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The Fréchet envelopes of vector-valued Smirnov classes

by

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Abstract. In this paper the Nevanlinna class $N(X)$ and the Smirnov class $N_*(X)$ of analytic functions on the unit disc in the complex plane with values in a Banach space X are studied. Representations of the Fréchet envelope of the Smirnov class by Fréchet spaces of analytic functions are found. The continuous linear functionals on $N_*(X)$ and multipliers of $N_*(X)$ into the Hardy spaces H_p $(0 < p \leqslant \infty)$ are described. The best possible estimates of the mean growth of functions and their Taylor coefficients are obtained. The proofs of our results are mainly soft and simplify earlier ones given in the scalar case by N. Yanagihara [22–24].

0. Introduction. Let $E = (E, \tau)$ be an F-space (i.e. a complete metrizable t.v.s.) with separating dual space and let τ^c be the locally convex topology associated to τ, i.e. the strongest locally convex topology on E which is weaker than τ. Obviously, if $\mathscr{B}$ is a base of neighbourhoods of zero for τ, then the family $\{\operatorname{conv} U\colon U \in \mathscr{B}\}$ is a base of neighbourhoods of zero for τ^c. Therefore, the topology τ^c produces the same space E' of continuous linear functionals as τ and has a countable base at zero. The dual space E' of (E, τ^c) separates the points of E, so τ^c is metrizable. The completion $\hat{E}$ of (E, τ^c) is a Fréchet space (i.e. locally convex F-space) which is called the *Fréchet envelope* of E.

Observe that (E, τ^c) is a Mackey space, i.e. τ^c coincides with the Mackey topology of the dual pair (E, E') (cf. [18], IV.3.4). Therefore, the Fréchet envelope of E is the unique, up to isomorphism, Fréchet space containing E as a dense subspace and such that the restriction map $T \to T|_E$ takes $\hat{E}'$ onto E'.

For many classes of concrete (function or sequence) spaces, it is possible to describe $\hat{E}$ as another concrete space. This has been done e.g. for the Hardy spaces H_p $(0 < p < 1)$ [8], the Bergman spaces $B_{p,\alpha}$ $(0 < p < 1,\ \alpha > -1)$ [19], the Orlicz function and sequence spaces [13, 14, 6], and the Lorentz sequence spaces $d(w, p)$ $(0 < p < 1)$ [15]. In 1973, N. Yanagihara [22, 23] identified the dual space and then the Fréchet envelope of the Smirnov class N_* of analytic functions on the unit disc on the complex plane.

The main purpose of this paper is to improve Yanagihara's representation result and to extend it to the vector-valued case, i.e. to the Smirnov class $N_*(X)$ of analytic functions with values in a complex Banach space X.

We give a simplified version of Yanagihara's construction. First of all we avoid many constructions of analytic functions. We show that it is possible to derive all necessary properties of functions belonging to the Smirnov class $N_*(X)$ from the corresponding well-known properties of functions from the Nevanlinna class $N(X)$. While proving the results, we use, instead of "hard analysis", elementary topological connections between these two important classes of analytic functions. Such an approach is motivated by the Shapiro and Shields theorem [20] which states that the Nevanlinna class N is a topological group and the Smirnov class N_* is the largest subspace of N which is a topological vector space in the relative topology. In Section 1 the simple proof of this theorem obtained by L. Drewnowski [5] is included. In addition, we collect preliminary definitions, notation and basic facts on topological vector groups. The notion of topological vector group is especially useful in our study.

In Section 2 the boundary behaviour of functions from the Nevanlinna and Smirnov classes is discussed. It is well known that, in contrast to the scalar case, there are bounded vector-valued analytic functions without radial limits on sets of positive measure. We show that the closure $N_0(X)$ in $N(X)$ of the space of all X-valued polynomials is the largest subspace of $N(X)$ consisting of functions having radial limits almost everywhere which is a t.v.s. in the relative topology.

In Section 3 we prove our main result (Theorem 3.6): the representation of the Fréchet envelope of $N_0(X)$.

As a simple application of this representation theorem we obtain in Section 4 the best possible estimates for the mean growth of functions from the Smirnov class $N_*(X)$ and their Taylor coefficients (see Propositions 4.1 and 4.2). N. Yanagihara [24] proved similar estimates for scalar-valued functions using the saddle point method of W. K. Hayman (see [11]). It is rather surprising that we obtain such "purely analytic" results without constructing any analytic function.

In the last part of this paper we describe the multipliers of $N_*(X)$ into H_p $(0 < p \leqslant \infty)$.

I wish to thank Professor Lech Drewnowski for a number of helpful discussions on this material.

1. Preliminaries. A (locally balanced) *topological vector group*, shortly t.v.g., is a vector space E (over $\mathbf{C}$) equipped with a topology τ such that addition is continuous, while scalar multiplication is continuous at the point $(0, 0)$ of $\mathbf{C} \times E$. Any t.v.g. is a topological group which has a base at zero

consisting of balanced sets. A metrizable and complete t.v.g. is called an *FG-space*.

In any t.v.g. $E = (E, \tau)$ the set

$$v(E) = \{x \in E\colon \tau\text{-}\lim_{t \to 0} tx = 0\}$$

is the largest linear subspace which is a t.v.s. in the relative topology. $v(E)$ is always closed in E (cf. [5]). Therefore, if E is an FG-space, then $v(E)$ is an F-space.

