[13] A. M. Mantero and A. Tonge, *The Schur multiplication in tensor algebras*, Studia Math. 68 (1980), 1–24.

[14] R. Meise and D. Vogt, *Holomorphic functions on nuclear sequence spaces*, Lecture Notes, Departamento de Teoría de Funciones, Universidad Complutense de Madrid, 1986.

[15] G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. 151 (1983), 181–208.

[16] —, *Factorisation of Linear Operators and Geometry of Banach Spaces*, Amer. Math. Soc. Reg. Conf. Ser. Math. 60 (1986).

[17] O. Toeplitz, *Über eine bei den Dirichletschen Reihen auftretende Aufgabe aus der Theorie der Potenzreihen von unendlichvielen Veränderlichen*, Göttinger Nachr. 1913, 417–432.

[18] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Math. Stud. 67, Amsterdam 1982.

Seán Dineen

UNIVERSITY COLLEGE DUBLIN
Belfield, Dublin 4, Ireland

Richard M. Timoney

SCHOOL OF MATHEMATICS
TRINITY COLLEGE DUBLIN
Dublin 2, Ireland

Received November 6, 1987
Revised version August 30, 1988 (2377)

STUDIA MATHEMATICA, T. XCIV (1989)

On an extrapolation theorem of Carleson–Sjölin with applications to a.e. convergence of Fourier series

by

FERNANDO SORIA* (Madrid)

Abstract. A weak type version of Yano's extrapolation theorem is presented which improves a result of Carleson and Sjölin about almost everywhere convergence of Fourier–Walsh series.

§ 0. Introduction. Let (Ω, μ) be an arbitrary measure space. A well-known result due to Yano [Y] (see also Zygmund [Z]) states that if $\mu(\Omega) < \infty$, T is a continuous linear operator from $L^p(\Omega)$ into $L^p(\Omega)$, $1 < p \leqslant p_0$, and for some fixed $m \geqslant 0$, T satisfies the estimate

$$\|T\chi_A\|_{L^p} \leqslant C(p-1)^{-m}\mu(A)^{1/p}, \tag{1}$$

for every measurable subset A of Ω and with C independent of A and p, then we can "extrapolate" and conclude that T acts continuously from $L(1+\log^+ L)(\Omega)$ into $L^1(\Omega)$.

It is not hard to see that the result remains true if T is assumed to be sublinear and estimate (1) is replaced by the weaker assumption

$$\|T\chi_A\|_{L^1} \leqslant C(p-1)^{-m}\mu(A)^{1/p}. \tag{2}$$

In this case, the condition "$\mu(\Omega) < \infty$" is not even needed. (See § 3 for a simple proof of this fact.)

In 1967, extending the fundamental work of Carleson [Ca], R. Hunt [H] found the following basic estimate for the maximal operator, S^*, associated to the partial sums of Fourier series:

$$t\mu\{x\in\Omega:\ S^*\chi_A > t\}^{1/p} \leqslant Cp^2(p-1)^{-1}\mu(A)^{1/p}, \quad 1 < p < \infty,\ t > 0, \tag{3}$$

where (Ω, μ) represents here the one-dimensional torus with the usual Lebesgue measure. This, combined with the inequality

$$\|f\|_{L^1} \leqslant p(p-1)^{-1}\sup_{t>0}[t\mu\{x:\ |f(x)| > t\}^{1/p}], \tag{4}$$

gives an estimate for S^* like (2) with $m = 2$ and, therefore, Yano's extrapolation theorem ensures the a.e. convergence of Fourier series for functions in $L(\log^+ L)^2$.

* Work supported in part by a grant of the NSF at the University of Chicago.

