[6] D. Catlin, *Global regularity of the $\bar{\partial}$-Neumann problem*, in: Proc. Sympos. Pure Math. 41, A.M.S., 1984, 39–49.
[7] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. 39 (1977), 129–141.
[8] G. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy–Riemann Complex*, Princeton Univ. Press, 1972.
[9] F. Forelli and W. Rudin, *Projections on spaces of holomorphic functions in balls*, Indiana Univ. Math. J. 24 (1974), 593–602.
[10] S. Janson, J. Peetre and R. Rochberg, *Hankel forms and the Fock space*, technical report, Uppsala 1986.
[11] N. P. Jewell, *Multiplication by the coordinate functions on the Hardy space of the unit sphere in $C^n$*, Duke Math. J. 44 (1977), 839–851.
[12] N. Kerzman and E. Stein, *The Szegö kernel in terms of Cauchy–Fantappiè kernels*, ibid. 45 (1978), 197–223.
[13] G. Komatsu, *Boundedness of the Bergman projector and Bell's duality theorem*, Tôhoku Math. J. 36 (1984), 453–467.
[14] E. Ligocka, *The regularity of the weighted Bergman projection*, in: Seminar of Deformation Theory 1982/84, Lecture Notes in Math. 1165, Springer, 1985, 197–203.
[15] —, *The Sobolev spaces of harmonic functions*, Studia Math. 84 (1986), 79–87.
[16] —, *The Hölder duality for harmonic functions*, ibid. 84 (1986), 269–277.
[17] —, *Estimates in Sobolev norms $\| \|_p^s$ for harmonic and holomorphic functions and interpolation between Sobolev and Hölder spaces of harmonic functions*, ibid. 86 (1987), 255–271.
[18] —, *On the space of Bloch harmonic functions and interpolation of spaces of harmonic and holomorphic functions*, ibid. 87 (1987), 223–238.
[19] —, *On duality and interpolation for spaces of polyharmonic functions*, ibid. 88 (1988), 139–163.
[20] J. Peetre, *A reproducing kernel*, Boll. Un. Mat. Ital. (6) 3A (1984), 373–382.
[21] —, *Hankel forms in weaker assumptions*, technical report, Lund 1987.
[22] W. Rudin, *Function Theory in the Unit Ball of $C^n$*, Springer, 1980.
[23] N. Sibony, *Une classe de domaines pseudoconvexes*, Duke Math. J. 55 (1987), 299–319.
[24] —, *Problème de la couronne pour des domaines pseudoconvexes à bord lisse*, Ann. of Math. 126 (1987), 675–682.
[25] H. Triebel, *Theory of Function Spaces*, Birkhäuser, 1983.

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# A multidimensional Wolff theorem

by

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**Abstract.** We prove a Wolff theorem for $n$-tuples of Banach spaces. First we establish a general Wolff theorem for Aronszajn–Gagliardo orbit and coorbit functors and later we specialize this general result to the case of Sparr's spaces. Furthermore, we show an alternative proof for this last situation which works also in the quasi-Banach case. Some remarks on multiparameter scales are given as well.

**0. Introduction.** The question of extending Wolff's theorem [10] to a multidimensional context appears in [7]. Accordingly, we prove in this paper such a result. In fact, once the "geometry" of the new situation is understood it is easy to adapt directly the proof of the general Wolff theorem in [7] for Aronszajn–Gagliardo orbit and coorbit functors. This is done in Sec. 1, where we also review the basic terminology connected with interpolation of several Banach spaces. Similarly, in Sec. 2 we review the essentials of Sparr's theory of real interpolation of several Banach spaces [8]. In Sec. 3 we then specialize the general result in Sec. 1 to this case. This first calls for realizing the $K$- and $J$-functors as suitable coorbits respectively orbits, similarly to the case of Banach couples [6]. However, this is possible, strictly speaking, only in the Banach case. Accordingly, we sketch in Sec. 4 an alternative approach, closer to Wolff's original proof [10] (the argument there really goes back to Stafney [9]), which works also in the quasi-Banach case. At an early stage of our investigation we had thought that our Wolff theorem could be applied in the context of multiparameter scales of interpolation spaces, in a similar way to that in the one parameter case (see [7], Sec. 4), but we soon met unexpected difficulties of geometric nature, which we have not been able to overcome. So perhaps a more refined result might be needed. In Sec. 5 we have included a brief sketch of what kind of applications we had in mind.

**1. A general Wolff theorem.** We begin by fixing the terminology (cf. [8]).

By a *Banach n-tuple* we mean a family $\bar{A} = \{A_1, \ldots, A_n\}$ consisting of $n$ Banach spaces $A_i$ ($i = 1, \ldots, n$) all continuously embedded in some Hausdorff topological vector space $\mathscr{A}$, in symbols: $A_i \subset \mathscr{A}$.

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We say that a Banach space $A$ is an *intermediate space* with respect to a given Banach $n$-tuple $\bar{A} = \{A_1, \ldots, A_n\}$ if we have the continuous embeddings

$$\Delta(\bar{A}) = A_1 \cap \ldots \cap A_n \subset A \subset A_1 + \ldots + A_n = \Sigma(\bar{A}).$$

Let $\bar{A} = \{A_1, \ldots, A_n\}$ and $\bar{B} = \{B_1, \ldots, B_n\}$ be two Banach $n$-tuples, $\mathscr{A}$ and $\mathscr{B}$ being the corresponding ambient spaces. If $T$ is a linear map from $\mathscr{A}$ into $\mathscr{B}$ we use the notation $T\colon \bar{A} \to \bar{B}$ to mean that the restriction of $T$ to each $A_i$ defines a bounded linear operator from $A_i$ into $B_i$, in symbols: $T\colon A_i \to B_i$. We also write

$$\|T\|_{\bar{A},\bar{B}} = \max_{1 \leqslant i \leqslant n} \|T\|_{A_i,B_i}$$

where $\|T\|_{A_i,B_i}$ is the norm of $T\colon A_i \to B_i$, i.e.

$$\|T\|_{A_i,B_i} = \sup_{a \neq 0} \|Ta\|_{B_i}/\|a\|_{A_i}.$$

Let $A$ be an intermediate space with respect to $\bar{A}$, and $B$ one with respect to $\bar{B}$. We say that $A$ and $B$ are (*relative*) *interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$ if the relation $T\colon \bar{A} \to \bar{B}$ always implies $T\colon A \to B$. In this case there exists a constant $C$ such that

$$\|T\|_{A,B} \leqslant C\|T\|_{\bar{A},\bar{B}} \equiv C \max_{1 \leqslant i \leqslant n} \|T\|_{A_i,B_i}. \tag{1}$$

Let $R = R(x_1, \ldots, x_n)$ be a function of $n$ nonnegative real arguments such that

(a) $0 \leqslant R(x_1, \ldots, x_n) < \infty$, $R(0, \ldots, 0) = 0$;

(b) $R(x_1, \ldots, x_n)$ is a nondecreasing function in each argument $x_i$ separately;

(c) $R(x_1, \ldots, x_n)$ is homogeneous of degree 1, i.e., $R(\lambda x_1, \ldots, \lambda x_n) = \lambda R(x_1, \ldots, x_n)$ $(\lambda \geqslant 0)$;

(d) $R(x_1, \ldots, x_n) \leqslant C\max_{1 \leqslant i \leqslant n} x_i$.

If (1) can be replaced by the (hopefully) sharper inequality

$$\|T\|_{A,B} \leqslant R(\|T\|_{A_1,B_1}, \ldots, \|T\|_{A_n,B_n}), \tag{2}$$

we say that we are dealing with *R-interpolation spaces*.

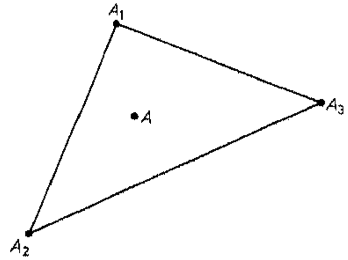
EXAMPLE 1.1. The most important case is if $R$ is a product of powers

$$R(x_1, \ldots, x_n) = x_1^{\theta_1} \ldots x_n^{\theta_n}$$

where $0 \leqslant \theta_1 \leqslant 1, \ldots, 0 \leqslant \theta_n \leqslant 1$, $\theta_1 + \ldots + \theta_n = 1$, or perhaps a constant multiple of such a function. In this case it is illuminating to associate the spaces



$A_1, \ldots, A_n$ (or $B_1, \ldots, B_n$) with the vertices of an $n$-simplex in an affine space and the space $A$ (or $B$) with a point in the same space with the barycentric coordinates $(\theta_1, \ldots, \theta_n)$ with respect to this $n$-simplex. Thus we get the following figure ($n = 3$).