An *FG-norm* on a linear space E is a map $q\colon E \to \bar{\mathbf{R}}_+$ satisfying: $q(x) = 0 \Leftrightarrow x = 0$, $q(x+y) \leqslant q(x)+q(y)$, $q(tx) \leqslant q(x)$ $(x, y \in E,\ t \in \mathbf{C},\ |t| \leqslant 1)$. For any FG-norm q on E there is a group topology $\tau = \tau(q)$ which has as a base at zero the family $\{V_c\colon c > 0\}$, where $V_c = \{x \in E\colon q(x) \leqslant c\}$. E equipped with the topology $\tau(q)$ induced by q is a metrizable t.v.g.

Throughout this paper we assume that $X = (X, \|\cdot\|)$ is a complex Banach space. By $H(X)$ we denote the space of all X-valued analytic functions on the unit disc $\mathbf{D} = \{z \in \mathbf{C}\colon |z| < 1\}$. For information about vector-valued analytic functions the reader is referred to [12].

We recall that $H(X)$ equipped with the compact-open topology $\varkappa$ is a Fréchet space.

For any function $f\colon \mathbf{D} \to X$ and $0 \leqslant r < 1$ we denote by $(f)_r$ the function on the unit circle $T = \partial \mathbf{D}$ defined by $(f)_r(w) = f(rw)$, $w \in T$.

The *Nevanlinna class* $N(X)$ is the space of all $f \in H(X)$ for which

$$|f| = \sup_{0 \leqslant r < 1} \int_T \log(1 + \|f(rw)\|)\, dm(w) < \infty,$$

where m is the normalized Lebesgue measure on T.

It is easy to see that the functional $f \mapsto |f|$ is an FG-norm on $N(X)$. Let ν denote the group topology induced by $|\cdot|$.

LEMMA 1.1. *For any $f \in N(X)$ and $z \in \mathbf{D}$*

$$(*) \qquad \|f(z)\| \leqslant \exp\left(\frac{2|f|}{1-|z|}\right) - 1.$$

Proof. In the scalar case the estimate $(*)$ is well known (cf. [16], II.3.1). In general, fix $f \in N(X)$ and $z \in \mathbf{D}$. Find $x^* \in X'$, $\|x^*\| = 1$, such that $|x^*(f(z))| = \|f(z)\|$. Then $|x^* f| \leqslant |f|$, so it suffices to apply $(*)$ to the scalar-valued function $x^* f$.

The above growth estimate implies that $\varkappa|_{N(X)} \leqslant \nu$ and all the balls $V_c = \{f \in N(X)\colon |f| \leqslant c\}$, $c > 0$, are $\varkappa$-closed in $H(X)$. Thus, the group $(N(X), \nu)$ is complete and has enough continuous linear functionals to separate points.

The *Smirnov class* $N_*(X)$ is the subspace of $N(X)$ consisting of all functions f for which the family of measures

$$\{\int_{(\cdot)} \log(1+\|f(rw)\|)\,dm(w):\ 0 \leqslant r < 1\}$$

is uniformly m-continuous, i.e.

$$\lim_{m(A)\to 0} \sup_r \int_A \log(1+\|f(rw)\|)\,dm(w) = 0.$$

The next result has been obtained by L. Drewnowski (cf. [5], Theorem 4.1). It implies the Shapiro–Shields theorem mentioned above. We include its proof for the convenience of the reader.

PROPOSITION 1.2. $N_*(X) = v(N(X)) = \{f \in N(X):\ \lim_{t\to 0} |tf| = 0\}$.

Proof. Let $f \in N(X)$ and assume $\lim_{t\to 0} |tf| = 0$. Given $\varepsilon > 0$, choose $t > 0$ so small that $|tf| < \varepsilon/2$. Now, if A is a measurable subset of T and $m(A)\log(1+t^{-1}) < \varepsilon/2$, then for every $0 \leqslant r < 1$

$$\int_A \log(1+\|(f)_r\|)\,dm \leqslant \int_A \log(1+t\|(f)_r\|)\,dm + \int_A \log(1+t^{-1})\,dm$$
$$< \varepsilon/2 + m(A)\log(1+t^{-1}) < \varepsilon,$$

which proves that $f \in N_*(X)$. Thus, $v(N(X)) \subset N_*(X)$.

Now let $f \in N_*(X)$. If $c > 0$ and $0 \leqslant r < 1$, let

$$A(r, c) = \{w \in T:\ \|(f)_r(w)\| > c\};$$

then $m(A(r, c))\log(1+c) \leqslant |f|$. Thus,

$$\lim_{c\to\infty} \sup_{0\leqslant r<1} m(A(r, c)) = 0.$$

If $0 < t \leqslant 1$, $c > 0$ and $0 \leqslant r < 1$, then $\log(1+t\|(f)_r\|) \leqslant \log(1+tc)$ on $T\setminus A(r, c)$ and $\log(1+t\|(f)_r\|) \leqslant \log(1+\|(f)_r\|)$ on $A(r, c)$. Therefore,

$$\int_T \log(1+\|t(f)_r\|)\,dm \leqslant \log(1+tc) + \int_{A(r,c)} \log(1+\|(f)_r\|)\,dm,$$

from which, using the uniform m-continuity of $\int_{(\cdot)} \log(1+\|(f)_r\|)\,dm$ $(0 \leqslant r < 1)$ we have $\lim_{t\to 0} |tf| = 0$. Thus, $N_*(X) \subset v(N(X))$, so the proof is finished.