In 1969, P. Sjölin [Sj] proved (3) for the maximal operator associated this time to the partial sums of Walsh series. However, he avoided the use of (4), which introduces an extra power in the dependence of the constant on p. Instead of that, he investigated, jointly with Carleson, the possibility of obtaining an extrapolation theorem, similar to Yano's, directly from the weak type estimate (3). Their result is the following:

THEOREM 1. *Assume that* $\mu(\Omega) < \infty$. *Let* T *be a continuous sublinear operator on* $L^p(\Omega)$ *which satisfies*

$$(5)\qquad t\mu\{x\in\Omega:\ |T\chi_A(x)| > t\}^{1/p} \leqslant C(p-1)^{-1}\mu(A)^{1/p},\qquad 1 < p \leqslant 2,\ t > 0,$$

for every measurable subset $A \subset \Omega$ *and with* C *independent of* A, p *and* t. *Then there exist two constants,* c_1 *and* c_2, *so that if*

$$J(f) = \int_\Omega |f|(\log^+|f|)(\log^+\log^+|f|)\,d\mu < 1/2,$$

we have

$$\mu\{x\in\Omega:\ |Tf(x)| > c_1 J(f)^{1/5}\} \leqslant c_2 J(f)^{1/5}.$$

In particular, T *maps* $L(\log^+ L)(\log^+\log^+ L)$ *continuously into weak-*L^1.

The immediate consequence of Theorem 1 is that the Fourier–Walsh series of every function in $L(\log^+ L)(\log^+\log^+ L)$ converges a.e.

In this paper we study the extrapolation properties of sublinear operators satisfying restricted weak type (p, p) estimates, with a given growth on the constants as $p \to 1$. After some preliminary definitions and notation in § 1, we give in § 2 a simple proof of a more general form of Carleson–Sjölin's extrapolation theorem which holds for wider classes of functions. These classes, as will be seen, naturally arise as "extrapolation spaces". Applications to a.e. convergence of Fourier series are presented.

Some previous work in this direction can be found in [S1] (see also [S2]) where applications to the theory of differentiation of integrals are given. In addition to more generality, in this paper we eliminate the artificial use of "special functions". (Using the terminology in [H-T], f is a *special function* if there exist a set $A \subset \Omega$ and another function $s(x)$, with $1/2 \leqslant |s(x)| \leqslant 1$, so that $f = s\chi_A$.) Special functions were first considered by Hunt and Taibleson [H-T], following a suggestion of E. Stein, to give a simple proof of the a.e. convergence of Fourier series for $L(\log^+ L)(\log^+\log^+ L)$-functions in the ring of integers of a local field, but this required the establishment of an estimate like (3) with χ_A replaced by an arbitrary special function.

In the last section we analyze in more detail the properties of one of the extrapolation spaces; namely, the one which arises in connection with restricted weak type (1, 1) estimates.

§ 1. Preliminaries. For a measurable function f we define its distribution function as

$$\lambda_f(t) = \mu\{x\in\Omega:\ |f(x)| > t\},\qquad t > 0.$$

Now, given a positive, increasing and concave function φ defined on $[0, \mu(\Omega)]$ with $\varphi(0) = 0$, we define the function space B_φ as the class of all measurable functions for which the functional

$$\|f\|_\varphi = \int_0^\infty \varphi(\lambda_f(t))\,dt$$

is finite (¹). The conditions imposed on φ make $\|\ \|_\varphi$ a norm and B_φ a Banach space. In fact, if $\varphi\in C^1([0, \mu(\Omega)])$ we have, by a "change of variables",

$$\|f\|_\varphi = \int_0^\infty f^*(s)\varphi'(s)\,ds,$$

where f^* is the nonincreasing rearrangement of f, and, therefore, we are under the conditions stated by G. G. Lorentz [L] in order for $\|\ \|_\varphi$ to be a norm.

In particular, if we set

$$\varphi_m(s) = s(1+\log^+ 1/s)^m,$$

for some $m \geqslant 0$, with $\mu(\Omega) \leqslant e^{1-m}$, then one can easily prove that B_{φ_m} is the familiar Orlicz–Lorentz space $L(1+\log^+ L)^m$. This is true even if $e^{1-m} < \mu(\Omega) \leqslant \infty$, but in this case $\|\ \|_{\varphi_m}$ is not a norm (since $\varphi_m(s)$ is not concave on $[0, \infty)$, except when $m = 0$). To avoid this technicality one could consider instead of φ_m its smallest increasing concave envelope.

