Ceci implique que $\forall n \geqslant 1,\ \exists m \geqslant 1,\ \exists c_n > 0$:

$$\sup_{C^l} \frac{|g(z)|}{\exp h_n(z)} \leqslant c_n \sup_{\|\mu\|_m^S \leqslant 1} |\langle g, R'\mu\rangle| = c_n \sup_{\|\mu\|_n^S \leqslant 1} \Big|\int_S g(\lambda)\, d\mu(\lambda)\Big| \leqslant c_n \sup_S \frac{|g(\lambda)|}{\exp h_m(\lambda)},$$

donc S est suffisant.

Le théorème est démontré.

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FACULTÉ DE MATHÉMATIQUES
INSTITUT DE PÉDAGOGIE No. 1
Hanoi, Vietnam

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Absolute bases, tensor products and a theorem of Bohr

by

SEÁN DINEEN and RICHARD M. TIMONEY (Dublin)

Abstract. For E a locally convex Montel space with basis, we show that if the monomials form an absolute basis for the space of holomorphic functions on E, then E must be nuclear. Using related methods, we give an affirmative result for a special case of Grothendieck's conjecture on tensor products and nuclearity. We also relate our methods to a polydisc version of a theorem of Bohr on power series of bounded functions.

In [2] Boland and Dineen proved that the monomials on a fully nuclear space with basis form an absolute basis for the space of holomorphic functions (with the compact-open topology). We present here a result in the converse direction (solving a question posed by Meise and Vogt [14, p. 39]).

Using similar methods, we also obtain results on a conjecture of Grothendieck [5] concerning tensor products and on a generalisation of a theorem of H. Bohr [1] from one to several variables.

By monomials we mean finite products of coordinate evaluations (coordinates with respect to a specified basis). The class of fully nuclear spaces which was considered in [2] includes all Fréchet nuclear spaces and all DFN-spaces. Our converse (Theorem 1.7) is stated for Montel spaces (with basis).

The main problem, which we refer to as the *basis problem*, leads naturally to a particular case of a conjecture of Grothendieck [5] concerning equality of the projective (π) and injective (ε) tensor products on locally convex spaces. Grothendieck's conjecture is known to be true for many classes of Banach spaces (see [15] for details) and for certain locally convex spaces [8], but Pisier [16, 15] has given a Banach space counterexample to the conjecture. The basis problem leads to a case where Grothendieck's conjecture is true (we give an elementary proof which works for some of the earlier results), but which is not covered by the results in [8, 15, 16].

Our proofs ultimately depend on a classical matrix inequality (Proposition 1.6). In the course of the reduction of the basis problem to enable us to apply this result, it became clear that a suitable generalisation of an equality of H. Bohr [1] from discs to polydiscs of large dimension would solve the basis problem. In the end, we found that such a generalisation is possible, but via a generalisation of Proposition 1.6 due to Mantero and Tonge [13].

This paper is organised as follows. In § 1, we give an overview of some of the background and show how to reduce the basis problem to the classical result (Proposition 1.6). In § 2, we explain the connection with Grothendieck's conjecture and show how the case of the conjecture which arises naturally can be proved using Proposition 1.6. In § 3, we explain how Bohr's inequality fits in to our context and obtain an asymptotic generalisation of Bohr's inequality to polydiscs in several variables.

Our references for locally convex spaces are [9, 7, 5, 18], for infinite-dimensional holomorphy [4] and for the geometry of Banach spaces [16, 10]. All locally convex spaces will be over the complex numbers (C) and will be assumed to be Hausdorff.

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1. The basis problem. We let $cs(E)$ denote the set of all continuous seminorms on a locally convex space E. By a (Schauder) *basis* for E we mean a sequence $(e_n)_n$ in E such that each $x \in E$ can be uniquely represented as a convergent series $x = \sum_{n=1}^{\infty} x_n e_n$ and such that the coordinate evaluations $x \mapsto x_n$: $E \to C$ are each continuous (see [2]).

$N^{(N)}$ will denote the collection of all finitely non-zero sequences of non-negative integers. For $(z_n)_n$ a sequence of complex numbers and $m \in N^{(N)}$, $|m| = \sum_j m_j$ and z^m denotes the (finite) product $\prod_j z_j^{m_j}$ (where $z_j^0 \equiv 1$).