COROLLARY 1.3. *$(N_*(X), v|_{N_*(X)})$ is an F-space with separating dual.*

Denote by $H_\infty(X)$ the space of all bounded, X-valued analytic functions on $\boldsymbol{D}$. We shall use in the sequel the following simple generalization of the classical F. and R. Nevanlinna theorem:

LEMMA 1.4. *An X-valued analytic function f belongs to $N(X)$ if and only if it may be written in the form $f = g/h$, where $g \in H_\infty(X)$, $h \in H_\infty$.*

2. The Fatou property. Let Y be a subset of $H(X)$. We say that Y has the *Fatou property* ($Y \in$(FP)) if every function $f \in Y$ has a radial limit $\lim_{r\to 1-} f(rw) = f^*(w)$ for almost every $w \in T$.

It is one of the most important and well-known theorems in the theory of H_p-spaces that the scalar Nevanlinna class N has the Fatou property. However, this result cannot be extended to vector-valued functions. Indeed, $f(z) = (z, z^2, z^3, \ldots)$ is a c_0-valued bounded analytic function without a radial limit at every point of T.

The following result is a consequence of Lemma 1.4 and the fact that $H_\infty \in$(FP).

PROPOSITION 2.1. *For any Banach space X the following assertions are equivalent:*

(a) $H_\infty(X) \in$(FP).
(b) $N_*(X) \in$(FP).
(c) $N(X) \in$(FP).

Assertion (a) means that X has the *analytic Radon–Nikodym property*. This property was defined for the first time by A. V. Bukhvalov [1]. Recently, A. V. Bukhvalov and A. A. Danilevich [2] have shown that every Banach space with the Radon–Nikodym property has the analytic Radon–Nikodym property. The converse is not true: $L_1(0, 1)$ is a counterexample.

For the definition of the usual Radon–Nikodym property and information about classes of Banach spaces having this property the reader is referred to [4].

Recently, G. A. Edgar has characterized the analytic Radon–Nikodym property in terms of the convergence of analytic martingales (see [9]).

THEOREM 2.2. *For any $f \in N_*(X)$ the following assertions are equivalent:*

(a) *f has a radial limit almost everywhere.*
(b) $\lim_{r\to 1-} |f - f(r\cdot)| = 0$.
(c) *$f \in \overline{P(X)}^v$, where $P(X)$ is the space of X-valued polynomials, i.e. functions of the form $g(z) = x_0 + x_1 z + \ldots + x_n z^n$, where $x_0, x_1, \ldots, x_n \in X$, $z \in \boldsymbol{D}$, $n \in \boldsymbol{N}$.*
(d) *$f \in \overline{X \otimes N_*}^v$, where $X \otimes N_*$ is the linear subspace of $N_*(X)$ spanned by the functions $x \otimes g$ $(x \otimes g(z) = g(z)x)$, $x \in X$, $g \in N_*$.*

Proof. (a)$\Rightarrow$(b). Suppose $\lim_{r\to 1-} f(rw) = f^*(w)$ exists a.e. Then f^* is Bochner measurable. By Fatou's lemma

$$\int_T \log(1+\|f^*\|)\,dm \leqslant \lim_{r\to 1-} \int_T \log(1+\|(f)_r\|)\,dm$$

(the integrals on the right-hand side of the above inequality increase, because the function $h(z) = \log(1+\|f(z)\|)$ is subharmonic; see [7], Theorem 1.6).

Using the uniform m-continuity of $\{\int_{(\cdot)}\log(1+\|(f)_r\|)\,dm:\ 0\leqslant r<1\}$ and Vitali's theorem we obtain

$$(*)\qquad \lim_{r\to 1-}\int_E \log(1+\|(f)_r\|)\,dm=\int_E \log(1+\|f^*\|)\,dm$$

for any measurable subset E of T.

Fix $\varepsilon>0$ and a sequence $(r_n)\subset[0,1)$, $r_n\to 1$. There is a measurable set $E_0\subset T$ such that

$$\int_{E_0}\log(1+\|(f)_{r_n}\|)\,dm\leqslant\varepsilon/2,\quad n\in N,\qquad \int_{E_0}\log(1+\|f^*\|)\,dm\leqslant\varepsilon/2,$$

and the sequence $((f)_{r_n})$ tends to f^* uniformly on $T\setminus E_0$. From this and $(*)$ we have

$$\limsup_{n\to\infty}|f-f(r_n\cdot)|=\limsup_{n\to\infty}\int_T\log(1+\|f^*-(f)_{r_n}\|)\,dm$$
$$\leqslant\lim_{n\to\infty}\int_{T\setminus E_0}\log(1+\|f^*-(f)_{r_n}\|)\,dm+\int_{E_0}\log(1+\|f^*\|)\,dm$$
$$+\limsup_{n\to\infty}\int_{E_0}\log(1+\|f)_{r_n}\|)\,dm\leqslant\varepsilon.$$

(b) $\Rightarrow$ (c). For any $r\in(0,1)$ the function $f(r\cdot)$ is analytic on $r^{-1}\boldsymbol{D}$. Hence, its Taylor series is uniformly convergent on $\boldsymbol{D}$, so also convergent in $N_*(X)$. Therefore, $f(r\cdot)\in\overline{P(X)}^{\nu}$. By (b), this implies $f\in\overline{P(X)}^{\nu}$.