Associated to B_φ we will consider the following subspace:

$$B_\varphi^* = \Big\{f:\ \|f\|_\varphi^* = \int_0^\infty \varphi(\lambda_f(t))\big(1+\log(\|f\|_\varphi/t\,\varphi(\lambda_f(t)))\big)\,dt < \infty\Big\}.$$

For every φ as above, $\|\ \|_\varphi^*$ is a quasi-norm and B_φ^* has a structure of a quasi-Banach space. To see this, observe that for $f\in B_\varphi^*$ we have

$$(6)\qquad \|f\|_\varphi^* \leqslant N\big(\{2^k\varphi(\lambda_f(2^k))\}_{k=-\infty}^{\infty}\big) \leqslant (2/(1-\log 2))\|f\|_\varphi^*,$$

where given a numerical sequence $a = \{a_k\}_k$ we have defined

$$N(a) = \sum_k |a_k|\big(1+\log\big(\sum_j |a_j|/|a_k|\big)\big).$$

Now, the inequality $\|f+g\|_\varphi^* \leqslant K(\|f\|_\varphi^* + \|g\|_\varphi^*)$, for some constant K, follows from (6) and the facts that $\lambda_{f+g}(2s) \leqslant \lambda_f(s)+\lambda_g(s)$, that φ is subadditive and that for two sequences $a = \{a_k\}$ and $b = \{b_k\}$ one has

$$N(\{a_k+b_k\}) \leqslant (1+\log 2)(N(a)+N(b)).$$

(¹) The notation in this paper is slightly different from that in [S1].

(See Taibleson–Weiss [T-W].)

Also, a simple change of variables using the fact that $\lambda_{cf}(t) = \lambda_f(|c|t)$ gives

$$\|cf\|_\varphi^* = |c|\|f\|_\varphi^*, \quad \text{for every scalar } c.$$

The proof of the completeness of B_φ^* is standard.

§ 2. Extrapolation results. In this section we will consider a fixed function φ defined on $[0, \mu(\Omega)]$ which is increasing, concave and satisfies $\varphi(0) = 0$ and $\varphi(s) \geqslant s$. Observe that the last condition implies in particular that $B_\varphi \subset L^1$.

We can now state

THEOREM 2. *Let φ be as above and define $\Phi(y) = e^{y-1}\varphi(e^{1-y})$. Assume that T is a sublinear operator satisfying the following estimate:*

$$(7) \qquad t\mu\{x\in\Omega\colon |T\chi_A(x)| > t\} \leqslant C\Phi((p-1)^{-1})\mu(A)^{1/p}, \quad 1 < p \leqslant 2,$$

$\forall t > 0$ and $\forall A$ with C independent of t, A and p. Then T maps continuously B_φ^* into weak-L^1, i.e., there exists an absolute constant C_0 such that

$$(8) \qquad t\mu\{x\in\Omega\colon |Tf(x)| > t\} \leqslant C_0\|f\|_\varphi^*, \quad \forall f\in B_\varphi^*.$$

Proof. Given a measurable set A of positive measure, we let $p = 1+(1+\log^+\mu(A)^{-1})^{-1}$. Then (7) implies in the case $\mu(A) \leqslant 1$

$$\begin{aligned} t\mu\{x\in\Omega\colon |T\chi_A(x)| > t\} &\leqslant C\Phi(1+\log\mu(A)^{-1})\mu(A)\mu(A)^{1/p-1} \\ &= C\varphi(\mu(A))\mu(A)^{1/p-1} \leqslant Ce\varphi(\mu(A)). \end{aligned}$$

In the case $\mu(A) \geqslant 1$, we have $p = 2$ and, hence,

$$t\mu\{x\in\Omega\colon |T\chi_A(x)| > t\} \leqslant C\Phi(1)\mu(A)^{1/2} \leqslant C\varphi(1)\mu(A) \leqslant C\varphi(1)\varphi(\mu(A)),$$

where we have used in the last inequality the hypothesis $s \leqslant \varphi(s)$. Thus, in both cases we obtain

$$(9) \qquad t\mu\left\{x\in\Omega\colon \left|T(\varphi(\mu(A))^{-1}\chi_A)(x)\right| > t\right\} \leqslant C',$$

where $C' = \max(Ce, C\varphi(1))$.