DEFINITION 1.1. A basis $(e_n)_n$ for a locally convex space E is said to be *absolute* if for each $p \in cs(E)$ there exists $q \in cs(E)$ satisfying

$$\sum_n |x_n| p(e_n) \leqslant q\Big(\sum_n x_n e_n\Big) \tag{1.1}$$

for all $\sum_n x_n e_n \in E$.

A *holomorphic function* on a locally convex space E is a continuous function f: $E \to C$ which is holomorphic on all finite-dimensional subspaces of E. The collection of all such functions will be denoted by $H(E)$ and we will consider it with the topology τ_0 of uniform convergence on compact subsets of E. For later reference we recall that Montel spaces E are reflexive *quasi-complete* (closed bounded subsets are complete) locally convex spaces in which all closed bounded sets are compact. Moreover, (E', τ_0) coincides with the strong dual E'_β (E' with the topology of uniform convergence on bounded subsets of E) and is also a Montel space.

If E has a basis $(e_n)_n$, then by monomials on E we mean the functions

$$\sum_n z_n e_n \mapsto z^m \qquad (m \in N^{(N)}).$$

The problem we consider (the basis problem) is then whether E must be nuclear if the monomials form an absolute basis for $(H(E), \tau_0)$. Notice that for any

space E with a basis, finite-dimensional considerations show that every $f \in H(E)$ can be represented pointwise as

$$f\Big(\sum_n z_n e_n\Big) = \sum_{m \in N^{(N)}} a_m z^m \tag{1.2}$$

for unique scalars a_m, $m \in N^{(N)}$, and z in the algebraic span of the basis. Thus our problem concerns the situation where the series (1.2) converges in τ_0 and moreover has the following property: for each $K \subset E$ compact there is a compact $K' \subset E$ and a constant $C > 0$ with

$$\sum_{m \in N^{(N)}} |a_m| \, \|z^m\|_K \leqslant C \Big\| \sum_{m \in N^{(N)}} a_m z^m \Big\|_{K'} \tag{1.3}$$

for each $\sum a_m z^m \in H(E)$. (Here $\|f\|_K = \sup\{|f(z)|: z \in K\}$.)

Using Taylor series expansions in several variables it follows that if the monomials form an absolute basis for $H(E)$, then the monomials of degree n form an absolute basis for $(P(^nE), \tau_0)$, where $P(^nE)$ denotes the space of (continuous) n-homogeneous polynomials on E. (Recall that $p \in P(^nE)$ if and only if $p(z) = \tilde{p}(z, z, \ldots, z)$ for a symmetric continuous n-linear mapping $\tilde{p}$ on E.) In particular, taking $n = 1$, we find that $E' = P(^1E)$ must have an absolute basis. In the case when E is a Montel space this will allow us to describe E' as a locally convex sequence space with seminorms given by a family of weights.

A *weight* is a sequence w of non-negative real numbers. We consider collections W of weights and, for convenience, we assume that if $w = (w_n)_n$, $w' = (w'_n)_n \in W$ then there exists $w'' = (w''_n)_n \in W$ satisfying $\max(w_n, w'_n) \leqslant w''_n$ (all n). We consider sequence spaces

$$\Lambda(W) = \Big\{(z_n)_n: \sum_n |z_n| w_n < \infty \text{ for all } w = (w_n)_n \in W\Big\}$$

topologised by the family of seminorms $\sum_n |z_n| w_n$.

LEMMA 1.2. *Let F be a locally convex space with an absolute basis $(f_n)_n$. Suppose F is quasi-complete and W consists of all the weights $(|p(f_n)|)_n$ with $p \in cs(F)$. Then the map*

$$\sum_n z_n f_n \mapsto (z_n)_n: \ F \to \Lambda(W)$$

is a linear topological isomorphism.

Proof. This is straightforward to verify. ∎

PROPOSITION 1.3. *Let E be a Montel space with basis $(e_n)_n$ and suppose that the monomials form an absolute basis for $(H(E), \tau_0)$. Then*

(i) *E'_β may be identified with $\Lambda(W)$, for some W.*
(ii) *E has a fundamental system of compact sets consisting of the "polydiscs"*

$$\Big\{\sum z_n e_n \in E: \ \sup_n |z_n| a_n \leqslant R\Big\}$$

with $(a_n^{-1})_n \in W$, $R > 0$. [Here we allow $a_n = \infty$, $a_n^{-1} = 0$, and use $0 \cdot \infty = 0$ in defining $|z_n| a_n$.]