(c) $\Rightarrow$ (d). This implication is obvious, because $P(X)\subset X\otimes N_*$.

(d) $\Rightarrow$ (b). It is well known that (b) holds for functions from N_*, hence also for any function $x\otimes g$ $(x\in X,\ g\in N_*)$. Now, it suffices to observe that the set $\{f\in N_*(X):\ \lim_{r\to 1-}|f-f(r\cdot)|=0\}$ is a closed subspace of $N_*(X)$.

In order to prove the remaining implication (b) $\Rightarrow$ (a) we need the following lemma.

LEMMA 2.3. *Let $g\in H_\infty(X)$. Suppose that there exists a sequence $(r_n)\subset[0,1)$, $r_n\to 1$, such that the sequence of functions $((g)_{r_n})$ is convergent almost everywhere on T. Then g has a radial limit $\lim_{r\to 1-}g(rw)$ for almost all $w\in T$.*

Proof. Fix $g\in H_\infty(X)$ and (r_n) as in the assumptions of the lemma. Let $g^*=\lim_{n\to\infty}(g)_{r_n}$ a.e. Of course $g^*\in L_\infty(T,m;X)$, so the function $[0,1]\ni t\mapsto g^*(e^{2\pi it})$ is Bochner integrable on $[0,1]$ with respect to the Lebesgue measure. Thus, for almost all $s\in[0,1]$ one has

$$(+)\qquad \lim_{h\to 0}h^{-1}\int_s^{s+h}g^*(e^{2\pi it})\,dt=g^*(e^{2\pi is}).$$

Let $P[g^*]$ be the Poisson integral of g^*, i.e.

$$P[g^*](rw_0)=\int_T P_r(w_0w^{-1})g^*(w)\,dm(w)\qquad\text{for }0\leqslant r<1,\ w_0\in T,$$

where $P_r(w)=(1-w^2)/|1-rw|^2$ is the Poisson kernel. The same proof as in the scalar case (see [7], Theorem 1.2) shows that $\lim_{r\to 1-}P[g^*](re^{2\pi is})$ exists at every point $s\in[0,1]$ for which $(+)$ is satisfied. However, by the continuity of $(g)_{r_n}$, $n=1,2,\ldots$, on T and the Dominated Convergence Theorem

$$g(rw_0)=\lim_{n\to\infty}g(rr_nw_0)=\lim_{n\to\infty}P[(g)_{r_n}](rw_0)$$
$$=P[g^*](rw_0)\qquad\text{for any }r\in[0,1)\text{ and }w_0\in T.$$

Thus, $\lim_{r\to 1-}g(rw)$ exists for almost all $w\in T$.

Proof of the implication (b) $\Rightarrow$ (a). Let $f\in N_*(X)$ be such that $|f-f(r\cdot)|\to 0$ $(r\to 1-)$. Fix any sequence $(r_n)\subset[0,1)$, $r_n\to 1$. Then

$$(**)\qquad |f(r_n\cdot)-f(r_m\cdot)|=\int_T\log(1+\|(f)_{r_n}-(f)_{r_m}\|)\,dm$$

for any $n,m\in N$.

We denote by $L_G(T,m;X)$ the Orlicz space defined by the function $G(t)=\log(1+t)$, i.e. the space of all Bochner measurable X-valued functions u on T for which $|u|_G=\int_T\log(1+\|u\|)\,dm<\infty$. It is well known that the metric $d(u,v)=|u-v|_G$ induces on $L_G(T,m;X)$ a complete vector topology. By $(**)$, $((f)_{r_n})$ is a Cauchy sequence in $L_G(T,m;X)$, so it converges to some function $f^*\in L_G(T,m;X)$. It follows that there is a subsequence of $((f)_{r_n})$ which converges to f^* a.e.

We have thus shown that there exists some sequence $(s_n)\subset[0,1)$, $s_n\to 1$, such that $\lim_{n\to\infty}f(s_nw)=f^*(w)$ for almost all $w\in T$. By Lemma 1.4, $f=g/h$ for some functions $g\in H_\infty(X)$, $h\in H_\infty$. As $H_\infty\in$(FP), the radial limit h^* of h exists almost everywhere. Thus, the limit $\lim_{n\to\infty}g(s_nw)=g^*(w)$ has to exist for almost all $w\in T$. By Lemma 2.3, $\lim_{r\to 1-}g(rw)=g^*(w)$ a.e. Finally, $g^*(w)/h^*(w)=\lim_{r\to 1-}g(rw)/h(rw)$ for almost all $w\in T$. The proof is finished.

