subset (L_1+L_\infty)^a$ then $\|x-x_n\|_{L_1+L_\infty}\to 0$ and the boundedness of T from $L_1(\mu)+L_\infty(\mu)$ into $L_1(\nu)+L_\infty(\nu)$ implies $\|Tx-Tx_n\|_{L_1+L_\infty}\to 0$. Thus $Tx_{n_k}\to Tx$ ν-a.e. By the Fatou lemma and Theorem 1 we have

$$\|Tx\|_\varphi \leqslant \liminf_{k\to\infty} \|Tx_{n_k}\|_\varphi \leqslant \max(M_1, M_2)\liminf_{k\to\infty} \|x_{n_k}\|_\varphi \leqslant \max(M_1, M_2)\|x\|_\varphi.$$

If $L_\varphi = L_{\varphi_0}+L_\infty$ with $L_{\varphi_0} \subset (L_1+L_\infty)^a$ then from the above T is bounded in L_{φ_0}. Therefore T, as a linear operator, is bounded in $L_{\varphi_0}+L_\infty = L_\varphi$.

Note that $C = 1$ if either $\mu\Omega < \infty$ or $\lim_{u\to 0^+} \varphi^{-1}(u) = 0$.

Consequently, we have the following result of Orlicz (1935).

THEOREM A. *Let $T\colon L_1(a, b)\to L_1(a, b)$ be a linear bounded operator with norm M_1 which is also bounded in $L_\infty(a, b)$ with norm M_2. Then T is bounded in $L_\varphi(a, b)$ with norm $M \leqslant \max(M_1, M_2)$.*

b) *Inequalities for rearrangements.* Let $T\colon S_0(\mu)\to S_0(m)$ be defined by $Tx = x_\mu^*$. Then from property (iii) of rearrangement $\int_0^\infty Tx(t)\,dt = \int_\Omega |x|\,d\mu$. We prove now that T is a nonexpansive mapping from $L_1(\mu)$ into $L_1(0, \infty)$ and from $L_\infty(\mu)$ into $L_\infty(0, \infty)$.

If $x, y\in L_1(\mu)$ then

$$\|Tx-Ty\|_1 = \int_0^\infty (Tx-Ty)_+\,dt+\int_0^\infty (Ty-Tx)_+\,dt$$
$$\leqslant \int_0^\infty \bigl[T(\max(|x|, |y|))-Ty\bigr]\,dt+\int_0^\infty \bigl[T(\max(|x|, |y|))-Tx\bigr]\,dt$$
$$= \int_\Omega [\max(|x|, |y|)-|y|]\,d\mu+\int_\Omega [\max(|x|, |y|)-|x|]\,d\mu$$
$$= \int_\Omega \bigl||x|-|y|\bigr|\,d\mu \leqslant \|x-y\|_1.$$

If $x, y\in L_\infty(\mu)$ then $|x| \leqslant \|x-y\|_\infty+|y|$, $|y| \leqslant \|x-y\|_\infty+|x|$. Thus $x^* \leqslant \|x-y\|_\infty+y^*$ and $y^* \leqslant \|x-y\|_\infty+x^*$. Hence

$$\|Tx-Ty\|_\infty = \|x^*-y^*\|_\infty \leqslant \|x-y\|_\infty.$$

Note that for $x, y\in S_0(\mu)$ it is possible to construct sequences x_n, y_n with the property (10). Indeed, let $x_n = x^{(n)}1_{A_n}$ and $y_n = y^{(n)}1_{A_n}$, where $A_n \nearrow \Omega$ and $\mu A_n < \infty$ (such a sequence of sets exists because μ is σ-finite). Then x_n, $y_n\in L_1(\mu)\cap L_\infty(\mu)$ and from property (1) of truncation and property (i) of rearrangement we get $|x_n-y_n| \leqslant |x^{(n)}-y^{(n)}| \leqslant |x-y|$ and $x_n^*-y_n^*\to x^*-y^*$ a.e.