We will also need the following simple result in measure theory due to E. Stein and N. Weiss [S-NW] (see [T-W] for the formulation presented here):

LEMMA 3. *Let $\{g_k\}$ be a sequence of functions uniformly bounded in weak-L^1, i.e.,*

$$t\mu\{x\in\Omega\colon |g_k(x)| > t\} \leqslant C'',$$

with C'' independent of k and t. Then, for every numerical sequence $\beta = \{\beta_k\}$, with $N(\beta) < \infty$, $g = \sum\beta_k g_k$ converges in weak-L^1 and, moreover,

$$t\mu\{x\in\Omega\colon |g(x)| > t\} \leqslant 6C''N(\beta).$$

The conclusion of Theorem 2 is now an immediate consequence of estimate (9), Lemma 3 and the following

LEMMA 4. *If $f(x) \geqslant 0$ belongs to B_φ^*, there exist a double sequence $\{E_{k,j}\}_{k\in\mathbf{Z},j\geqslant 1}$ of measurable sets and a double sequence $\beta = \{\beta_{k,j}\}_{k\in\mathbf{Z},j\geqslant 1}$ of positive numbers such that*

$$(10) \qquad f(x) = \sum_{k\in\mathbf{Z}}\sum_{j\geqslant 1}\beta_{k,j}\varphi(\mu(E_{k,j}))^{-1}\chi_{E_{k,j}}(x) \quad a.e.$$

and, moreover,

$$(11) \qquad N(\beta) \leqslant C'''\|f\|_\varphi^*$$

for some universal constant C'''.

Proof. For a given function $g \geqslant 0$, we will write $E_k(g) = \{x\colon 2^k \leqslant g(x) < 2^{k+1}\}$ and $g^* = g - \sum_{k\in\mathbf{Z}} 2^k\chi_{E_k(g)}$. In particular, $0 \leqslant g^*(x) \leqslant g(x)/2$ a.e.

Now, fix $f\in B_\varphi^*$. Define $f_0 = f$, $f_1 = f^*$ and, inductively, $f_{n+1} = f_n^*$, $n = 1, 2, \ldots$ Clearly,

$$(12) \qquad f = f_{n+1} + \sum_{1\leqslant j\leqslant n}\sum_{k\in\mathbf{Z}} 2^k\chi_{E_{k,j}},$$

where we have set $E_{k,j} = E_k(f_j)$. Also, $f_{n+1}(x) \leqslant f_n(x)/2$ and, by induction, $f_{n+1}(x) \leqslant 2^{-(n+1)}f(x)$ a.e. $\forall n$. Hence, (12) implies (10) with $\beta_{k,j} = 2^k\varphi(\mu(E_{k,j}))$.

In order to prove (11), we will show below that if $\beta_j = \{\beta_{k,j}\}_{k\in\mathbf{Z}}$, then

$$(13) \qquad N(\beta) \leqslant 2N(\{N(\beta_j)\}_{j\in\mathbf{N}}).$$

Assuming this first, we deduce from (6)

$$\begin{aligned} N(\beta) &\leqslant 2N(\{(2/(1-\log 2))\|f_j\|_\varphi^*\}_j) \\ &\leqslant 2N(\{(2/(1-\log 2))2^{-j}\|f\|_\varphi^*\}_j) \\ &= (4/(1-\log 2))N(\{2^{-j}\}_j)\|f\|_\varphi^*, \end{aligned}$$

where we have used in the last inequality the fact that each B_φ^* is a lattice space. This gives (11) with $C''' = (4/(1-\log 2))\sum_{j\geqslant 1} 2^{-j}(1+j\log 2)$. ■