Theorem 2.2 and Proposition 1.2 imply

COROLLARY 2.4. *The closure in $N(X)$ of the space $P(X)$ of all polynomials is the largest subspace of $N(X)$ with the Fatou property which is a t.v.s. in the relative topology.*

We shall denote $\overline{P(X)}^{\nu}$ by $N_0(X)$.

3. The Fréchet envelope of $N_0(X)$. If $f\in H(X)$, then its nth Taylor coefficient is denoted by $\hat f(n)$.

For any $f\in H(X)$ and $c>0$ we define

$$\|f\|_c=\sum_{n=0}^{\infty}\|\hat f(n)\|\exp(-c\sqrt{n}).$$

Of course $\|\cdot\|_c$ is a norm on $F_c(X)=\{f\in H(X):\ \|f\|_c<\infty\}$. It is easy to see that for any sequence $(x_n) \subset X$ such that $\sum_{n=0}^{\infty} \|x_n\| \exp(-c\sqrt{n}) < \infty$, the series $\sum_{n=0}^{\infty} x_n z^n$ is convergent on D. This implies that

(3.1) the mapping $f \mapsto (\hat{f}(n))$ is an isometry of $(F_c(X), \|\cdot\|_c)$ *onto* $L_1(N, \mu_c; X)$, where μ_c is the finite measure on 2^N defined by $\mu_c(\{n\}) = \exp(-c\sqrt{n})$.

Therefore,

(3.2) $(F_c(X), \|\cdot\|_c)$ is a Banach space for every $c > 0$.

Let $F(X) = \bigcup_{c>0} F_c(X)$. Every functional $\|\cdot\|_c$ is an FG-norm on $F(X)$. Denote by φ_c the group topology induced on $F(X)$ by $\|\cdot\|_c$. Let $\varphi = \sup\{\varphi_c: c > 0\}$, i.e. φ is the weakest group topology which is stronger than φ_c for every $c > 0$. For any $0 < c < d$ we have $\|\cdot\|_c \geqslant \|\cdot\|_d$, so $\varphi_d \subset \varphi_c$. This implies that $\varphi = \sup\{\varphi_w: w \text{ positive and rational}\}$. Therefore, $(F(X), \varphi)$ is a metrizable topological vector group which has a base at zero consisting of convex sets. By (3.2), φ is complete, and thus $(F(X), \varphi)$ is an FG-space.

We denote by $F_*(X)$ the largest subspace of $F(X)$ which is a t.v.s. in the relative topology. $F_*(X)$ is a Fréchet space and, obviously,

$$F_*(X) = v(F(X)) = \bigcap_{c>0} F_c(X).$$

Observe that $F_*(X)$ is not locally bounded. Indeed, the following general fact is easily verified:

FACT. *Suppose that $(X_n, \|\cdot\|_n)$, $n = 1, 2, \ldots$, is a sequence of Banach spaces such that:*

1. *X_{n+1} is continuously included in X_n, $n = 1, 2, \ldots$*
2. *$\bigcap_n X_n$ is dense in X_n for $n = 1, 2, \ldots$*

Then $\bigcap_n X_n$ equipped with the natural projective topology (i.e. the weakest vector topology for which all inclusions $\bigcap_n X_n \hookrightarrow X_m$, $m = 1, 2, \ldots$, are continuous) is locally bounded if and only if there exists $n_0 \in N$ such that $X_n = X_{n_0}$ for $n \geqslant n_0$.

Of course, $F_c(X) \neq F_d(X)$ if $c \neq d$.

PROPOSITION 3.1. *For any Banach space X:*

(a) *$N_*(X) \subset F_*(X)$ and the inclusion mapping is continuous.*
(b) *$N_0(X)$ is dense in $F_*(X)$.*

The proof is based on the folllowing lemma, which can be obtained in the same way as in the scalar case (see [16], II.11.2).

LEMMA 3.2. *For every $c > 0$ there is a positive constant k_c such that*

$$\|\hat{f}(n)\| \leqslant k_c \exp(4\sqrt{cn}), \quad n = 0, 1, \ldots,$$

if $f \in H(X)$ satisfies

$$\|f(z)\| \leqslant \exp\left(\frac{c}{1-|z|}\right), \quad z \in D.$$

Proof of Proposition 3.1. (a) From Lemmas 1.1 and 3.2 it follows directly that for every $c > 0$ the FG-norm $\|\cdot\|_c$ is bounded on any neighbourhood of zero $\{f \in N(X): |f| \leqslant (c')^2/32\}$, where $0 < c' < c$. Thus, $N(X) \subset F(X)$ and the inclusion mapping is continuous. Therefore,

$$N_*(X) = v(N(X)) \subset v(F(X)) = F_*(X).$$

(b) Define $T: F_*(X) \to Y = \bigcap_{c>0} L_1(N, \mu_c; X)$ by $Tf = (\hat{f}(n))$, where μ_c is defined in (3.1). We equip Y with the natural projective topology. Then it is easily verified that T is an isomorphism which maps the space $P(X)$ of polynomials onto the dense subspace of Y consisting of all finitely supported sequences. Therefore, $P(X)$ (thus also $N_0(X)$) is dense in $F_*(X)$.