Proof. (i) follows from Lemma 1.2 and our earlier remarks. For (ii) we use the fact that a subset $K \subset E$ is bounded if and only if its polar is a barrel (i.e. closed, absolutely convex and absorbent). The Montel property of E and the fact that E'_β is barrelled (i.e. every barrel is a neighbourhood of the origin) allow us to deduce (ii). ■

LEMMA 1.4. *With the hypothesis and notation of Proposition* 1.3, *if* $(a_n^{-1})_n \in W$ *then there exist* $(b_n^{-1}) \in W$ *and a constant* $C > 0$ *satisfying*

$$(1.4)\quad \sum_{m\in N^k} \|a_m z^m\|_{\{|z_n| \leqslant r_n\}} = \sum_{m\in N^k} |a_m| r^m \leqslant C \sup\{|\sum_{m\in N^k} a_m z^m|: \max_j |z_j| \leqslant 1\}$$

for all k *where* $(a_m)_{m\in N^{(N)}}$ *is any set of scalars and* $r_n = (b_n/a_n)$ *for all* n.

Proof. This is immediate from Proposition 1.3 and (1.3). ■

Restricting (1.4) to coefficients $(a_m)_m$ with $a_m = 0$ for $|m| \neq 2$ we obtain the following:

LEMMA 1.5. *With the hypotheses and notation of Proposition* 1.3, *if* $(a_n^{-1})_n \in W$ *then there exist* $(b_n^{-1})_n \in W$ *and a constant* $C > 0$ *satisfying*

$$\sum_{j,k=1}^{n} \frac{b_j b_k}{a_j a_k} |\alpha_{jk}| \leqslant C \sup\{|\sum_{j,k} \alpha_{jk} z_j z_k|: \max_j |z_j| \leqslant 1\}$$

for all symmetric $n \times n$ *matrices* $(\alpha_{jk})_{jk}$ *and all* n.

As is customary, we use l_∞^n for C^n with the l_∞-norm $\|\cdot\|_\infty$. If $A = (a_{jk})_{j,k=1}^n$ is an $n \times n$ symmetric matrix then

$$\|A\|_{\infty,\infty} = \sup\{|\sum_{j,k=1}^{n} a_{jk} x_j y_k|: x, y \in l_\infty^n, \|x\|_\infty \leqslant 1, \|y\|_\infty \leqslant 1\}$$

denotes the norm of A as a bilinear form on l_∞^n. Let $i = \sqrt{-1}$.

PROPOSITION 1.6. *For* $a_{jk} = \exp(2\pi ijk/n)$, $1 \leqslant j, k \leqslant n$, $A = (a_{jk})_{jk}$ *we have*

$$\|A\|_{\infty,\infty} \leqslant n^{3/2}.$$

See Hardy and Littlewood [6, (5.1.5)] for a proof of this inequality. An earlier result of Toeplitz [17] would be sufficient for our purpose. For n a power of 4, he constructs a symmetric $n \times n$ matrix A with entries $a_{jk} = \pm 1$ which satisfies $\|A\|_{\infty,\infty} \sim n^{3/2}$.

THEOREM 1.7. *Let E be a Montel space with basis* $(e_n)_n$ *and suppose that the monomials form an absolute basis for* $(H(E), \tau_0)$. *Then* E'_β *is nuclear.*

Proof. Applying Lemma 1.5 together with Proposition 1.6, we find that given $(a_n^{-1}) \in W$ there exist $(b_n^{-1})_n \in W$ and $C > 0$ so that

$$\sum_{j,k=1}^{n} \frac{b_{m_j} b_{m_k}}{a_{m_j} a_{m_k}} \leqslant C n^{3/2}$$

holds for all sequences $1 \leqslant m_1 < \ldots < m_n$, all n. Letting $(\beta_j)_{j=1}^\infty$ denote the non-increasing rearrangement of b_j/a_j, it follows that

$$(\sum_{j=1}^{n} \beta_j)^2 \leqslant C n^{3/2}.$$

Consequently $\beta_n \leqslant C/n^{1/4}$ and $\sum_{n=1}^\infty (b_n/a_n)^8 < \infty$.

Using the Cauchy–Schwarz inequality we see that there exists $(c_n^{-1})_n \in W$ with $\sum_n c_n/a_n < \infty$. The Grothendieck–Pietsch criterion for nuclearity [5, 18, 7] now implies that E'_β is nuclear. ■

Remark 1.8. If E is assumed to be metrizable, then E is nuclear if and only if E'_β is. Thus nuclearity of E (as well as E'_β) can be deduced from the hypotheses of Theorem 1.7 together with the assumption that one of E or E'_β is metrizable.