From the above and Remark 2 we have

THEOREM B. *For each Orlicz function φ and for any $x, y\in S_0(\mu)$*

$$I_\varphi(x^*-y^*) \leqslant I_\varphi(x-y).$$

This inequality is proved in [3] for the case $L_p(0, 1)$ and in [4] for the case when $\Omega = \boldsymbol{R}^n$ with Lebesgue measure. Note that in [4] this inequality is claimed to hold for $x, y\in S(\boldsymbol{R}^n)$. This is a misunderstanding because if $x, y\in S(\boldsymbol{R}^n)\setminus S_0(\boldsymbol{R}^n)$, then $x^* = y^* = \infty$ and the left side of the inequality is not defined.

THEOREM C (Hardy, Littlewood and Pólya). *Let $x, y\in L_1(\mu)+L_\infty(\mu)$. Then $\int_0^u x^*(s)\,ds \leqslant \int_0^u y^*(s)\,ds$ for any $0 < u < \mu\Omega$ if and only if $I_\varphi(x) \leqslant I_\varphi(y)$ for any Orlicz function φ.*

The proof that the inequality for rearrangements implies that for I_φ is the same as the second proof of Theorem 1. On the other hand, if $I_\varphi(x) \leqslant I_\varphi(y)$ for any Orlicz function then $\int_\Omega (|x|-t)_+\,d\mu \leqslant \int_\Omega (|y|-t)_+\,d\mu$ for any $t > 0$, because $\varphi_t(u) = (u-t)_+$ is an Orlicz function.

Let $0 < u < \mu\Omega$ and $t = y^*(u)$. Then

$$\int_0^u [x^*(s)-t]\,ds \leqslant \int_0^{\mu\Omega} [x^*(s)-t]_+\,ds = \int_0^\infty d_{(x^*-t)_+}(\lambda)\,d\lambda$$

$$= \int_0^\infty d_{x^*}(t+\lambda)\,d\lambda = \int_0^\infty d_x(t+\lambda)\,d\lambda = \int_0^\infty d_{(|x|-t)_+}(\lambda)\,d\lambda$$

$$= \int_\Omega (|x|-t)_+\,d\mu \leqslant \int_\Omega (|y|-t)_+\,d\mu = \int_0^{\mu\Omega} [y^*(s)-t]_+\,ds = \int_0^u [y^*(s)-t]\,ds,$$

and the proof is complete.

From Theorems B and C easily follows

THEOREM D (Lorentz–Shimogaki inequality [13]). *If* $x, y \in L_1(\mu)+L_\infty(\mu)$, *then for* $0 < u < \mu\Omega$

$$\int_0^u (x^*-y^*)^*(s)\,ds \leqslant \int_0^u (x-y)^*(s)\,ds.$$

c) *Modulus of continuity of equimeasurable functions.* Let us now confine our attention to periodic functions on $[0,1]$ with period 1. Given an $x \in L_\varphi(0, 1)$, the expression

$$\omega_\varphi(t, x) = \sup_{0 \leqslant h \leqslant t} \|x(\cdot+h)-x(\cdot)\|_\varphi$$

is called the *integral modulus of continuity*, in L_φ, of x. Using Brudnyi's result in [3] for $L_p(0, 1)$ spaces and Orlicz's theorem we prove

THEOREM E (Brudnyi inequality). *If* $x \in L_\varphi(0, 1)$ *then*

$$\omega_\varphi(t, x^*) \leqslant 3\omega_\varphi(t, x).$$

Proof. Theorem B (or Orlicz's theorem, i.e., the special case of our Theorem 2) implies that the operator T: $L_1(0, 1) \to L_1(0, 1)$ defined by $Tx = x^*$ is nonexpansive in $L_\varphi(0, 1)$.

The Pólya–Szegö inequality means that $\|(Tx)'\|_\varphi \leqslant \|x'\|_\varphi$ for $x \in \dot{L}_\varphi := \{x \in L_\varphi;\ x \in AC,\ x' \in L_\varphi\}$ (see [20]; see also [6], Th. 4.1 or [7], Th. 1).