LEMMA 3.3. *For any $c > 0$, let*

$$f_c(z) = \exp\left(\frac{c}{2}\frac{1+z}{1-z}\right), \quad z \in D.$$

Then $f_c \in N$, $|f_c| = c$, and

$$\inf_n |\hat{f}_c(n)| \exp(-\sqrt{cn}) > 0.$$

The above estimate of the Taylor coefficients of f_c is well known (cf. [16], II.11.2).

LEMMA 3.4. *Let $V_c = \{f \in H(X): |f| \leqslant c\}$, $c > 0$. Then*

$$V_c = \overline{V_c \cap N_0(X)}^{\varkappa}.$$

Proof. From the definition of the FG-norm $|\cdot|$ it follows directly that V_c is $\varkappa$-closed in $H(X)$. Therefore, $\overline{V_c \cap N_0(X)}^{\varkappa} \subset V_c$. If $f \in V_c$, then $f(r\cdot) \in V_c \cap N_0(X)$ for every $0 \leqslant r < 1$. Moreover, $f(r\cdot) \to f$ $(\varkappa)$ as $r \to 1$. Thus, $V_c \subset \overline{V_c \cap N_0(X)}^{\varkappa}$, which completes the proof.

LEMMA 3.5. *Fix $c > 0$. Let (a_n) be a sequence of positive numbers and (x_n^*) a sequence of continuous linear functionals on X such that for any $f \in V_c \cap N_0(X)$*

$$(*) \qquad |x_n^*(\hat{f}(n))| \leqslant a_n, \quad n = 0, 1, \ldots$$

Then

$$\|x_n^*\| \leqslant K a_n \exp(-\sqrt{cn}), \quad n = 0, 1, \ldots,$$

where K is a positive constant which depends only on c.

Proof. Fix $x \in X$, $\|x\| = 1$. Let $h_c = x \otimes f_c$, where f_c is as in Lemma 3.3. Then $h_c \in V_c$ and by Lemma 3.4 there exists a sequence $(g_k) \subset V_c \cap N_0(X)$ which is $\varkappa$-convergent to h_c. Therefore, $\lim_{k\to\infty} \hat{g}_k(n) = \hat{h}_c(n)$ for $n = 0, 1, \ldots$ Using (*) and the continuity of x_n^* we have

$$|x_n^*(\hat{h}_c(n))| = \lim_{k\to\infty} |x_n^*(\hat{g}_k(n))| \leqslant a_n, \quad n = 0, 1, \ldots$$

However, $\hat{h}_c(n) = (x \otimes f_c)\hat{}\,(n) = \hat{f}_c(n)\,x$, so by Lemma 3.3

$$|x_n^*(x)| \leqslant a_n |\hat{f}_c(n)|^{-1} \leqslant K a_n \exp(-\sqrt{cn}), \quad n = 0, 1, \ldots,$$

for some positive constant K which depends only on c. This completes the proof, because $x \in X$, $\|x\| = 1$, was chosen arbitrarily.

Theorem 3.6. *For any Banach space X, $F_*(X)$ is the Fréchet envelope of $N_0(X)$.*

Proof. By Proposition 3.1 the inclusion maps continuously $N_0(X)$ onto a dense subspace of $F_*(X)$. Thus, in order to prove the theorem it is enough to show that every ν-continuous linear functional on $N_0(X)$ is also φ-continuous.

Fix $T \in (N_0(X))'$. Then there is a $c > 0$ such that

(a) $$|Tf| \leqslant 1 \quad \text{for any } f \in V_c \cap N_0(X).$$

Define $x_n^*(x) = T(x \otimes z^n)$, $x \in X$, $n \in N$. Of course $x_n \in X'$, $n = 0, 1, \ldots$ We shall show that

(b) $$\sup_n \|x_n^*\| \exp(c'\sqrt{n}) < \infty \quad \text{for some } c' > 0.$$

Indeed, observe that for any $\zeta \in \boldsymbol{D}$ and $f \in N_0(X)$ the series $f(\zeta z) = \sum_{n=0}^{\infty} \hat{f}(n) \zeta^n z^n$ is uniformly convergent on $\boldsymbol{D}$, so it is convergent in $N_0(X)$ as well. Therefore,

(c) $$T(f(\zeta\cdot)) = \sum_{n=0}^{\infty} T(\hat{f}(n) \otimes z^n)\zeta^n = \sum_{n=0}^{\infty} x_n^*(\hat{f}(n))\zeta^n$$

for every $\zeta \in \boldsymbol{D}$. This implies that for every $f \in V_c \cap N_0(X)$ the function f^T defined on $\boldsymbol{D}$ by $f^T(\zeta) = T(f(\zeta\cdot))$ is analytic and bounded by 1. Thus,

(d) $$|(f^T)\hat{}\,(n)| = |x_n^*(\hat{f}(n))| \leqslant 1 \quad \text{for } f \in V_c \cap N_0(X),\ n = 0, 1, \ldots$$

Now we obtain (b) using Lemma 3.5 with $a_n = 1$, $n = 0, 1, \ldots$

For any $f \in N_0(X)$ we have $|f - f(r\cdot)| \to 0$ $(r \to 1-)$. This, (b), and (c) imply that

$$|Tf| = \lim_{r\to 1} |T(f(r\cdot))| \leqslant \sum_{n=0}^{\infty} |x_n^*(\hat{f}(n))|$$
$$\leqslant \sum_{n=0}^{\infty} \|x_n^*\| \, \|\hat{f}(n)\| \leqslant K' \sum_{n=0}^{\infty} \|\hat{f}(n)\| \exp(-c'\sqrt{n})$$

for some positive constants K', c' and every $f \in N_0(X)$. Therefore, T is continuous with respect to the topology induced on $N_0(X)$ by $F_*(X)$.