Proof of (13). By a homogeneity argument we may assume without loss of generality that $\sum_{j\geqslant 1} N(\beta_j) = 1$. This also implies $\sum_{j\geqslant 1}\sum_{k\in\mathbf{Z}}\beta_{k,j} \leqslant 1$. Therefore,

$$\begin{aligned} N(\beta) &\leqslant \sum_j\sum_k\beta_{k,j}(1+\log\beta_{k,j}^{-1}) \\ &= \sum_j\sum_k\beta_{k,j}\Big(1+\log\big(\sum_{k'}\beta_{k',j}/\beta_{k,j}\big)-\log\big(\sum_{k'}\beta_{k',j}\big)\Big) \\ &\leqslant \sum_j\Big(N(\beta_j)+\varphi_1\big(\sum_k\beta_{k,j}\big)\Big) \leqslant \sum_j\big(N(\beta_j)+\varphi_1(N(\beta_j))\big) \\ &\leqslant 2\sum_j\varphi_1(N(\beta_j)) = 2N(\{N(\beta_j)\}_{j\in\mathbf{N}}). \ ■ \end{aligned}$$

Observe that if $\varphi = \varphi_m$, then $\Phi((p-1)^{-1}) = (p-1)^{-m}$ and, therefore, in the case $\mu(\Omega) < \infty$, $m = 1$ and $\varphi = \varphi_1$, condition (5) automatically implies (7). This fact, together with the following result on the relationship between classes B_φ^* and some other function spaces "close" to B_φ show how Theorem 2 generalizes the conclusion in Theorem 1:

PROPOSITION 5. *If f is a measurable function with finite support such that*

$$\int_0^\infty \varphi(\lambda_f(t))\bigl(1+\log^+\log^+(\lambda_f(t)^{-1})\bigr)\,dt < \infty, \tag{14}$$

then $f \in B_\varphi^*$.

In particular, if $\varphi = \varphi_m$ for some $m \geqslant 0$, (14) is equivalent to

$$\int_0^\infty \lambda_f(t)(1+\log^+ t)^m (1+\log^+\log^+ t)\,dt < \infty.$$

Thus, Proposition 5 gives as a trivial consequence:

COROLLARY 6. $(L(\log^+ L)^m(\log^+\log^+ L))_{\text{locally}} \subset (B_{\varphi_m}^*)_{\text{locally}}$.

The above inclusion is proper as has been shown in [S1]. Indeed, for every $\varepsilon > 0$ there are functions in $B_{\varphi_m}^*$ with compact support which do not belong to $L(\log^+ L)^m(\log^+\log^+ L)^\varepsilon$.

Proof of Proposition 5. Let A be the support of f. Observe that if (14) holds, $f \in B_\varphi$. We will assume, without loss of generality, that $\|f\|_\varphi = 1$ and write

$$\|f\|_\varphi^* = \Bigl(\int_{E_1} + \int_{E_2} + \int_{E_3}\Bigr)\varphi_1\bigl(t\varphi(\lambda_f(t))\bigr)\,dt/t = I_1 + I_2 + I_3,$$

where $E_1 = [0, e]$, $E_2 = \{t > e:\ t\varphi(\lambda_f(t)) > (\log(\lambda_f(t))^{-1})^{-2}\}$, $E_3 = (e, \infty) - E_2$. Then

$$I_1 \leqslant \int_0^e \varphi_1(t\mu(A))\,dt/t < \infty,$$

$$I_2 \leqslant \int_e^\infty \varphi(\lambda_f(t))\bigl(1+2\log^+\log^+(\lambda_f(t)^{-1})\bigr)\,dt < \infty, \qquad \text{from (14)}.$$

Notice also that $\lambda_f(t) \leqslant \|f\|_1/t \leqslant 1/t$ since $\|f\|_1 \leqslant \|f\|_\varphi$. Hence, if $t \in E_3$ we have $\varphi(\lambda_f(t)) \leqslant (t(\log t)^2)^{-1}$, and, therefore,

$$I_3 \leqslant \int_e^\infty (1+2\log\log t)(t(\log t)^2)^{-1}\,dt < \infty. \ \blacksquare$$