We require further the obvious analogues of the two Aronszajn–Gagliardo functors [1].

1°. Let $\bar{A} = \{A_1, \ldots, A_n\}$ be a "fixed" Banach $n$-tuple and $\bar{B} = \{B_1, \ldots, B_n\}$ a "variable" one. Let further $A$ be a "fixed" intermediate space with respect to $\bar{A}$. Then $G(\bar{A}; A)(\bar{B})$ consists of all sums $\sum_{m=1}^{\infty} T_m a_m \in \Sigma(\bar{B})$ where $a_m \in A$, $T_m\colon \bar{A} \to \bar{B}$ and $\sum_{m=1}^{\infty} \|T_m\|_{\bar{A},\bar{B}} \|a_m\|_A < \infty$. This space becomes a Banach space when normed by

$$\|b\|_G = \inf\{\sum_{m=1}^{\infty} \|T_m\|_{\bar{A},\bar{B}} \|a_m\|_A\colon b = \sum_{m=1}^{\infty} T_m a_m\}.$$

and it is the minimal space $B$ such that $A$ and $B$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$.

2°. Dually, let us "fix" instead the $n$-tuple $\bar{B}$ and an intermediate space $B$ with respect to $\bar{B}$. Then we define $H(\bar{B}; B)(\bar{A})$ as the collection of all elements $a \in \Sigma(\bar{A})$ such that $Ta \in B$ for all linear maps $T\colon \bar{A} \to \bar{B}$. The norm in this space is given by $\|a\|_H = \sup\{\|Ta\|_B\colon \|T\|_{\bar{A},\bar{B}} \leqslant 1\}$. It is the maximal Banach space $A$ such that $A$ and $B$ are interpolation spaces with respect to $\bar{A}$ and $\bar{B}$.

To proceed to Wolff's theorem let us look at the following situation: Let $\bar{A} = \{A_1, \ldots, A_n\}$ and $\bar{B} = \{B_1, \ldots, B_n\}$ be two given Banach $n$-tuples and let $X_1, \ldots, X_n$ be $n$ intermediate spaces with respect to $\bar{A}$ and $Y_1, \ldots, Y_n$ be $n$ intermediate spaces with respect to $\bar{B}$. We assume that for some integer $k$ $(1 \leqslant k \leqslant n)$ the following holds:

If $j \in I' = [1, k]$ then $X_j$ and $Y_j$ are $R_j$-interpolation spaces with respect to the Banach $n$-tuples

$$\{\bar{A}_1, \bar{X}_2\} = \{A_1, \ldots, A_k, X_{k+1}, \ldots, X_n\} \quad \text{and}$$

$$\{\bar{B}_1, \bar{Y}_2\} = \{B_1, \ldots, B_k, Y_{k+1}, \ldots, Y_n\};$$

If $j \in I'' = [k+1, n]$ then $X_j$ and $Y_j$ are $R_j$-interpolation spaces with respect to the Banach $n$-tuples

$$\{\bar{X}_1, \bar{A}_2\} = \{X_1, \ldots, X_k, A_{k+1}, \ldots, A_n\} \quad \text{and}$$
$$\{\bar{Y}_1, \bar{B}_2\} = \{Y_1, \ldots, Y_k, B_{k+1}, \ldots, B_n\}.$$

Here $R_j = R_j(x_1, \ldots, x_n)$ are $n$ functions subject to the previous conditions (a)–(d) about which we further assume the following:

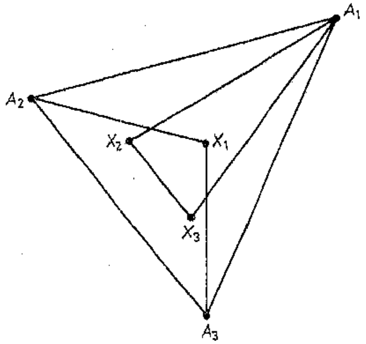
(e) If $j \in I'$ then

$$\lim_{x\to 0} R_j(\underbrace{x, \ldots, x}_{k \text{ times}}, \underbrace{1, \ldots, 1}_{n-k \text{ times}}) = 0;$$

and if $j \in I''$ then

$$\lim_{x\to 0} R_j(\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{x, \ldots, x}_{n-k \text{ times}}) = 0.$$

EXAMPLE 1.2. If the $R_j$ are power functions then, using Example 1.1, we arrive at the following picture ($n = 3$, $k = 1$):




Now we state our key lemma generalizing the corresponding result in [7].

LEMMA 1.3. *There exists a constant $C < \infty$ such that for any linear operator $T$: $\Sigma(\bar{A}) \to \Delta(\bar{B})$ we have*

$$\max_{1\le j\le n} \|T\|_{X_j, Y_j} \le C \max_{1\le i\le n} \|T\|_{A_i, B_i} \equiv C\|T\|_{\bar{A}, \bar{B}}.$$

Proof. For simplicity introduce the notation $N_j = \|T\|_{X_j, Y_j}$, $M_i = \|T\|_{A_i, B_i}$ $(i, j = 1, \ldots, n)$. Then by hypothesis we have

$$N_j \le R_j(M_1, \ldots, M_k, N_{k+1}, \ldots, N_n), \quad j \in I',$$
$$N_j \le R_j(N_1, \ldots, N_k, M_{k+1}, \ldots, M_n), \quad j \in I''.$$

without loss of generality we may assume that $\max_{1\le i\le n} M_i = 1$. Set further $N^* = \max_{1\le j\le n} N_j$. Then $N^* = N_j$ for some $j$. Suppose, for instance, that $j \in I'$. Then



$$N^* = N_j \le R_j(1, \ldots, 1, N^*, \ldots, N^*) \quad \text{or}$$
$$1 \le R_j(1/N^*, \ldots, 1/N^*, 1, \ldots, 1).$$

Our hypothesis on $R_j$ shows that $1/N^*$ must be bounded away from 0, i.e., $N^* \le C$ for some $C$ independent of $T$. The case $j \in I''$ is treated similarly. ∎

Now, as in [7], we must try to get rid of the assumption that $T$ maps into the intersection. We begin with the case of the functor $G$.

THEOREM 1.4. *Let the spaces $A_1, \ldots, A_n, X_1, \ldots, X_n, B_1, \ldots, B_n, Y_1, \ldots, Y_n$ have the same meaning as before. Assume that for $j \in I'$ the functor $G(\{\bar{A}_1, \bar{X}_2\}; X_j)(-)$ yields $R_j$-interpolation spaces and that for $j \in I''$ the functor $G(\{\bar{X}_1, \bar{A}_2\}; X_j)(-)$ yields $R_j$-interpolation spaces, where the functions $R_j$ are subject to the same hypotheses* (a)–(e) *as before. Assume further that for each $a \in X_j$ with $1 \le j \le n$ there exists a linear operator $P$: $\Sigma(\bar{A}) \to \Delta(\bar{A})$ such that*

$$\|Pa - a\|_{X_j} \le \tfrac{1}{2}\|a\|_{X_j},$$
$$\max_{1\le j\le n} \|P\|_{A_j, A_j} \le C \quad (\textit{independent of } a).$$

*Then if*

$$G(\{\bar{A}_1, \bar{X}_2\}; X_j)(\{\bar{B}_1, \bar{Y}_2\}) \subset Y_j \quad \textit{for } j \in I',$$
$$G(\{\bar{X}_1, \bar{A}_2\}; X_j)(\{\bar{Y}_1, \bar{B}_2\}) \subset Y_j \quad \textit{for } j \in I'',$$

*we have for each $1 \le j \le n$*

$$G(\bar{A}; X_j)(\bar{B}) \subset Y_j.$$

The proof is patterned on the one in [7]. Let $T$: $\bar{A} \to \bar{B}$ and $a \in X_j$ with $j \in I'$ be given. We have to prove that $Ta \in Y_j$. With no loss of generality we may assume that $\|T\|_{\bar{A}, \bar{B}} \le 1$. Let $P$ be the operator in the hypothesis of the theorem. By Lemma 1.3

$$\|TP\|_{X_j, Y_j} \le C, \qquad \|TPa\|_{Y_j} \le C\|a\|_{X_j}.$$

Let $a_1 = Pa$ and $a' = a - Pa$ so that

$$a = a_1 + a', \qquad \|a'\|_{X_j} \le \tfrac{1}{2}\|a\|_{X_j}, \qquad \|Ta_1\|_{Y_j} \le C\|a\|_{X_j}.$$

Continue the same procedure with $a'$ instead of $a$: We find $a_2$, $a''$ such that

$$a' = a_2 + a'', \qquad \|a''\|_{X_j} \le \tfrac{1}{4}\|a\|_{X_j}, \qquad \|Ta_2\|_{Y_j} \le \tfrac{1}{2}C\|a\|_{X_j}.$$

By induction we produce thus a sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$a = \sum_{n=1}^{\infty} a_n, \qquad \|Ta_n\|_{Y_j} \le \frac{1}{2^{n-1}} C\|a\|_{X_j}.$$

This clearly shows that $Ta \in Y_j$. ∎

Next we give the dual result with $H$.