Now, for $x \in L_\varphi$ and $t > 0$, consider the Steklov average function

$$x_t(u) = t^{-1}\int_u^{u+t} x(s)\,ds = t^{-1}\int_0^t x(s+u)\,ds.$$

Then

$$\|x_t - x\|_\varphi \leqslant t^{-1}\int_0^t \|x(\cdot+s)-x\|_\varphi\,ds \leqslant t^{-1}\int_0^t \omega_\varphi(s, x)\,ds \leqslant \omega_\varphi(t, x),$$

$x_t \in \dot{L}_\varphi$ and $\|x_t'\|_\varphi = t^{-1}\|x(\cdot+t)-x\|_\varphi \leqslant t^{-1}\omega_\varphi(t, x)$. Hence

$$\omega_\varphi(t, Tx) \leqslant \omega_\varphi(t, Tx-Tx_t)+\omega_\varphi(t, Tx_t) \leqslant 2\|Tx-Tx_t\|_\varphi + t\|(Tx_t)'\|_\varphi$$

$$\leqslant 2\|x-x_t\|_\varphi + t\|x_t'\|_\varphi \leqslant 2\omega_\varphi(t, x)+\omega_\varphi(t, x) = 3\omega_\varphi(t, x).$$

This proves the theorem.

Acknowledgements. Some results of this paper were presented in 1979 at Professor W. Orlicz's seminar. The final version of this paper was written at the Department of Mathematics of the Central University of Venezuela in January–April 1988. The author is indebted to the Department for its hospitality and to the University for its financial support. The author is also grateful to Dr. W. Wnuk for his help in the proof of Proposition 1, to Professor C. E. Finol for the discussions he had with him and to the referee for some remarks and suggestions.

References

[1] H. Brezis and W. A. Strauss, *Semi-linear second-order elliptic equations in L^1*, J. Math. Soc. Japan 25 (1973), 565–590.
[2] F. Browder, *Remarks on nonlinear interpolation in Banach spaces*, J. Funct. Anal. 4 (1969), 390–403.
[3] Yu. A. Brudnyi, *Moduli of continuity and rearrangements*, Mat. Zametki 18 (1975), 63–66 (in Russian).
[4] G. Chiti, *Rearrangements of functions and convergence in Orlicz spaces*, Applicable Anal. 9 (1979), 23–27.
[5] M. G. Crandall and L. Tartar, *Some relations between nonexpansive and order preserving mappings*, Proc. Amer. Math. Soc. 78 (1980), 385–390.
[6] G. F. Duff, *Differences, derivatives, and decreasing rearrangements*, Canad. J. Math. 19 (1967), 1153–1178.
[7] —, *A general integral inequality for the derivative of an equimeasurable rearrangement*, ibid. 28 (1976), 793–804.
[8] A. M. Garsia, *Topics in Almost Everywhere Convergence*, Markhan, Chicago 1970.
[9] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, AMS, Providence 1982.
[10] U. Krengel, *Ergodic Theorems*, de Gruyter Stud. Math. 6, Berlin–New York 1985.
[11] U. Krengel and M. Lin, *Order preserving nonexpansive operators in L_1*, Israel J. Math. 58 (1987), 170–192.
[12] J. L. Lions, *Some remarks on variational inequalities*, in: Proc. Internat. Conf. Functional Analysis and Related Topics (Tokyo 1969), Univ. of Tokyo Press, Tokyo 1970, 269–282.
[13] G. G. Lorentz and T. Shimogaki, *Interpolation theorems for operators in function spaces*, J. Funct. Anal. 2 (1968), 31–51.
[14] L. Maligranda, *Interpolation of Lipschitz operators for the pairs of spaces (L^1, L^∞) and (l^1, c_0)*, Comment. Math. Prace Mat. 21 (1979), 327–339.
[15] —, *Calderón–Lozanovskii spaces and interpolation of operators*, Semesterbericht Funktionalanalysis, Tübingen 8 (1985), 83–92.
[16] —, *On interpolation of nonlinear operators*, Comment. Math. Prace Mat. 28 (1988), 85–107.

