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A characterization of weakly amenable Banach algebras

by

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Abstract. Let $\mathfrak{A}$ be a commutative Banach algebra. Then $\mathfrak{A}$ is called *weakly amenable* if each bounded module derivation into any commutative Banach $\mathfrak{A}$-module is necessarily zero. We give necessary and sufficient conditions for $\mathfrak{A}$ to be weakly amenable in terms of the projective tensor product $\mathfrak{A}\hat{\otimes}\mathfrak{A}$. These conditions correspond to the conditions for amenability given in Theorem II.21 of [10], and Theorem 3.11 of [3]. We use our characterization of weak amenability to describe weak amenability for a large class of weighted discrete convolution algebras.

Introduction. The notion of weak amenability was introduced by W. G. Bade, P. C. Curtis, Jr., and H. G. Dales in [1]. A Banach algebra $\mathfrak{A}$ is called *weakly amenable* if it is commutative and whenever X is a commutative Banach $\mathfrak{A}$-module and $D\colon \mathfrak{A}\to X$ is a bounded derivation, then D is necessarily zero. The question whether bounded derivations are necessarily zero has also been considered in [5] and [6]. As the name of the concept suggests, weak amenability is derived from the stronger concept of amenability introduced by B. E. Johnson (see [8]) and a principal aim of the paper [1] was to exhibit classes of weakly amenable Banach algebras which are not amenable.

Amenability of Banach algebras has been one of the major themes in the homology theory of Banach algebras developed by A. Ya. Khelemskii and others, expounded in the survey article [10]. The characterization of amenability obtained here has recently been investigated by P. C. Curtis, Jr., and R. J. Loy in [3], replacing the homology techniques by arguments based on B. E. Johnson's original approach. It is our objective in this paper to demonstrate the corresponding characterization of weak amenability.

Entirely crucial for amenability is the existence of bounded approximate identities in certain closed ideals, naturally defined from the algebra in question. A rather straightforward observation (see e.g. Proposition 29.7 of [2]) states that a Banach algebra $\mathfrak{A}$ has a right bounded approximate identity if and only if $\mathfrak{A}^{**}$, equipped with the Arens multiplication, has a right unit. Most of the results on the characterization of amenability are based on variations of this observation, reflecting the fact that amenability is a rather algebraic concept, being characterized by the existence of elements with certain algebraic properties. In this paper we prove that the rôle for amenability of "I has a (one-sided) bounded approximate identity" is replaced by "linear combinations of products of elements from I are dense in I", where I is some closed (left) ideal. This suggests that weak amenability is a concept which to a large extent reflects topological properties of the algebra.

In Section 1 we establish some notation and set up the framework in which we shall be working. In Section 2 we form new weakly amenable Banach algebras from old ones. In Section 3 we give the main characterization and in Section 4 we apply this to a class of discrete convolution algebras.

1. The derivation rule. Throughout $\mathfrak{A}$ will be a Banach algebra and $\mathfrak{A}^\# = \mathbf{C}1 \oplus \mathfrak{A}$ the Banach algebra obtained by formally adjoining an identity, denoted by $\mathbf{1}$, to $\mathfrak{A}$. The notion of a Banach $\mathfrak{A}$-bimodule is defined as in [2] and we shall always assume that module actions are bounded by 1. If X is a Banach $\mathfrak{A}$-bimodule, we define an *X-derivation* to be a linear map D: $\mathfrak{A} \to X$ satisfying the derivation rule

$$\text{(D)} \qquad D(ab) = a \cdot D(b) + D(a) \cdot b$$

for all $a, b \in \mathfrak{A}$. In this paper we shall only consider bounded derivations. An instance of special importance is the inner derivations $\delta_x(a) = x \cdot a - a \cdot x$, defined for each $x \in X$.