THEOREM 1.5. *We make the same assumption on $H$ as in the previous theorem* *for G (the existence of $R_j$). We also assume that for each $1 \leqslant j \leqslant n$ there exists a family of linear maps $P_\alpha$: $\Sigma(\bar{B}) \to \Delta(\bar{B})$ such that*

$$\max_{1\leqslant j\leqslant n} \|P_\alpha\|_{B_j,B_j} \leqslant 1,$$

$$\sup_\alpha \|P_\alpha b\|_{Y_j} < \infty \;\Rightarrow\; b \in Y_j \quad \textit{for } b \in \Sigma(\bar{B}).$$

*Then if*

$$H(\{\bar{B}_1, \bar{Y}_2\};\, Y_j)(\{\bar{A}_1, \bar{X}_2\}) \supset X_j \quad \textit{for } j \in I',$$

$$H(\{\bar{Y}_1, \bar{B}_2\};\, Y_j)(\{\bar{X}_1, \bar{A}_2\}) \supset X_j \quad \textit{for } j \in I'',$$

*we have for $1 \leqslant j \leqslant n$*

$$H(\bar{B};\, Y_j)(\bar{A}) \supset X_j.$$

Proof. Let $T$: $\bar{A} \to \bar{B}$. Then $P_\alpha T$: $\Sigma(\bar{A}) \to \Delta(\bar{B})$ for any $\alpha$, so, by Lemma 1.3, $\|P_\alpha T\|_{X_j,Y_j} \leqslant C$. If $a \in X_j$ we get $\sup_\alpha \|P_\alpha Ta\|_{Y_j} < \infty$ implying $Ta \in Y_j$. ∎

**2. Brief review of Sparr's theory.** In what follows, we shall also work with $n$-tuples of positive real numbers. Given two $n$-tuples, $\bar{t} = (t_1, \ldots, t_n)$ and $\bar{s} = (s_1, \ldots, s_n)$, and $\beta \in \boldsymbol{R}$, we write

$$-\bar{t} = (-t_1, \ldots, -t_n), \qquad \beta\bar{t} = (\beta t_1, \ldots, \beta t_n),$$

$$\bar{t}\bar{s} = (t_1 s_1, \ldots, t_n s_n), \qquad |t| = t_1 \ldots t_n, \qquad 2^{\bar{t}} = (2^{t_1}, \ldots, 2^{t_n}).$$

Moreover, we denote by $\hat{t}$ the $(n+1)$-tuple defined by $\hat{t} = (0, t_1, \ldots, t_n)$. The symbol $S^n$ [resp. $\bar{S}^n$] stands for the set of all $n$-tuples $\bar{\theta} = (\theta_1, \ldots, \theta_n)$ of numbers in $(0, 1)$ [resp. $[0, 1]$] such that $\sum_{j=1}^n \theta_j = 1$. Also $(n-1)$-tuples of integer numbers $\bar{\nu} = (\nu_1, \ldots, \nu_{n-1}) \in \boldsymbol{Z}^{n-1}$ will be of special interest for us.

We now recall the definitions and some properties of $K$- and $J$-spaces described by Sparr in [8].

The $K$- and $J$-functionals are defined by

$$K(\bar{t}, a; \bar{A}) = \inf\Big\{\sum_{j=1}^n t_j \|a_j\|_{A_j}:\ a = \sum_{j=1}^n a_j,\ a_j \in A_j\Big\},$$

$$J(\bar{t}, a; \bar{A}) = \max_{1\leqslant j\leqslant n} \{t_j \|a\|_{A_j}\}.$$

Note that for each $n$-tuple $\bar{t}$, the functional $K(\bar{t}, \cdot)$ [resp. $J(\bar{t}, \cdot)$] is a norm in $\Sigma(\bar{A})$ [resp. $\Delta(\bar{A})$], any two of them being equivalent.

Let $1 \leqslant q \leqslant \infty$ and $\bar{\theta} \in S^n$. The space $\bar{A}_{\bar{\theta},q;K}$ consists of all elements $a \in \Sigma(\bar{A})$ having a finite norm

$$\|a\|_{\bar{\theta},q;K} = \Big(\sum_{\bar{\nu}\in \boldsymbol{Z}^{n-1}} \big(|2^{-\bar{\nu}\bar{\theta}}| K(2^{\hat{\nu}}, a)\big)^q\Big)^{1/q}.$$

The space $\bar{A}_{\bar{\theta},q;J}$ is formed by all elements $a \in \Sigma(\bar{A})$ which can be represented as

(1) $$a = \sum_{\bar{\nu}\in \boldsymbol{Z}^{n-1}} u_{\bar{\nu}} \qquad (\text{convergence in } \Sigma(\bar{A}))$$

where $u_{\bar{\nu}} \in \Delta(\bar{A})$ and



(2) $$|||(u_{\bar{\nu}})|||_{\bar{\theta},q;J} = \Big(\sum_{\bar{\nu}} \big(|2^{-\bar{\nu}\bar{\theta}}| J(2^{\hat{\nu}}, u_{\bar{\nu}})\big)^q\Big)^{1/q} < \infty.$$

The norm in $\bar{A}_{\bar{\theta},q;J}$ is given by $\|a\|_{\bar{\theta},q;J} = \inf |||(u_{\bar{\nu}})|||_{\bar{\theta},q;J}$ where the infimum is taken over all sequences $(u_{\bar{\nu}})$ satisfying (1) and (2).

These spaces are generalizations of the classical real interpolation space for a Banach couple (see [3]). It is possible to give a continuous definition for them, using integrals instead of sums, which is equivalent to the former one. We should mention that, besides Sparr, some other authors have also been interested in similar spaces (see [8] for references).

Unlike the case of Banach couples, where $\bar{A}_{\theta,q;K}$ and $\bar{A}_{\theta,q;J}$ coincide to within equivalence of norms, we only have now the continuous embedding $\bar{A}_{\bar{\theta},q;J} \subsetneq \bar{A}_{\bar{\theta},q;K}$. The converse inclusion fails in general. Counterexamples can be found in [8] and [4] (see also [2]).

A sufficient condition on $\bar{A}$ for equality between $K$- and $J$-spaces is the following.

CONDITION $\mathscr{F}(\bar{A})$. *There is a constant $\gamma(\bar{A})$, depending only on $\bar{A}$, such that for every $a \in \Sigma(\bar{A})$ for which*

$$\sum_{\bar{\nu}\in \boldsymbol{Z}^{n-1}} \min(1, 2^{-\nu_1}, \ldots, 2^{-\nu_{n-1}}) K(2^{\hat{\nu}}, a) < \infty$$

*there exists a representation $a = \sum_{\bar{\nu}} u_{\bar{\nu}}$ (convergence in $\Sigma(\bar{A})$) of $a$ such that*

$$J(2^{\hat{\nu}}, u_{\bar{\nu}}) \leqslant \gamma(\bar{A}) K(2^{\hat{\nu}}, a).$$

When $K$- and $J$-spaces are equal, we denote any of them by the simpler notation $\bar{A}_{\bar{\theta},q}$.

We shall also need the following stability result established in [8].

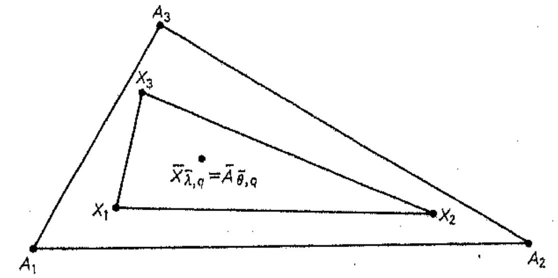
REITERATION THEOREM. *Let $\bar{A} = (A_1, \ldots, A_n)$ and $\bar{X} = (X_1, \ldots, X_n)$ be two Banach $n$-tuples, and suppose that*

$$\bar{A}_{\bar{\theta}_j,1;J} \subsetneq X_j \subsetneq \bar{A}_{\bar{\theta}_j,\infty;K}$$

*where $\bar{\theta}_j \in \bar{S}^n$ $(1 \leqslant j \leqslant n)$ and $\boldsymbol{R}^n$ is spanned by $\{\bar{\theta}_j\}_{j=1}^n$. Let $\bar{\lambda} \in S^n$ and write $\bar{\theta} = \sum_{j=1}^n \lambda_j \bar{\theta}_j$. If $\mathscr{F}(\bar{A})$ is satisfied, then for $1 \leqslant q \leqslant \infty$ we have*

$$\bar{X}_{\bar{\lambda},q} = \bar{A}_{\bar{\theta},q} \quad (\textit{equivalent norms}).$$

In particular, the theorem applies when $X_j = \bar{A}_{\bar{\theta}_j,q_j}$, $1 \leqslant q_j \leqslant \infty$, $1 \leqslant j \leqslant n$. The following figure illustrates this situation ($n = 3$):



As in the former examples, $\bar{\theta}_j, \bar{\lambda}, \bar{\theta}$ have the meaning of barycentric coordinates.