Corollary 3.7. *If a Banach space X has the analytic Radon–Nikodym property, then $(N_*(X))\hat{} = F_*(X)$.*

It is not known whether Corollary 3.7 is true without the assumption that X has the analytic Radon–Nikodym property. This question is clearly equivalent to the following

Problem 3.8. Let X be a Banach space. Is the subspace $N_0(X)$ weakly dense in $N_*(X)$?

We recall that if X does not have the analytic Radon–Nikodym property then $N_0(X)$ is a proper closed subspace of $N_*(X)$.

We conclude this section by showing another representation of $(N_0(X))\hat{}$.

It is straightforward to check that

(3.3) for any positive continuous function Φ on $[0, 1)$ with $\Phi(r) \to 0$ as $r \to 1$, the space

$$A(\Phi, X) = \{f \in H(X) \colon \|f\|_\Phi = \sup_{z \in \boldsymbol{D}} \|f(z)\| \, \Phi(|z|) < \infty\}$$

equipped with the norm $\|\cdot\|_\Phi$ is a Banach space.

Moreover,

(3.4) $$A_0(\Phi, X) = \{f \in H(X) \colon \lim_{|z|\to 1} \|f(z)\| \, \Phi(|z|) = 0\}$$

is a closed subspace of $A(\Phi, X)$ containing the space $P(X)$ of polynomials as a dense subspace.

Indeed, if $f \in A_0(\Phi, X)$ then $f(r\cdot) \to f$ in $A_0(\Phi, X)$ as $r \to 1$. Since each $f(r\cdot)$ can be uniformly approximated on $\boldsymbol{D}$ by polynomials, $P(X)$ is dense in $A_0(\Phi, X)$.

For any $c > 0$ and $0 \leqslant r < 1$ we define

$$\Phi_c(r) = \exp\left(-\frac{c}{1-r}\right).$$

Let $F^1(X) = \bigcup_{c>0} A(\Phi_c, X)$ and let ψ be the supremum of the group topologies defined on $F^1(X)$ by all FG-norms $\|\cdot\|_{\Phi_c}$, $c > 0$. Then $(F^1(X), \psi)$ is an FG-space and, by Lemmas 1.1 and 3.2, $N(X) \subset F^1(X) \subset F(X)$ and the inclusion mappings are continuous. Now we easily check that

$$N_*(X) \subset v(F^1(X)) = \bigcap_{c>0} A(\Phi_c, X) = \bigcap_{c>0} A_0(\Phi_c, X) \subset F_*(X),$$

so $F_*^1(X) = \bigcap_{c>0} A(\Phi_c, X)$ is a Fréchet space containing $N_0(X)$ as a dense subspace and inducing on $N_0(X)$ the Mackey topology (cf. Theorem 3.6). This implies

PROPOSITION 3.9. *For any Banach space X the spaces $F_*(X)$ and $F_*^1(X)$ coincide set-theoretically and topologically.*

4. Applications

(a) *Taylor coefficients*

PROPOSITION 4.1. *For any $f \in F_*(X)$*

$$(*) \qquad \|\hat{f}(n)\| = O\left(\exp\left(o(\sqrt{n})\right)\right).$$

Furthermore, the estimate $(*)$ is best possible: given any positive sequence (c_n) tending to zero, there exists $f \in N_0(X)$ such that

$$\|\hat{f}(n)\| \neq O\left(\exp(c_n\sqrt{n})\right).$$

Proof. The proof of $(*)$ for functions from $F_*(X)$ is very simple and may be omitted.

Suppose that there exists a sequence (c_n) tending to zero such that

$$(4.1) \qquad \sup_n \|\hat{f}(n)\| \exp(-c_n\sqrt{n}) < \infty \quad \text{for every } f \in N_0(X).$$

Let $G = \{f \in H(X): |||f||| = \sum_{n=0}^{\infty} \|\hat{f}(n)\| \exp(-2c_n\sqrt{n}) < \infty\}$. Using the same arguments as in (3.1) and (3.2) we can prove that $(G, |||\cdot|||)$ is a Banach space. The space $P(X)$ of polynomials is dense in G. It is easily verified that $G \subset F_*(X)$. Moreover, by (4.1) we have $N_0(X) \subset G$. From the closed graph theorem follows the continuity of both the inclusion mappings $N_0(X) \subset G \subset F_*(X)$. This implies that G generates on $N_0(X)$ the Mackey topology, because $(N_0(X))^\frown = F_*(X)$ (cf. Theorem 3.6). Therefore, $G = F_*(X)$. However, this is impossible, because $F_*(X)$ is not locally bounded. Consequently, there exists $f \in N_0(X)$ such that $\sup_n \|\hat{f}(n)\| \exp(-c_n\sqrt{n}) = \infty$.