Remarks. 1. It is worth while to point out that in the finite case, say $\mu(\Omega) \leqslant 1$, and $\varphi = \varphi_m$, $B_{\varphi_m}^*$ is in some sense the largest extrapolation space for sublinear operators satisfying estimate (7) in Theorem 2. In fact, conditions (7) and (8) are in this case equivalent, for if $f = \chi_A$ then (8) gives

$$\begin{aligned} t\mu\{x\in\Omega:\ |T\chi_A(x)| > t\} &\leqslant C_0\|\chi_A\|_{\varphi_m}^* \leqslant 2C_0\mu(A)\bigl(1+\log(1/\mu(A))\bigr)^m \\ &\leqslant C_0\mu(A)(mp/(p-1))^m (e/\mu(A))^{(p-1)/p} \\ &\leqslant C_m(p-1)^{-m}\mu(A)^{1/p}, \qquad 1 < p \leqslant 2. \end{aligned}$$

2. Given a quasi-Banach function space B, with q-norm $\|\ \|_B$, and containing the simple functions we say that a sublinear operator T, mapping simple functions into measurable ones, is of *weak type* (B, L^1) if the following inequality holds:

$$t\mu\{x\in\Omega:\ |Tf(x)| > t\} \leqslant C\|f\|_B, \qquad \forall f \text{ simple}$$

(and C independent of f and t). We say that T is of *restricted weak type* (B, L^1) if the above holds for characteristic functions of sets.

With these definitions we can summarize some of the previous results in the following

THEOREM 7. *Fix $m \geqslant 0$. The following are equivalent for T sublinear:*

(a) *T is of restricted weak type (L^p, L^1), $1 < p \leqslant 2$, with constant $(p-1)^{-m}$, i.e.,*

$$t\mu\{x\in\Omega:\ |T\chi_A(x)| > t\} \leqslant C_1(p-1)^{-m}\mu(A)^{1/p}.$$

(b) *T is of restricted weak type (B_{φ_m}, L^1), i.e.,*

$$t\mu\{x\in\Omega:\ |T\chi_A(x)| > t\} \leqslant C_2\|\chi_A\|_{\varphi_m} = C_2\varphi_m(\mu(A)).$$

(c) *T is of weak type $(B_{\varphi_m}^*, L^1)$, i.e.,*

$$t\mu\{x\in\Omega:\ |Tf(x)| > t\} \leqslant C_3\|f\|_{\varphi_m}^*.$$

Proof. (a)$\Rightarrow$(c) is just Theorem 2. (c)$\Leftrightarrow$(b) follows from the fact that

$$\|\chi_A\|_{\varphi_m}^* = \int_0^\infty \varphi_m(\mu(A))(1+\log 1/t)\,dt = 2\varphi_m(\mu(A)) = 2\|\chi_A\|_{\varphi_m}.$$

Finally, (c)$\Rightarrow$(a) comes from the observation made in Remark 1. $\blacksquare$

From the equivalence of (b) and (c) it is natural to consider $B_{\varphi_m}^*$ as "restricted-$L(\log^+ L)^m$".

§ 3. Applications. As has been implicit throughout this paper, one of the first consequences of Theorem 2 is the following:

COROLLARY 8. *The Fourier series of each function in $B_{\varphi_1}^*$ converges a.e.*

This was also announced in [S1] but the extrapolation theorem in that paper required the use of "special functions".

The arguments in Theorem 2 can also be used to give a simple proof of Yano's theorem, in fact of the following stronger result:

PROPOSITION 9. *With the notation of Theorem 2, if*

$$\|T\chi_A\|_{L^1} \leqslant C\Phi\big((p-1)^{-1}\big)\,\mu(A)^{1/p}, \qquad \forall A \subset \Omega, \tag{16}$$

then

$$\|Tf\|_{L^1} \leqslant C'\|f\|_{\varphi},$$

in particular, when $\varphi = \varphi_m$ for some $m \geqslant 0$ conditions (16) *and* (2) *are equivalent.*

Proof. As in Theorem 2, (16) implies $\big\|T\big(\varphi(\mu(A))^{-1}\chi_A\big)\big\|_{L^1} \leqslant C_0$, and therefore, decomposing $f \geqslant 0$ as in Lemma 4 we obtain

$$\|Tf\|_{L^1} \leqslant C_0 \sum_j \sum_k 2^k \varphi\big(\lambda_{f_j}(2^k)\big) \leqslant 2C_0\|f\|_{\varphi}.$$