Let us finish this section by pointing out that both functors $\bar{A}_{\bar{\theta},q;K}$ and $\bar{A}_{\bar{\theta},q;J}$ produce $R$-interpolation spaces with $R(x_1, \ldots, x_n) = Cx_1^{\theta_1} \ldots x_n^{\theta_n}$.

**3. A Wolff theorem for Sparr spaces.** In order to derive from results in Section 1 a Wolff theorem for $\bar{A}_{\bar{\theta},q}$ spaces, we shall require a description of $K$- and $J$-spaces as spaces obtained by using the two Aronszajn–Gagliardo functors introduced in Section 1.

Subsequently, we shall work with scalar sequence spaces over $\boldsymbol{Z}^{n-1}$. For $1 \leqslant q \leqslant \infty$, we put

$$l_q = \{(\xi_{\bar{v}}):\ \|(\xi_{\bar{v}})\|_{l_q} = \big(\sum_{\bar{v}\in \boldsymbol{Z}^{n-1}} |\xi_{\bar{v}}|^q\big)^{1/q} < \infty\}.$$

Given $\bar{\eta} \in \boldsymbol{R}^{n-1}$, we define

$$l_q(2^{-\bar{\eta}}) = \{(\xi_{\bar{v}}):\ \|(\xi_{\bar{v}})\|_{l_q(2^{-\bar{\eta}})} = \|(2^{-\bar{v}\bar{\eta}}|\xi_{\bar{v}}|)\|_{l_q} < \infty\}.$$

It is clear that for every $\bar{v} = (v_1, \ldots, v_{n-1}) \in \boldsymbol{Z}^{n-1}$, the sequence

$$e_{\bar{v}} = (\lambda_{\bar{\tau}}) \quad \text{where} \quad \lambda_{\bar{\tau}} = \begin{cases} 1 & \text{if } \bar{\tau} = \bar{v}, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to all $l_q$-spaces. In the particular case when $\bar{v} = (0, \ldots, 0, 1, 0, \ldots, 0)$ has all coordinates equal to zero except for the $j$th which is equal to 1, we write $e_{\bar{v}} = \bar{e}_j$. Moreover, given an $n$-tuple of positive real numbers $\bar{t} = (t_1, \ldots, t_n)$, the symbol $\tilde{t}$ stands for the $(n-1)$-tuple $(t_2, \ldots, t_n)$.

Next we give the announced descriptions. The proofs are similar to those in the case of a Banach couple ($n = 2$), see [6].

THEOREM 3.1 (Orbit theorem). *Let* $1 \leqslant q \leqslant \infty$, $\bar{\theta} \in S^n$ *and let* $\bar{A} = \{A_1, \ldots, A_n\}$ *be a Banach n-tuple. Then*

$$G(\{l_1, l_1(2^{-\bar{e}_1}), \ldots, l_1(2^{-\bar{e}_{n-1}})\}; l_q(2^{-\bar{\theta}}))(\bar{A}) = \bar{A}_{\bar{\theta},q;J} \quad (\textit{equal norms}).$$

Proof. Let $T$ be any operator from

$$\bar{l}_1 = \{l_1, l_1(2^{-\bar{e}_1}), \ldots, l_1(2^{-\bar{e}_{n-1}})\}$$

into $\bar{A}$. Write $w_{\bar{v}} = Te_{\bar{v}} \in \Delta(\bar{A})$, $\bar{v} \in \boldsymbol{Z}^{n-1}$. Then we have $T(\xi_{\bar{v}}) = \sum_{\bar{v}} \xi_{\bar{v}} w_{\bar{v}}$ for $(\xi_{\bar{v}}) \in \Sigma(\bar{l}_1)$. In addition

$$J(2^{\bar{v}}, w_{\bar{v}}) = \max(\|Te_{\bar{v}}\|_{A_1}, 2^{v_1}\|Te_{\bar{v}}\|_{A_2}, \ldots, 2^{v_{n-1}}\|Te_{\bar{v}}\|_{A_n}) \leqslant \|T\|_{\bar{l}_1,\bar{A}}.$$

Therefore, if $(\xi_{\bar{v}}) \in l_q(2^{-\bar{\theta}})$, the element $a = T(\xi_{\bar{v}})$ can be represented by the sum $\sum_{\bar{v}} \xi_{\bar{v}} w_{\bar{v}}$ where each $\xi_{\bar{v}} w_{\bar{v}}$ belongs to $\Delta(\bar{A})$ and

$$\begin{aligned} |||(\xi_{\bar{v}} w_{\bar{v}})|||_{\bar{\theta},q;J} &= \Big(\sum_{v_1,\ldots,v_{n-1}} (2^{-v_1\theta_2} \ldots 2^{-v_{n-1}\theta_n} J(2^{\bar{v}}, \xi_{\bar{v}} w_{\bar{v}}))^q\Big)^{1/q} \\ &\leqslant \|T\|_{\bar{l}_1,\bar{A}}\Big(\sum_{v_1,\ldots,v_{n-1}} (2^{-v_1\theta_2} \ldots 2^{-v_{n-1}\theta_n}|\xi_{\bar{v}}|)^q\Big)^{1/q} \\ &= \|T\|_{\bar{l}_1,\bar{A}}\|(\xi_{\bar{v}})\|_{l_q(2^{-\bar{\theta}})}. \end{aligned}$$



Thus

$$(1) \qquad \|a\|_{\bar{\theta},q;J} \leqslant \|T\|_{\bar{l}_1,\bar{A}}\|(\xi_{\bar{v}})\|_{l_q(2^{-\bar{\theta}})}.$$

Let us now take any $a \in G(\bar{l}_1; l_q(2^{-\bar{\theta}}))(\bar{A})$. Then $a$ can be written as $a = \sum_{n=1}^{\infty} T_n \lambda^{(n)}$ where

$$\sum_{n=1}^{\infty} \|T_n\|_{\bar{l}_1,\bar{A}}\|\lambda^{(n)}\|_{l_q(2^{-\bar{\theta}})} < \infty.$$

Consequently, using (1), we obtain

$$\|a\|_{\bar{\theta},q;J} \leqslant \sum_{n=1}^{\infty} \|T_n\lambda^{(n)}\|_{\bar{\theta},q;J} \leqslant \sum_{n=1}^{\infty} \|T_n\|_{\bar{l}_1,\bar{A}}\|\lambda^{(n)}\|_{l_q(2^{-\bar{\theta}})}.$$

This gives the continuous embedding $G(\bar{l}_1; l_q(2^{-\bar{\theta}}))(\bar{A}) \hookrightarrow \bar{A}_{\bar{\theta},q;J}$.

Reciprocally, given any $a \in \bar{A}_{\bar{\theta},q;J}$ and $\varepsilon > 0$, we can find a representation $a = \sum_{\bar{v}} u_{\bar{v}}$ such that

$$|||(u_{\bar{v}})|||_{\bar{\theta},q;J} = \Big(\sum_{v_1,\ldots,v_{n-1}} (2^{-v_1\theta_2} \ldots 2^{-v_{n-1}\theta_n} J(2^{\bar{v}}, u_{\bar{v}}))^q\Big)^{1/q} < \|a\|_{\bar{\theta},q;J} + \varepsilon.$$

Put $\lambda_{\bar{v}} = J(2^{\bar{v}}, u_{\bar{v}})$ and define $T$ by $T(\xi_{\bar{v}}) = \sum_{\bar{v}} \lambda_{\bar{v}}^{-1} \xi_{\bar{v}} u_{\bar{v}}$. Then $\|T\|_{\bar{l},\bar{A}} \leqslant 1$ and $T$ takes the value $a$ on the sequence $(\lambda_{\bar{v}}) \in l_q(2^{-\bar{\theta}})$. So $a \in G(\bar{l}_1; l_q(2^{-\bar{\theta}}))(\bar{A})$ and

$$\|a\|_G \leqslant \|T\|_{\bar{l}_1,\bar{A}}\|(\lambda_{\bar{v}})\|_{l_q(2^{-\bar{\theta}})} < \|a\|_{\bar{\theta},q;J} + \varepsilon. \ \blacksquare$$

Dually, for the $K$-spaces we have:

THEOREM 3.2 (Coorbit theorem). *Let* $1 \leqslant q \leqslant \infty$, $\bar{\theta} \in S^n$ *and let* $\bar{A} = \{A_1, \ldots, A_n\}$ *be a Banach n-tuple. Then*

$$H(\{l_\infty, l_\infty(2^{-\bar{e}_1}), \ldots, l_\infty(2^{-\bar{e}_{n-1}})\}; l_q(2^{-\bar{\theta}}))(\bar{A}) = \bar{A}_{\bar{\theta},q;K} \quad (\textit{equal norms}).$$

Proof. Let $\bar{l}_\infty = \{l_\infty, l_\infty(2^{-\bar{e}_1}), \ldots, l_\infty(2^{-\bar{e}_{n-1}})\}$ and let $T$ be any operator from $\bar{A}$ into $\bar{l}_\infty$ having norm less than or equal to 1. Then $T$ can be written as

$$Ta = (\langle f_{\bar{v}}, a\rangle), \quad \bar{v} \in \boldsymbol{Z}^{n-1},$$

where $f_{\bar{v}}$ belongs to the dual space of $\Sigma(\bar{A})$ and

$$\|f_{\bar{v}}\|_{A_1'} \leqslant 1, \quad \|f_{\bar{v}}\|_{A_2'} \leqslant 2^{v_1}, \ldots, \|f_{\bar{v}}\|_{A_n'} \leqslant 2^{v_{n-1}}.$$

Hence, given $a \in \Sigma(\bar{A})$ and any representation $a = \sum_{j=1}^{n} a_j$ with $a_j \in A_j$, we obtain

$$|\langle f_{\bar{v}}, a\rangle| \leqslant \sum_{j=1}^{n} |\langle f_{\bar{v}}, a_j\rangle| \leqslant \|a_1\|_{A_1} + 2^{v_1}\|a_2\|_{A_2} + \ldots + 2^{v_{n-1}}\|a_n\|_{A_n}.$$

Thus

$$|\langle f_{\bar{v}}, a\rangle| \leqslant K(2^{\bar{v}}, a), \quad \bar{v} \in \boldsymbol{Z}^{n-1}.$$

This implies $\bar{A}_{\bar\theta,q;K} \subset H(\bar{l}_\infty; l_q(2^{-\bar\theta}))(\bar{A})$ with $\|a\|_H \leqslant \|a\|_{\bar\theta,q;K}$.

Conversely, given any $a \in H(\bar{l}_\infty; l_q(2^{-\bar\theta}))(\bar{A})$ and any $\bar\nu \in \mathbf{Z}^{n-1}$, applying the Hahn–Banach theorem we can find $f_{\bar\nu} \in \Sigma(\bar{A})'$ such that

$$\langle f_{\bar\nu}, a\rangle = K(2^{\bar\nu}, a),$$

$$\|f_{\bar\nu}\|_{A_1'} \leqslant 1, \quad \|f_{\bar\nu}\|_{A_2'} \leqslant 2^{\nu_1}, \ldots, \|f_{\bar\nu}\|_{A_n'} \leqslant 2^{\nu_{n-1}}.$$

Therefore, $Tx = (\langle f_{\bar\nu}, x\rangle)$ defines an operator $T: \bar{A} \to \bar{l}_\infty$ with $\|T\|_{\bar{A},\bar{l}_\infty} \leqslant 1$ and

$$\|a\|_{\bar\theta,q;K} = \Big(\sum_{\nu_1,\ldots,\nu_{n-1}} (2^{-\nu_1\theta_2}\ldots 2^{-\nu_{n-1}\theta_n} K(2^{\bar\nu}, a))^q\Big)^{1/q}$$

$$= \|Ta\|_{l_q(2^{-\bar\theta})} \leqslant \|a\|_H. \ \blacksquare$$

Remark 3.3. In the definition of the spaces $\bar{A}_{\bar\theta,q;J}$ and $\bar{A}_{\bar\theta,q;K}$ the base 2 does not have any special role. This observation and a reasoning similar to those above allow us to derive the following more general formulae:

(2) $$G(\{l_1(2^{\bar\alpha_1}), \ldots, l_1(2^{\bar\alpha_n})\}; l_q(2^{\bar\theta}))(\bar{A}) = \bar{A}_{\bar\lambda,q;J},$$

(3) $$H(\{l_\infty(2^{\bar\alpha_1}), \ldots, l_\infty(2^{\bar\alpha_n})\}; l_q(2^{\bar\theta}))(\bar{A}) = \bar{A}_{\bar\lambda,q;K}$$

(with equivalence of norms).

Here $\bar\alpha_j = (\alpha_{jl}) \in S^n$ $(1 \leqslant j \leqslant n)$, with $\mathbf{R}^n$ being spanned by $\{\bar\alpha_j\}_{j=1}^n$, and $\bar\theta = (\theta_j)$ and $\bar\lambda = (\lambda_j)$ both belong to $S^n$ and $\theta_j = \sum_{l=1}^n \lambda_l \alpha_{lj}$.

Now we are ready to establish a Wolff theorem for Sparr spaces. In what follows, we assume that $X_1, \ldots, X_n$ are $n$ intermediate spaces with respect to the $n$-tuple $\bar{A}$. Moreover, we continue writing $\{\bar{A}_1, \bar{X}_2\} = \{A_1, \ldots, A_k, X_{k+1}, \ldots, X_n\}$ and $\{\bar{X}_1, \bar{A}_2\} = \{X_1, \ldots, X_k, A_{k+1}, \ldots, A_n\}$.

THEOREM 3.4. *Assume that $\bar\lambda_j = (\lambda_{jl})$ and $\bar\theta_j = (\theta_{jl})$ belong to $S^n$ for $1 \leqslant j \leqslant n$, and that there exists some integer $k$ $(1 \leqslant k \leqslant n)$ such that*

(4) $$\theta_{jl} = \lambda_{jl} + \sum_{m=k+1}^{n} \lambda_{jm}\theta_{ml} \quad (1 \leqslant j \leqslant k,\ 1 \leqslant l \leqslant k),$$

(5) $$\theta_{jl} = \sum_{m=k+1}^{n} \lambda_{jm}\theta_{ml} \quad (1 \leqslant j \leqslant k,\ k+1 \leqslant l \leqslant n),$$

(6) $$\theta_{jl} = \sum_{m=1}^{k} \lambda_{jm}\theta_{ml} \quad (k+1 \leqslant j \leqslant n,\ 1 \leqslant l \leqslant k),$$

(7) $$\theta_{jl} = \lambda_{jl} + \sum_{m=1}^{k} \lambda_{jm}\theta_{ml} \quad (k+1 \leqslant j \leqslant n,\ k+1 \leqslant l \leqslant n),$$

*and*

(8) *$\mathbf{R}^n$ is spanned by $\{\bar{e}_1, \ldots, \bar{e}_k, \bar\theta_{k+1}, \ldots, \bar\theta_n\}$ and also by $\{\bar\theta_1, \ldots, \bar\theta_k, \bar{e}_{k+1}, \ldots, \bar{e}_n\}$.*



*Then the following holds:*

(i) *If $\{\bar{A}_1, \bar{X}_2\}_{\bar\lambda_j,1;J} \subset X_j$ $(1 \leqslant j \leqslant k)$ and $\{\bar{X}_1, \bar{A}_2\}_{\bar\lambda_j,1;J} \subset X_j$ $(k+1 \leqslant j \leqslant n)$, then $\bar{A}_{\bar\theta_j,1;J} \subset X_j$ for $1 \leqslant j \leqslant n$.*

(ii) *If $\{\bar{A}_1, \bar{X}_2\}_{\bar\lambda_j,\infty;K} \supset X_j$ $(1 \leqslant j \leqslant k)$ and $\{\bar{X}_1, \bar{A}_2\}_{\bar\lambda_j,\infty;K} \supset X_j (k+1 \leqslant j \leqslant n)$, then $\bar{A}_{\bar\theta_j,\infty;K} \supset X_j$ for $1 \leqslant j \leqslant n$.*

Proof. Put $Y_1 = l_1$, $Y_2 = l_1(2^{\bar{e}_1}), \ldots, Y_n = l_1(2^{\bar{e}_{n-1}})$ and $W_1 = l_1(2^{\bar\theta_1}), \ldots, W_n = l_1(2^{\bar\theta_n})$. By Remark 3.3(2) and the hypothesis of (i) we have

$$G(\{\bar{Y}_1, \bar{W}_2\}; W_j)\{\bar{A}_1, \bar{X}_2\} \subset X_j \quad (1 \leqslant j \leqslant k),$$

$$G(\{\bar{W}_1, \bar{Y}_2\}; W_j)\{\bar{X}_1, \bar{A}_2\} \subset X_j \quad (k+1 \leqslant j \leqslant n).$$

Furthermore, $\{\bar{Y}, \bar{W}\}$ satisfies the approximation hypothesis required in Theorem 1.4 (take for $P$ a finite rank operator $(\xi_{\bar\nu}) \to (\xi_{\bar\nu})_{|\nu_j| \leqslant N}$). Therefore, Theorem 1.4 and (2) give that $\bar{A}_{\bar\theta_j,1;J} \subset X_j$ for $1 \leqslant j \leqslant n$.