2. Tensor products. For a locally convex space F, $F \otimes F$ denotes the algebraic tensor product consisting of (finite) linear combinations of elements $x \otimes y$. Elements of $F \otimes F$ may be viewed as bilinear forms on $F' \otimes F'$ via

$$x \otimes y(x', y') = x'(x) y'(y).$$

If $p \in cs(F)$ then the seminorms $p \otimes_\pi p$ and $p \otimes_\varepsilon p$ on $F \otimes F$ are given by

$$p \otimes_\pi p(u) = \inf\{\sum_i p(x_i) p(y_i): u = \sum x_i \otimes y_i\},$$
$$p \otimes_\varepsilon p(u) = \sup\{u(x', y'): x', y' \in B_p^\circ\},$$

where $B_p^\circ = \{x' \in F': |x'(x)| \leqslant p(x) \text{ for all } x \in F\}$. $F \otimes_\pi F$ denotes $F \otimes F$ with the topology generated by the seminorms $p \otimes_\pi p$, $p \in cs(F)$. $F \otimes_\varepsilon F$ is similarly defined.

We denote by $F \otimes^s F$ the subspace of $F \otimes F$ spanned by the elements $x \otimes x$ and we use $F \otimes_\pi^s F$ and $F \otimes_\varepsilon^s F$ to indicate $F \otimes^s F$ with the π- and ε-topologies.

We now consider the case where $F = E'_\beta$ and E is (as in the previous section) a Montel space with basis and with the property that the monomials form an absolute basis for $(H(E), \tau_0)$. Then we can identify $E'_\beta \otimes_\varepsilon^s E'_\beta$ with a subspace of $(P(^2E), \tau_0)$ since, for $u \in E' \otimes^s E'$,

$$\sup\{u(z; z): z \in B_p^\circ\} \leqslant p \otimes_\varepsilon p(u) \leqslant 2 \sup\{|u(z, z)|: z \in B_p^\circ\}.$$

Using the fact that $l_1^n \otimes_\pi l_1^n$ is isometric to $l_1^{(n^2)}$, (1.3) for $|m| = 2$ and Proposition 1.3(ii) we can deduce that our hypotheses on E imply

$$E'_\beta \otimes_\pi^s E'_\beta = E'_\beta \otimes_\varepsilon^s E'_\beta.$$

Consequently, an alternative route to Theorem 1.7 would be via the following result.

PROPOSITION 2.1. *Let $F = \Lambda(W)$. F must be nuclear if either one of the following conditions holds:*

(i) $F \otimes_\pi^s F = F \otimes_\varepsilon^s F$.
(ii) $F \otimes_\pi F = F \otimes_\varepsilon F$.

Proof. (i) follows essentially from Proposition 1.6, as in the proof of Theorem 1.7, and (ii) follows from (i). ∎

Case (ii) of Proposition 2.1 is a special case of a conjecture of Grothendieck [5]. The conjecture (now known not to be true in general [16]) was that any locally convex space F for which $F \otimes_\pi F = F \otimes_\varepsilon F$ must be nuclear. The result above seems not to be a consequence of previous positive results on the conjecture (see [8, 15] for these).

3. Bohr's theorem. Equation (1.4) led us to consider the following inequality of H. Bohr [1] (see also [3, p. 445]); if $f \in H^\infty(\Delta)$, the space of bounded holomorphic functions on the open unit disc Δ in C, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} |a_n| r^n \leqslant \sup\{|\sum_{n=0}^{\infty} a_n z^n| : |z| \leqslant 1\} \tag{3.1}$$

holds for $r \leqslant 1/3$. Moreover, if $r > 1/3$ then there exists an $f \in H^\infty(\Delta)$ for which inequality (3.1) fails.

Thus inequality (3.1) places restrictions on r and we were led to try generalising (3.1) to multidimensional polydiscs. Specifically, we considered the following.

PROBLEM 3.1. For each k describe the set B_k of all k-tuples $r = (r_1, \ldots, r_k)$ of non-negative real numbers which satisfy the inequality

$$\sum_{m \in N^k} |a_m| r^m \leqslant \| \sum_{m \in N^k} a_m z^m \|_{\Delta^k} \tag{3.2}$$

for all $f(z) = \sum_{m \in N^k} a_m z^m \in H^\infty(\Delta^k)$ where Δ^k is the unit polydisc in C^k.