[17] W. Orlicz, *Ein Satz über die Erweiterung von linearen Operationen*, Studia Math. 5 (1935), 127–140; reprinted in his Collected Papers, PWN, Warszawa 1988, 309–322.
[18] —, *On a class of operations over the space of integrable functions*, Studia Math. 14 (1954), 302–309; reprinted in Collected Papers, PWN, Warszawa 1988, 806–813.
[19] J. Peetre, *Interpolation of Lipschitz operators and metric spaces*, Mathematica (Cluj) 12 (1970), 325–334.
[20] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Ann. of Math. Stud. 27, Princeton 1951.

INSTYTUT MATEMATYCZNY PAN
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Mielżyńskiego 27/29, 61-725 Poznań, Poland

DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD CENTRAL DE VENEZUELA
Apartado 40645, Caracas 1040-A, Venezuela

Received June 1, 1988
Revised version October 3, 1988
(2448)

STUDIA MATHEMATICA, T. XCV (1989)

Continuity of derivations from radical convolution algebras

by

W. G. BADE (Berkeley, Calif.) and H. G. DALES (Leeds)

Abstract. We study the question whether or not every derivation from a radical weighted convolution algebra $A = L^1(\boldsymbol{R}^+, \omega)$, or from certain related algebras, into a Banach A-module is necessarily continuous. We show that such a derivation must have striking continuity properties.

Introduction. This paper studies continuity properties of module derivations from radical convolution algebras defined on the non-negative real numbers and on its subsemigroups.

First we describe our algebras and recall some standard notions which may be found in the text of Bonsall and Duncan [6].

Let $\boldsymbol{R}^+ = [0, \infty)$. A *radical weight* on $\boldsymbol{R}^+$ is a continuous, positive, submultiplicative function ω for which $\omega(0) = 1$ and $\lim_{t\to\infty}\omega(t)^{1/t} = 0$. We denote by $L^1(\boldsymbol{R}^+, \omega)$ the set of all equivalence classes of Borel measurable functions on $\boldsymbol{R}^+$ for which $\|f\| = \int_{\boldsymbol{R}^+}|f(t)|\omega(t)\,dt < \infty$. The set of all Radon measures μ on the Borel sets of $\boldsymbol{R}^+$ for which $\|\mu\| = \int_{\boldsymbol{R}^+}\omega(t)|\mu|(dt) < \infty$ will be denoted by $M(\boldsymbol{R}^+, \omega)$. Under convolution multiplication, $M(\boldsymbol{R}^+, \omega)$ is a local Banach algebra having for its identity the unit point mass at zero, while $L^1(\boldsymbol{R}^+, \omega)$ is a radical Banach algebra which is a closed ideal in $M(\boldsymbol{R}^+, \omega)$. Let $\boldsymbol{S}$ be a subgroup of $\boldsymbol{R}$, and let $\boldsymbol{S}^+ = \boldsymbol{S} \cap \boldsymbol{R}^+$. Then $l^1(\boldsymbol{S}^+, \omega)$ is the subalgebra of $M(\boldsymbol{R}^+, \omega)$ consisting of those discrete measures $f = \sum\{f(s)\delta_s\colon s\in\boldsymbol{S}^+\}$ for which $\|f\| = \sum\{|f(s)|\omega(s)\colon s\in\boldsymbol{S}^+\} < \infty$.

Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be an $\mathfrak{A}$-module (so that the module operations satisfy $a\cdot x = x\cdot a$ $(a\in\mathfrak{A},\ x\in\mathfrak{M})$). Then $\mathfrak{M}$ is a *Banach $\mathfrak{A}$-module* if $\mathfrak{M}$ is a Banach space and if the module operation $(a, x)\mapsto a\cdot x$ satisfies

$$\|a\cdot x\| \leqslant \|a\|\,\|x\| \quad (a\in\mathfrak{A},\ x\in\mathfrak{M}).$$

For example, one can regard $\mathfrak{A}$ as a Banach $\mathfrak{A}$-module over itself, where, of course, the module operation is the product in $\mathfrak{A}$. Then the dual $\mathfrak{A}'$ of $\mathfrak{A}$ becomes the *dual module* with respect to the operation $(a, \lambda)\mapsto a\cdot\lambda$, where

$$(a\cdot\lambda)(b) = \lambda(ab) \quad (a, b\in\mathfrak{A},\ \lambda\in\mathfrak{A}').$$