The proof of (ii) can be carried out in a similar way by using now Theorem 1.5 and (3). $\blacksquare$

Combining this result with the Reiteration Theorem we obtain:

COROLLARY 3.5. *Let $\{\bar\lambda_j\}_{j=1}^n$ and $\{\bar\theta_j\}_{j=1}^n$ satisfy conditions (4)–(8), let $1 \leqslant q_j \leqslant \infty$, and suppose that condition $\mathscr{F}(\bar{A})$ is also satisfied. If $\{\bar{A}_1, \bar{X}_2\}_{\bar\lambda_j,q_j} = X_j$ $(1 \leqslant j \leqslant k)$ and $\{\bar{X}_1, \bar{A}_2\}_{\bar\lambda_j,q_j} = X_j$ $(k+1 \leqslant j \leqslant n)$, then $\bar{A}_{\bar\theta_j,q_j} = X_j$ for $1 \leqslant j \leqslant n$.*

Remark 3.6. The meaning of conditions (4)–(7) will be discussed in the next section.

It is conceivable, but not quite clear, that the above result can be extended to the spaces of Fernandez (see e.g. [5]), which have a theory to a large extent parallel to Sparr's theory.

**4. The quasi-Banach case.** In this section we develop an alternative approach to the main results of the previous section (Thm. 3.4 plus its corollary), which is applicable also in the quasi-Banach case. For notational simplicity though we write out the details only in the Banach case. As was mentioned in the introduction, the method of proof has been adapted from [9] and [10].

Let thus $\bar{A} = (A_1, \ldots, A_n)$ be a given Banach $n$-tuple. Let further $X_1, \ldots, X_n$ be intermediate spaces with respect to $\bar{A}$ about which we assume that

(1) $$\|a\|_{X_j} \leqslant c \prod{}' \|a\|_{A_l}^{\lambda_{jl}} \prod{}'' \|a\|_{X_l}^{\lambda_{jl}}$$

for $a \in (\bigcap{}' A_l) \cap (\bigcap{}'' X_l)$ and $j \in I'$,

$$(2)\qquad \|a\|_{X_j} \leqslant c \prod{}' \|a\|_{X_l}^{\lambda_{jl}} \prod{}'' \|a\|_{A_l}^{\lambda_{jl}}$$
$$\text{for } a \in \left(\bigcap{}' X_l\right) \cap \left(\bigcap{}'' A_l\right) \text{ and } j \in I'',$$

$$(3)\qquad K(\bar{t}, a; A_1, \ldots, A_k, X_{k+1}, \ldots, X_n) \leqslant c \prod t_l^{\lambda_{jl}} \|a\|_{X_j}$$
$$\text{for } a \in X_j \text{ and } j \in I',$$

$$(4)\qquad K(\bar{t}, a; X_1, \ldots, X_k, A_{k+1}, \ldots, A_n) \leqslant c \prod t_l^{\lambda_{jl}} \|a\|_{X_j}$$
$$\text{for } a \in X_j \text{ and } j \in I'',$$

where $0 < \lambda_{jl} < 1$, $\sum_{l=1}^{n} \lambda_{jl} = 1$ (thus $\Lambda = (\lambda_{jl})$ is a strictly stochastic matrix).

Here and in the sequel we use the following notation: $I' = [1, k]$, $I'' = [k+1, n]$, $\prod'$ stands for a product extended over $I'$, $\prod''$ for one extended over $I''$, the symbols $\bigcap'$, $\bigcap''$, and below $\sum'$, $\sum''$ being used in a similar sense. Also $c$ is a constant whose meaning changes from time to time.

We want to prove that (1) and (2) imply

$$(5)\qquad \|a\|_{X_j} \leqslant c \prod \|a\|_{A_l}^{\theta_{jl}} \quad \text{for } a \in \bigcap A_l,$$

and that (3) and (4) imply

$$(6)\qquad K(\bar{t}, a; \bar{A}) \leqslant c \prod t_l^{\theta_{jl}} \|a\|_{X_j} \quad \text{for } a \in X_j$$

with $\theta_{jl} \geqslant 0$, $\sum_{l=1}^{n} \theta_{jl} = 1$ (thus $\Theta = (\theta_{jl})$ too is a stochastic matrix).

The relation between $\Lambda$ and $\Theta$ is the following: Let us identify our $2n$ spaces $A_1, \ldots, A_n, X_1, \ldots, X_n$ with points of an affine space. Then the numbers $\lambda_{jl}$ $(l = 1, \ldots, n)$ are thought of as the barycentric coordinates of $X_j$ with respect to $A_1, \ldots, A_k, X_{k+1}, \ldots, X_n$ if $j \in I'$ and with respect to $X_1, \ldots, X_k, A_{k+1}, \ldots, A_n$ if $j \in I''$, while the numbers $\theta_{jl}$ $(l = 1, \ldots, n)$ are thought of as the barycentric coordinates of $X_j$ with respect to $A_1, \ldots, A_n$.

Thus, symbolically, we may write, say, for $j \in I'$

$$X_j = \Sigma' \lambda_{jl} A_l + \Sigma'' \lambda_{jl} X_l \quad \text{where for each } l \quad X_l = \sum \theta_{lm} A_m.$$

Thus we find

$$(7)\qquad \theta_{jl} = \lambda_{jl} + \sum{}'' \lambda_{jm} \theta_{ml} \quad (j \in I',\ l \in I'),$$

$$(8)\qquad \theta_{jl} = \sum{}'' \lambda_{jm} \theta_{ml} \quad (j \in I',\ l \in I'').$$

Similarly, we find

$$(9)\qquad \theta_{jl} = \sum{}' \lambda_{jm} \theta_{ml} \quad (j \in I'',\ l \in I')$$

$$(10)\qquad \theta_{jl} = \lambda_{jl} + \sum{}' \lambda_{jm} \theta_{ml} \quad (j \in I'',\ l \in I'').$$

Let us write the matrices $\Lambda$ and $\Theta$ as block matrices (corresponding to the partition $[1, n] = I' \cup I''$):

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}.$$



Then we may write the relations (7)–(10) more compactly as

$$(7')\qquad \Theta_{11} = \Lambda_{11} + \Lambda_{12}\Theta_{21},$$

$$(8')\qquad \Theta_{12} = \Lambda_{12}\Theta_{22},$$

$$(9')\qquad \Theta_{21} = \Lambda_{21}\Theta_{11},$$

$$(10')\qquad \Theta_{22} = \Lambda_{22} + \Lambda_{21}\Theta_{12}.$$

Elimination between (7′) and (9′) yields

$$(7'')\qquad \Theta_{11} = \Lambda_{11} + \Lambda_{12}\Lambda_{21}\Theta_{11}.$$

Thus we find

$$(11)\qquad \Theta_{11} = (I - \Lambda_{12}\Lambda_{21})^{-1}\Lambda_{11} = \Lambda_{11} + (\Lambda_{12}\Lambda_{21})\Lambda_{11} + (\Lambda_{12}\Lambda_{21})^2\Lambda_{11} + \ldots$$
(Neumann series),

and a similar relation for $\Theta_{22}$.

We thus view these relations (7)–(10) or (7′)–(10′) as providing the rigorous meaning to what was stated in informal wording in the penultimate paragraph.