(b) *Mean growth*

PROPOSITION 4.2. *For any $f \in F_*^1(X)$ $(= F_*(X))$*

$$(**) \qquad \|f(z)\| = O\left(\exp\frac{o(1)}{1-|z|}\right) \quad \text{as } |z| \to 1.$$

The estimate $(**)$ is best possible: for any positive continuous function ω on $[0, 1)$ with $\lim_{r\to 1}\omega(r) = 0$, there exists $f \in N_0(X)$ such that

$$\|f(z)\| \neq O\left(\exp\frac{\omega(|z|)}{1-|z|}\right) \quad \text{as } |z| \to 1.$$

Proof. The estimate $(**)$ is easily seen. In the proof that this estimate is best possible we use the same idea as in Proposition 4.1, so we omit details.

Fix any positive continuous function ω on $[0,1)$ with $\lim_{r\to 1}\omega(r) = 0$.

Suppose that

$$(4.2) \qquad \sup_{z\in D} \|f(z)\| \exp\left(-\frac{\omega(|z|)}{1-|z|}\right) < \infty \quad \text{for any } f \in N_0(X).$$

Let

$$\Phi(r) = \exp\left(-\frac{2\omega(r)}{1-r}\right) \quad \text{for } r \in [0, 1).$$

By (3.4) and (4.2) the space $A_0(\Phi, X)$ defined by Φ is a Banach space containing $N_0(X)$ as a dense subspace. It is easy to check that $A_0(\Phi, X) \subset F_*^1(X)$. Using Proposition 3.9 we conclude, as in the proof of Proposition 4.1, that $A_0(\Phi, X) = F_*^1(X)$. This contradicts the fact that $F_*^1(X)$ is not locally bounded.

(c) *Multipliers from $N_*(X)$ into H_p $(0 < p \leq \infty)$.* A sequence (x_n^*) of continuous linear functionals on X is said to be a *multiplier* of $N_*(X)$ into H_p $(0 < p \leq \infty)$ if $\sum_{n=0}^{\infty} x_n^*(\hat{f}(n)) z^n \in H_p$ whenever $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in N_*(X)$.

Any multiplier $T = (x_n^*)$ induces a linear operator

$$\tilde{T}: \sum_{n=0}^{\infty} \hat{f}(n) z^n \mapsto \sum_{n=0}^{\infty} x_n^*(\hat{f}(n)) z^n$$

from $N_*(X)$ into H_p. The closed graph theorem tells us that $\tilde{T}$ is continuous.

PROPOSITION 4.3. *Let X be a Banach space. For any sequence $T = (x_n^*) \subset X'$ the following assertions are equivalent:*

(a) *T is a multiplier of $N_*(X)$ into H_p for every $0 < p \leq \infty$.*
(b) *T is a multiplier of $N_*(X)$ into H_p for some $0 < p \leq \infty$.*
(c) *$\sup_n \|x_n^*\| \exp(c\sqrt{n}) < \infty$ for some $c > 0$.*

Proof. Obviously, (a) implies (b).

(b) ⇒ (c). Suppose that $T = (x_n^*)$ is a multiplier of $N_*(X)$ into H_p. We may assume that $0 < p < 1$. The induced operator $\tilde{T}$ is continuous, so there exists a positive constant c such that

$$(4.3) \qquad \|\tilde{T}f\|_p \leq 1 \quad \text{for any } f \in N_0(X) \cap V_c,$$

where $V_c = \{f \in H(X): |f| \leq c\}$.

It was proved by Hardy and Littlewood (see [7], Theorem 6.4) that there is a $C > 0$ such that

$$(4.4) \qquad |\hat{g}(n)| \leq Cn^{1/p-1}\|g\|_p \quad \text{for any } g \in H_p.$$

By (4.3) and (4.4) we obtain

$$(4.5) \qquad |x_n^*(\hat{f}(n))| \leq Cn^{1/p-1} \quad \text{for any } f \in N_0(X) \cap V_c.$$

Using Lemma 3.5 we have

$$\|x_n^*\| \leqslant C' n^{1/p-1} \exp(-\sqrt{cn}) \leqslant C'' \exp(-c'\sqrt{n})$$

for some $C', C'', c' > 0$.

(c) $\Rightarrow$ (a). It is easily verified that if (c) holds and $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in N_*(X)$, then the function $(\tilde{T}f)(z) = \sum_{n=0}^{\infty} x_n^*(\hat{f}(n)) z^n$ is analytic on $\boldsymbol{D}$ and continuous on $\bar{\boldsymbol{D}}$. Thus, $\tilde{T}f \in H_p$ for any $0 < p \leqslant \infty$.

Remark 4.4. Proposition 4.1 and 4.2 simplify the proofs and extend Theorems 1, 2 and 3, 4 in [24], while Proposition 4.3 is an extension of Theorem 2 in [23].

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Added in proof (June 1989). After this paper had been completed the author showed (*Vector-valued H_p-spaces; copies of l^∞ and c_0. Duality when* $0 < p < 1$, preprint) that $P(X)$ is always weakly dense in $N_*(X)$ (see Problem 3.8). Consequently, for every Banach space X, $F_*(X)$ is the Fréchet envelope of $N_*(X)$.