Using Lemma 1.5 we could show that there exists a $\sigma > 0$ (in fact any $\sigma > 4$) and $M > 0$ (both independent of k) such that

$$\sum_{j=1}^{k} r_j^\sigma \leqslant M$$

for all $(r_1, \ldots, r_k) \in B_k$, and then an application of the Grothendieck–Pietsch criterion would solve the basis problem.

While we are unable to give a complete solution to Problem 3.1, the following is the best result we could obtain.

THEOREM 3.2. *For $r \in B_n$, $\sum_{j=1}^{n} r_j \leqslant 2\sqrt{n}$.*

LEMMA 3.3 ([13]). *For each $n > 1$, each $J \leqslant 2$ and each $\alpha > 0$ there exists a J-fold (symmetric) tensor*

$$t = \sum_{k_1, \ldots, k_J = 1}^{n} \varepsilon_k e_{k_1} \otimes \ldots \otimes e_{k_J}$$

with $\varepsilon_k = \pm 1$ (all k) which has ε-norm $\|t\|$ in $l_1^n \otimes \ldots \otimes l_1^n$ satisfying

$$\|t\| \leqslant 2^{J+1} [J n^{J+1} \log(1+4J)]^{1/2} + \alpha.$$

Proof. This follows from the proof of Theorem 1.1 of [13] (noting that the constant K used there may be taken to be J). ∎

Proof of Theorem 3.2. The ε-norm of the symmetric tensor t dominates $\|f\|_{\Delta^n}$ where f is the homogeneous polynomial on C^n

$$f(z) = \sum_k \varepsilon_k z^k.$$

From (1.6), it follows that each $r \in B_n$ satisfies

$$(\sum_{k=1}^{n} r_k)^J \leqslant \|t\|.$$

Letting $\alpha \to 0$, taking Jth roots and letting $J \to \infty$ gives the desired result. ∎

We remark that it is possible to modify Theorem 3.2 slightly so as to give alternative proofs of Theorem 1.7.

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Seán Dineen
UNIVERSITY COLLEGE DUBLIN
Belfield, Dublin 4, Ireland

Richard M. Timoney
SCHOOL OF MATHEMATICS
TRINITY COLLEGE DUBLIN
Dublin 2, Ireland

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On an extrapolation theorem of Carleson–Sjölin with applications to a.e. convergence of Fourier series

by

FERNANDO SORIA* (Madrid)

Abstract. A weak type version of Yano's extrapolation theorem is presented which improves a result of Carleson and Sjölin about almost everywhere convergence of Fourier–Walsh series.

§ 0. Introduction. Let (Ω, μ) be an arbitrary measure space. A well-known result due to Yano [Y] (see also Zygmund [Z]) states that if $\mu(\Omega) < \infty$, T is a continuous linear operator from $L^p(\Omega)$ into $L^p(\Omega)$, $1 < p \leqslant p_0$, and for some fixed $m \geqslant 0$, T satisfies the estimate

$$\|T\chi_A\|_{L^p} \leqslant C(p-1)^{-m}\mu(A)^{1/p}, \tag{1}$$

for every measurable subset A of Ω and with C independent of A and p, then we can "extrapolate" and conclude that T acts continuously from $L(1+\log^+ L)(\Omega)$ into $L^1(\Omega)$.

It is not hard to see that the result remains true if T is assumed to be sublinear and estimate (1) is replaced by the weaker assumption

$$\|T\chi_A\|_{L^1} \leqslant C(p-1)^{-m}\mu(A)^{1/p}. \tag{2}$$

In this case, the condition "$\mu(\Omega) < \infty$" is not even needed. (See § 3 for a simple proof of this fact.)

In 1967, extending the fundamental work of Carleson [Ca], R. Hunt [H] found the following basic estimate for the maximal operator, S^*, associated to the partial sums of Fourier series:

$$t\mu\{x \in \Omega: S^*\chi_A > t\}^{1/p} \leqslant Cp^2(p-1)^{-1}\mu(A)^{1/p}, \quad 1 < p < \infty,\ t > 0, \tag{3}$$

where (Ω, μ) represents here the one-dimensional torus with the usual Lebesgue measure. This, combined with the inequality

$$\|f\|_{L^1} \leqslant p(p-1)^{-1}\sup_{t>0}[t\mu\{x: |f(x)| > t\}^{1/p}], \tag{4}$$

gives an estimate for S^* like (2) with $m = 2$ and, therefore, Yano's extrapolation theorem ensures the a.e. convergence of Fourier series for functions in $L(\log^+ L)^2$.

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