Consider now (1) and (2). If we substitute the inequalities (2) into one of the inequalities (1), we obtain an inequality of the form

$$\|a\|_{X_j} \leqslant c \prod_l{}' \|a\|_{A_l}^{\lambda_{jl}} \prod_m{}'' \prod_l{}' \|a\|_{X_l}^{\lambda_{jm}\lambda_{ml}} \prod_m{}'' \prod_l{}'' \|a\|_{A_l}^{\lambda_{jm}\lambda_{ml}}$$
$$= c \prod_l{}' \|a\|_{A_l}^{\lambda_{jl}} \prod_l{}'' \|a\|_{A_l}^{\Sigma_m'' \lambda_{jm}\lambda_{ml}} \prod_l{}' \|a\|_{X_l}^{\Sigma_m'' \lambda_{jm}\lambda_{ml}} \quad (j \in I').$$

An analogous formula holds for $j \in I''$. In exponential notation, writing $\|a\|_{X_j} = e^{u_j}$, $\|a\|_{A_l} = e^{v_l}$, putting

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

(and forgetting about the constants) we may write this as

$$u_1 \leqslant \Lambda_{11}v_1 + \Lambda_{12}\Lambda_{22}v_2 + \Lambda_{12}\Lambda_{21}u_1,$$
$$u_2 \leqslant \Lambda_{21}\Lambda_{11}v_1 + \Lambda_{22}v_2 + \Lambda_{21}\Lambda_{12}u_2$$

(this time $u_1$ and $u_2$ are the components of $u$ written as a block vector; the same for $v_1$ and $v_2$). Let us iterate the first inequality a couple of times:

$$u_1 \leqslant \Lambda_{11}v_1 + \Lambda_{12}\Lambda_{22}v_2 + \Lambda_{12}\Lambda_{21}\Lambda_{11}v_1 + \Lambda_{12}\Lambda_{21}\Lambda_{12}\Lambda_{22}v_2 + (\Lambda_{12}\Lambda_{21})^2 u_1$$
$$= (I + \Lambda_{12}\Lambda_{21})\Lambda_{11}v_1 + \Lambda_{12}(I + \Lambda_{21}\Lambda_{12})\Lambda_{22}v_2 + (\Lambda_{12}\Lambda_{21})^2 u_1,$$
$$u_1 \leqslant \Lambda_{11}v_1 + \Lambda_{12}\Lambda_{22}v_2 + \Lambda_{12}\Lambda_{21}\Lambda_{11}v_1 + \Lambda_{12}\Lambda_{21}\Lambda_{12}\Lambda_{22}v_2$$
$$+ (\Lambda_{12}\Lambda_{21})^2\Lambda_{11}v_1 + (\Lambda_{12}\Lambda_{21})^2\Lambda_{12}\Lambda_{22}v_2 + (\Lambda_{12}\Lambda_{21})^3 u_1$$
$$= (I + \Lambda_{12}\Lambda_{21} + (\Lambda_{12}\Lambda_{21})^2)\Lambda_{11}v_1$$
$$+ \Lambda_{12}(I + \Lambda_{21}\Lambda_{12} + (\Lambda_{21}\Lambda_{12})^2)\Lambda_{22}v_2 + (\Lambda_{12}\Lambda_{21})^3 u_1$$

and so forth. This convinces us that in the limit we find

$$u_1 \leqslant (I+A_{12}A_{21}+(A_{12}A_{21})^2+\ldots)A_{11}v_1$$
$$+A_{12}(I+A_{21}A_{12}+(A_{21}A_{12})^2+\ldots)A_{22}v_2.$$

Comparison with (7′)–(10′) (see also (11)) shows that this is the same as

$$u_1 \leqslant \Theta_{11}v_1+\Theta_{12}v_2.$$

In the same way we find

$$u_2 \leqslant \Theta_{21}v_1+\Theta_{22}v_2.$$

This, apparently, establishes (5).

Next we turn our attention to (3) and (4). Let $x\in X_j$ for some fixed $j\in I'$. In order to establish (6) it suffices to prove that $x$ can be written in the form

(12) $$x = a+\sum' y_l$$

where $a\in\sum A_i$ and $y_l\in X_l$ with

(13) $$K(\bar{t}, a; \bar{A}) \leqslant cT_j\|x\|_{X_j},$$

(14) $$\sum' T_l\|y_l\|_{X_l} \leqslant \tfrac{1}{2}T_j\|x\|_{X_j}.$$

Here the vector $\bar{t} = (t_1, \ldots, t_n)$ is fixed throughout the discussion and we write

$$T_l = \prod_m t_m^{\theta_{lm}}.$$

Indeed, iterating the construction (12)–(14) once we obtain for each $l\in I'$

$$y_l = b_l+\sum' z_{lm}$$

where $b_l\in\sum A_i$ and $z_{lm}\in X_m$ with

$$K(\bar{t}, b_l; \bar{A}) \leqslant cT_l\|y_l\|_{X_l},$$
$$\sum' T_m\|z_{lm}\|_{X_m} \leqslant \tfrac{1}{2}T_l\|y_l\|_{X_l}.$$

Putting $a' = a+\sum_l' b_l$, $x_m' = \sum_l' z_{lm}$ we then have

(12′) $$x = a'+\sum' x_m'$$

with the estimates

(13′) $$K(\bar{t}, a'; \bar{A}) \leqslant c\big(T_j\|x\|_{X_j}+\sum' T_l\|y_l\|_{X_l}\big) \leqslant c(1+\tfrac{1}{2})T_j\|x\|_{X_j},$$

(14′) $$\sum_m' T_m\|x_m'\|_{X_m} \leqslant \sum_m'\sum_l' T_m\|z_{lm}\|_{X_m} \leqslant \tfrac{1}{2}\sum_l' T_l\|y_l\|_{X_l} \leqslant \frac{1}{2^2}T_j\|x\|_{X_j}.$$

After one more iteration we have

(12″) $$x = a''+\sum_m' x_m'',$$

(13″) $$K(\bar{t}, a''; \bar{A}) \leqslant c\left(1+\frac{1}{2}+\frac{1}{2^2}\right)T_j\|x\|_{X_j},$$

(14″) $$\sum' T_m\|x_m''\|_{X_m} \leqslant \frac{1}{2^3}T_j\|x\|_{X_j}.$$



The resulting sequences of $x$'s clearly tend to zero and the $a$ sequence tends to $x$. Therefore in the limit the estimate (6) evolves.

It remains to prove (12)–(14). From (3) and (4) it follows that we may write

$$x = \sum' a_l+\sum'' x_l, \quad \sum' s_l\|a_l\|_{A_l}+\sum'' s_l\|x_l\|_{X_l} \leqslant cS\|x\|_{X_j},$$
$$x_l = \sum' x_{lm}+\sum'' a_{lm}, \quad \sum' r_{lm}\|x_{lm}\|_{X_m}+\sum'' r_{lm}\|a_{lm}\|_{A_m} \leqslant cR_l\|x_l\|_{X_l},$$

for an arbitrary choice of the positive numbers $s_l$ $(l = 1, \ldots, n)$ and $r_{lm}$ $(l = k+1, \ldots, n,\ m = 1, \ldots, n)$, with

$$S = \prod_l s_l^{\lambda_{jl}}, \quad R_l = \prod_m r_{lm}^{\lambda_{lm}}.$$

If we can choose these numbers in such a way that

(15) $$\begin{cases} s_l^{-1}S \leqslant t_l^{-1}T_j & (l\in I'),\\ r_{lm}^{-1}R_ls_l^{-1}S \leqslant \dfrac{1}{2c^2}T_jT_m^{-1} & (l\in I'',\ m\in I'),\\ r_{lm}^{-1}R_ls_l^{-1}S \leqslant c_1t_m^{-1}T_j & (l\in I'',\ m\in I'') \end{cases}$$

where $c$ is the constant given by (3) and (4) and $c_1$ is any other constant, then we clearly obtain (12)–(14) taking

$$a = \sum_l' a_l+\sum_l''\sum_m'' a_{lm}, \quad y_m = \sum_l'' x_{lm}.$$

To see that the system of inequalities can be solved, we rewrite them first, as before, in exponential form, putting $t_l = e^{u_l}$, $s_l = e^{v_l}$ and $r_{lm} = e^{w_{lm}}$. Then it is easily seen that everything boils down to solving the following system of linear equations:

(16) $$\begin{cases} -v_l+V = -u_l+U_j & (l\in I'),\\ -w_{lm}+W_l-v_l+V = U_j-U_m & (l\in I'',\ m\in I'),\\ -w_{lm}+W_l-v_l+V = U_j-u_m & (l\in I'',\ m\in I''), \end{cases}$$

where we have also written

$$U_m = \sum \theta_{ml}u_l, \quad V = \sum \lambda_{jl}v_l, \quad W_l = \sum_m \lambda_{lm}w_{lm}.$$

(Indeed, if (16) can be solved then we can solve (15) but with constants one. To get the right constant in the middle inequality we just have to replace each $r_{lm}$ by $2c^2r_{lm}$, this for $l\in I''$, $m\in I'$, modifying the remaining $r_{lm}$ in an arbitrary manner but keeping the products $R_l$ fixed.) Notice that the number of equations is $k+(n-k)k+(n-k)^2 = k+(n-k)n$, whereas the number of unknowns is $n+(n-k)n$, a surplus of $n-k$. However, there are relations among the equations (16) and we must show that the latter are precisely those which arise from the relations between the $\lambda$'s and the $\theta$'s, i.e., the relations (7)–(10) or (7′)–(10′). We remark first that the condition for the solvability of the system

$$-w_{lm}+W_l = d_m \quad \text{(some constants)} \quad (m = 1, \ldots, n)$$

is precisely $\sum \lambda_{lm} d_m = 0$. Thus we obtain from the two last equations in (16)

(17) $$-v_l + V = U_j - \sum_m{}' \lambda_{lm} U_m - \sum_m{}'' \lambda_{lm} u_m \quad (l \in I'').$$

Next we treat in exactly the same way the system of the form

$$-v_l + V = e_l \quad (l = 1, \ldots, n)$$

resulting from (17) and the first equation (16). We find that we must have

(18) $$\sum_l \theta_{jl} u_l = \sum_l{}' \lambda_{jl} u_l + \sum_l{}'' \sum_m{}' \lambda_{jl} \lambda_{lm} U_m + \sum_m{}'' \sum_l{}' \lambda_{jl} \lambda_{lm} u_m.$$

Now this relation (18) must hold for any choice of $u_l$ (with $U_m = \sum_l \theta_{ml} u_l$). Thus we obtain the matrix relations

(19) $$\Theta_{11} = \Lambda_{11} + \Lambda_{12} \Lambda_{21} \Theta_{11},$$

(20) $$\Theta_{12} = \Lambda_{12} \Lambda_{21} \Theta_{12} + \Lambda_{12} \Lambda_{22}.$$

Here (19) is identical with (7″), while (20) is (8′) combined with (10′).