This yields Proposition 9 since every function is a linear combination of positive functions. ∎

§ 4. **"Restricted-L^1".** For reasons which will be more clear later, it is interesting to single out the class $B^*_{\varphi_0}$, "restricted-L^1". Recall that $\varphi_0(t) = t$ and, therefore, $B_{\varphi_0} = L^1$. Also, observe that conditions (7) and (8) read in this case

$$t\mu\{x \in \Omega:\ |T\chi_A(x)| > t\} \leqslant C\mu(A), \tag{7'}$$

$$t\mu\{x \in \Omega:\ |Tf(x)| > t\} \leqslant C' \int_0^\infty \lambda_f(t)\big(1 + \log(\|f\|_1/t\lambda_f(t))\big)\,dt. \tag{8'}$$

(7') can be rephrased by saying that T is of restricted weak type (1, 1) as defined by Stein–Weiss [S-GW].

An interesting feature of $B^*_{\varphi_0}$ was investigated in [S1] where it was proved that $B^*_{\varphi_0}$ is the rearrangement-invariant envelope of certain "block spaces" introduced by M. Taibleson and G. Weiss in connection with a.e. convergence of Fourier series.

Another feature of $B^*_{\varphi_0}$ that we would like to present here deals with the following characterization of A_1 weights for the Hardy–Littlewood maximal function M:

PROPOSITION 10. *Let $\Omega = \boldsymbol{R}^n$, $w(x)$ a positive function defined on $\boldsymbol{R}^n$ and set $\mu = w(x)\,dx$. Then the following are equivalent:*

(a) *w belongs to Muckenhoupt's class A_1.*

(b) *M maps $B^*_{\varphi_0} = B^*_{\varphi_0}(\boldsymbol{R}^n, w\,dx)$ continuously into weak-$L^1(w\,dx)$.*

The proof is easy and will be omitted. (See e.g. [G-R] to get some familiarity with the theory of weights.) This proposition represents a restatement of something which is well known. It is worth while, nevertheless, to compare this with a result in [C-H-K] which states the equivalence between M being bounded from $L^{p,1}(w\,dx)$ into weak-$L^p(w\,dx)$ and the condition

$$|E|/|Q| \leqslant C\big(w(E)/w(Q)\big)^{1/p}, \qquad 1 < p < \infty,$$

which must hold for every cube Q, every subset $E \subset Q$ and an absolute constant C. The limiting case $p = 1$ is just condition A_1. Thus, $B^*_{\varphi_0}$ appears again as the limiting space of the classes $L^{p,1}$, restricted-L^p.

Finally, regarding the closeness of $B^*_{\varphi_0}$ to $L^1 = B_{\varphi_0}$, we see from the results in § 2 that $L(\log^+\log^+ L)$ is (locally) contained in $B^*_{\varphi_0}$. Also, recall that Lorentz' spaces $L^{1,\beta}$, $0 < \beta < \infty$, are defined as

$$\{f:\ \|f\|_{1,\beta} = \big(\beta\int_0^\infty \lambda_f(t)^\beta t^{\beta-1}\,dt\big)^{1/\beta} < \infty\}.$$

Thus, $L^{1,1} = L^1$. With this definition in mind we also have

PROPOSITION 11. *If $0 < \beta < 1$, then $L^{1,\beta} \subset B^*_{\varphi_0}$.*

Proof. Define the space of numerical sequences $l^* = \{b = \{b_k\}: N(b) < \infty\}$. Given a function f we consider the sequence $a = a_f = \{a_k\}_{k\in\mathbf{Z}}$, where $a_k = 2^k\lambda_f(2^k)$. Then it is clear that $f \in B^*_{\varphi_0} \Leftrightarrow a_f \in l^*$. Also, $f \in L^{1,\beta} \Leftrightarrow a_f \in l^\beta$.

The proposition follows from the fact that for $0 < \beta < 1$, $l^\beta \subset l^*$. Indeed, a simple argument gives $N(b) \leqslant \big((2-\beta)/(1-\beta)\big)\|b\|_{l^\beta}$. ∎

References

[Ca] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. 116 (1966), 135–157.