Clearly all this together establishes (6) for $j \in I'$. The case $j \in I''$ is of course entirely parallel.

**5. On multiparameter scales of interpolation spaces.** Let $\Omega$ be an open subset in an affine space of dimension $n-1$ and let $\varphi$ be a function on $\Omega$ with values in $[1, \infty]$ such that $1/\varphi$ is affine (i.e. $1/\varphi(x) = (1-\theta)/\varphi(x_1) + \theta/\varphi(x_2)$ if $x, x_1, x_2 \in \Omega$, $0 \leqslant \theta \leqslant 1$ and $x = (1-\theta)x_1 + \theta x_2$). Let $\mathscr{A}$ be a Hausdorff topological vector space. By an *interpolation scale* (over $\Omega$ contained in $\mathscr{A}$) we mean a family of Banach spaces $\{A_x\}_{x\in\Omega}$ all continuously embedded in $\mathscr{A}$ which is closed for interpolation:

$$A_x = (A_{x_1}, \ldots, A_{x_n})_{\theta_1,\ldots,\theta_n;\varphi(x)}$$

whenever $x, x_1, \ldots, x_n \in \Omega$, $x \in \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_n\}$, $x = \theta_1 x_1 + \ldots + \theta_n x_n$, $0 < \theta_1, \ldots, \theta_n < 1$.

We remark that it is often useful, adopting a geometric picture, to view a scale as a kind of vector bundle over the set $\Omega$.

We first address ourselves to the question whether "overlapping interpolation scales can be pasted together": Let $\{A_x\}_{x\in\Omega}$ and $\{A'_x\}_{x\in\Omega'}$ be two interpolation scales contained in $\mathscr{A}$ over open sets $\Omega$ and $\Omega'$ with $\Omega \cap \Omega' \neq \emptyset$. We consider the family $\{A''_x\}_{x\in\Omega\cup\Omega'}$ where

$$A''_x = \begin{cases} A_x & \text{if } x \in \Omega, \\ A'_x & \text{if } x \in \Omega'. \end{cases}$$

QUESTION 5.1. Is this an interpolation scale?

We must show that

(1) $$A''_x = (A_{x_1}, \ldots, A_{x_k}, A'_{x_{k+1}}, \ldots, A'_{x_n})_{\theta_1,\ldots,\theta_n;\varphi(x)}$$



whenever $1 \leqslant k \leqslant n$, $x_1, \ldots, x_k \in \Omega$, $x_{k+1}, \ldots, x_n \in \Omega'$, $x \in \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_n\}$, $x = \theta_1 x_1 + \ldots + \theta_n x_n$, $0 < \theta_1, \ldots, \theta_n < 1$.

Let us make the assumption that $x \in \Omega \cap \Omega'$. Then we can construct $y_1, \ldots, y_n \in \Omega \cap \Omega'$ with $x \in \operatorname{int}\operatorname{conv}\{y_1, \ldots, y_n\}$ and such that

$$y_j \in \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\} \quad \text{if } 1 \leqslant j \leqslant k,$$

$$y_j \in \operatorname{int}\operatorname{conv}\{y_1, \ldots, y_k, x_{k+1}, \ldots, x_n\} \quad \text{if } k+1 \leqslant j \leqslant n.$$

If we also make the assumption that condition $\mathscr{F}$ is satisfied for any Banach $n$-tuple formed with spaces from the family $\{A''_x\}_{x\in\Omega\cup\Omega'}$ then we can apply our Wolff theorem (Cor. 3.5.). A renewed appeal to the Reiteration Theorem establishes (1) in this case.

In the general case we have *not* been able to prove (1).

Another problem with which we have busied ourselves is the question whether one can "fill in holes". Let $\{A_x\}_{x\in\Omega}$ be any interpolation scale over the open set $\Omega$ and consider the convex hull $\operatorname{conv}\Omega$ of $\Omega$. For $x \in \operatorname{conv}\Omega$ we set

$$A_x = (A_{x_1}, \ldots, A_{x_n})_{\theta_1,\ldots,\theta_n;\varphi(x)}$$

where now $x_1, \ldots, x_n \in \Omega$ and as before $x \in \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_n\}$, $x = \theta_1 x_1 + \ldots + \theta_n x_n$, $0 < \theta_1, \ldots, \theta_n < 1$. Two questions arise now:

1) Is this definition independent of the choice of $x_1, \ldots, x_n$?
2) Is the Banach family $\{A_x\}_{x\in\operatorname{conv}\Omega}$ an interpolation scale?

Question 2 we cannot solve (this is the problem just discussed) but let us say a few words about question 1.

Let us put $S = \operatorname{conv}\{x_1, \ldots, x_n\}$, with "$S$" for "simplex". The set of $n$-simplexes forms a manifold $\mathscr{M}$ of dimension $n^2$ with a natural topology. (In fact, it is an open subset of $\boldsymbol{R}^{n^2}$.) Let $E$ be the subset of $\mathscr{M}$ consisting of all simplexes $S \in \mathscr{M}$ such that $x \in \operatorname{int} S$. It is clear that $E$ is open.

Let us make the following assumption:

(∗) *$E$ is connected.*

If (∗) is fulfilled then we may argue as follows. As we do not know that the definition is independent of the choice of $x_1, \ldots, x_n$, let us, to be on the safe side, temporarily use the notation $A_s$ in place of $A_x$ if $x \in \operatorname{int} S$. Then the set of simplexes $S$ such that $A_S = A_{S_0}$ for a fixed $S_0$ is open. This follows from another application of the Reiteration Theorem (we are willing to admit that hypothesis $\mathscr{F}$ is fulfilled). We recall the following simple fact from topology.

LEMMA 5.2. *Let $X$ be a connected topological space and $Y$ a discrete space. If the map $f\colon X \to Y$ is continuous then $f(x) \equiv y$ for some $y \in Y$.*

The lemma and the hypothesis (∗) imply that $A_S$ depends only on $x$. Therefore we are henceforth free to use the notation $A_x$.

However, simple counterexamples show that (*) is not always true, although, on the other hand, it may hold for nonconvex sets. In such case we do not know how to proceed.

### References


[1] N. Aronszajn and E. Gagliardo, *Interpolation spaces and interpolation methods*, Ann. Mat. Pura Appl. 68(1965), 51–118.

[2] I. U. Asekritova, *A real method of interpolation for finite collections of Banach spaces*, in: Issled. Teor. Funkts. Mnogikh Veshchestv. Perem., Yaroslavl' 1981, 9–17 (in Russian).

[3] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren Math. Wiss. 223, Springer, Berlin 1976.

[4] M. Cwikel and S. Janson, *Real and complex interpolation methods for finite and infinite families of Banach spaces*, Adv. in Math. 66(1987), 234–290.

[5] D. L. Fernandez, *Interpolation of $2^d$ Banach spaces and the Calderón spaces $X(E)$*, Proc. London Math. Soc. 56(1988), 143–162.

[6] S. Janson, *Minimal and maximal methods of interpolation*, J. Funct. Anal. 44(1981), 50–73.

[7] S. Janson, P. Nilsson, J. Peetre and M. Zafran, *Notes on Wolff's note on interpolation spaces*, Proc. London Math. Soc. 48(1984), 283–299.

[8] G. Sparr, *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. 99(1974), 247–316.

[9] J. Stafney, *Analytic interpolation of certain multiplier spaces*, Pacific J. Math. 32(1970), 241–248.

[10] T. Wolff, *A note on interpolation spaces*, in: Proc. Conf. on Harmonic Analysis, Minneapolis 1981, Lecture Notes in Math. 908, Springer, Berlin 1982, 199–204.



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