[C-H-K] H.-M. Chung, R. Hunt and D. Kurtz, *The Hardy–Littlewood maximal function on L(p, q) spaces with weights*, Indiana Univ. Math. J. 31 (1982), 109–120.

[G-R] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.

[H] R. Hunt, *On the convergence of Fourier series*, in: Proc. Conf. at Southern Illinois Univ., 1967, 235–255.

[H-T] R. Hunt and M. Taibleson, *Almost everywhere convergence of Fourier series on the ring of integers of a local field*, SIAM J. Math. Anal. 2 (4) (1971), 607–625.

[L] G. G. Lorentz, *On the theory of spaces Λ*, Pacific J. Math. 1 (1951), 411–429.

[Sj] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh–Fourier series*, Ark. Math. 7 (1969), 551–570.

[S 1] F. Soria, *Note on differentiation of integrals and the halo conjecture*, Studia Math. 81 (1985), 29–36.

[S 2] —, *Characterizations of classes of functions generated by blocks and associated Hardy spaces*, Indiana Univ. Math. J. 34 (1985), 463–492.

[S-NW] E. Stein and N. Weiss, *On the convergence of Poisson integrals*, Trans. Amer. Math. Soc. 140 (1969), 35–54.

[S-GW] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[T-W] M. Taibleson and G. Weiss, *Certain function spaces connected with almost everywhere convergence of Fourier series*, in: Conf. on Harmonic Analysis in Honor of A. Zygmund, Wadsworth, 1982, 95–113.

[Y] S. Yano, *Notes on Fourier analysis (XXIX): An extrapolation theorem*, J. Math. Soc. Japan 3 (1951), 296–305.

[Z] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, 1959.

FACULTAD DE CIENCIAS
MATEMÁTICAS
UNIVERSIDAD AUTÓNOMA DE MADRID
28049 Madrid, Spain

Received December 7, 1987
Revised version August 8, 1988 (2390)

STUDIA MATHEMATICA, T. XCIV (1989)

On Muckenhoupt's classes of weight functions

by

INGEMAR WIK (Umeå)

Abstract. It is proved that if ω is in the weight class A_p on a cube Q, then ω^*, the nonincreasing rearrangement, lies in A_p on the interval $[0, |Q|]$. This gives another proof that ω then also lies in the more restricted class $A_{p-\varepsilon}$ for some $\varepsilon > 0$. An estimate of ε is given as well as a new characterization of A_p. The doubling condition $\omega(Q) \geqslant c\omega(2Q)$ is strictly weaker than the condition that $\omega \in A_\infty$. A new counterexample, comparatively simple, is given to demonstrate this fact.

1. Notations and introduction. The *Lebesgue measure* of a set E in $\mathbf{R}^n$ is denoted by $|E|$. ω will always be a locally integrable, nonnegative, real-valued function on $\mathbf{R}^n$ and we use the notation

$$\omega(E) = \int_E \omega(x)\,dx.$$

We denote by ω_* and ω^* the *nondecreasing* and *nonincreasing rearrangement* respectively of the function ω. E' is the *set of density points* of E. The *mean value* of a function f on a set E is written

$$⨍ f(x)\,dx = |E|^{-1} \int_E f(x)\,dx.$$

The *Muckenhoupt classes* A_p, $p > 1$, were introduced in [4]. A_p is defined as the class of locally integrable nonnegative functions ω that satisfy

$$(1)\qquad ⨍_Q \omega\,dx \Big(⨍_Q \omega^{-1/(p-1)}\,dx\Big)^{p-1} < A$$

for every cube Q and some constant A. We then say that *ω belongs to A_p with constant A*. If the cubes Q are restricted to lie within a fixed cube Q_0 we say that *ω belongs to A_p in Q_0*.

The paper consists of a proof that $\omega \in A_p$ in Q implies that ω^* (or equivalently ω_*) belongs to A_p on $[0, |Q|]$ (Theorem 1). Using a couple of elementary function-theoretic lemmas we proceed to prove the well-known fact that $\omega \in A_p$ implies $\omega \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. An estimate of ε comes as bonus (Theorem 2).As a corollary we obtain another