The following point of view will be fruitful for the discussion to follow. We make X a Banach $\mathfrak{A}^\#$-bimodule in the obvious way: $\mathbf{1} \cdot x = x \cdot \mathbf{1} = x$ for all $x \in X$. Then we get a sequence of linear maps:

$$\text{(S)}: \quad 0 \to X \xrightarrow{j} B(\mathfrak{A}^\#, X) \xrightarrow{q} B(\mathfrak{A}^\#, X)/\mathrm{Im}\, j \to 0$$

where $B(\mathfrak{A}^\#, X)$ is the set of bounded linear operators from $\mathfrak{A}^\#$ to X and for each $x \in X$ the operator $j(x)$ is defined as $a \to x \cdot a$, $a \in \mathfrak{A}^\#$. Since $x \cdot \mathbf{1} = x$ we see that j is an isometry. Thus it is apparent that (S) is a short exact sequence of Banach spaces, i.e. the kernel of each map is the image of the preceding map. But more is true. If we make $B(\mathfrak{A}^\#, X)$ a Banach $\mathfrak{A}$-bimodule by defining

$$a \cdot S: b \to a \cdot S(b): \mathfrak{A}^\# \to X \quad \text{and} \quad S \cdot a: b \to S(ab): \mathfrak{A}^\# \to X$$

for all $a \in \mathfrak{A}$ and all $S \in B(\mathfrak{A}^\#, X)$, we see that j and q are bimodule homomorphisms.

We shall now restate the definition (D) in terms of the short exact sequence (S). If S: $\mathfrak{A} \to X$ is any bounded linear map, we extend S to a map $S^\#$: $\mathfrak{A}^\# \to X$ by setting $S^\#(\mathbf{1}) = 0$.

PROPOSITION 1.1. *Let S: $\mathfrak{A} \to X$ be a bounded linear map and let* (S) *be defined as above. Then the following are equivalent*:

(i) *S is a derivation.*
(ii) $j \circ S = \delta_{S^\#}$.
(iii) $q \circ \delta_{S^\#} = 0$.

Proof. (ii) states that for all $a \in \mathfrak{A}$ we have $j(Sa) = S^\# \cdot a - a \cdot S^\#$, i.e. for all $a \in \mathfrak{A}$ and all $b \in \mathfrak{A}^\#$ we have $Sa \cdot b = S^\#(ab) - a \cdot S^\#(b)$. In particular, this holds for $b \in \mathfrak{A}$ so that S is a derivation. Conversely, if (i) holds then $S^\#$ is a derivation from $\mathfrak{A}^\#$ to X, so that for all $a, b \in \mathfrak{A}^\#$ we have $S^\#(a) \cdot b = S^\#(a \cdot b) - a \cdot S^\#(b)$. In particular, (ii) holds.

Since (S) is exact, the implication (ii) $\Rightarrow$ (iii) is automatic. Hence assume (iii). By exactness we get for each $a \in \mathfrak{A}$ a (unique) $x_a \in X$ so that for all $b \in \mathfrak{A}^\#$ we have $x_a \cdot b = S^\#(ab) - a \cdot S^\#(b)$. Applying this to $b = \mathbf{1}$ we get $x_a = Sa$ so that (ii) holds. ∎

Remarks. (a) Note that if j has a left inverse which preserves commutators, i.e. a left inverse Θ of j which is linear and satisfies $\Theta(a \cdot T - T \cdot a) = a \cdot \Theta(T) - \Theta(T) \cdot a$ for all $a \in \mathfrak{A}$ and $T \in B(\mathfrak{A}^\#, X)$, then every derivation from $\mathfrak{A}$ into X is inner.

(b) The statement (iii) has the somewhat peculiar consequence that S is a derivation if and only if the range of the inner derivation $\delta_{S^\#}$ is covered by ranges of derivations, i.e. $\delta_{S^\#}(\mathfrak{A}) \subseteq \{\delta_{D^\#}(\mathfrak{A}) \mid D \text{ an } X\text{-derivation from } \mathfrak{A}\}$. A much deeper result of R. V. Kadison [9] states that if $\mathfrak{A}$ is a von Neumann algebra, then a bounded linear map S: $\mathfrak{A} \to \mathfrak{A}$ is a derivation if and only if for each $a \in \mathfrak{A}$ there is a derivation D_a: $\mathfrak{A} \to \mathfrak{A}$ with $D_a(a) = S(a)$.

Of special interest is the module $\mathfrak{A}^*$. The module action of $\mathfrak{A}$ on $\mathfrak{A}^*$ is defined as $\langle a_1, a_2 \cdot f \rangle = \langle a_1 a_2, f \rangle$ ($a_1, a_2 \in \mathfrak{A}$; $f \in \mathfrak{A}^*$). In [1] it was observed that a commutative Banach algebra is weakly amenable if and only if the only $\mathfrak{A}^*$-derivation from $\mathfrak{A}$ is the zero derivation, and in [10] and [3] it was proved that $\mathfrak{A}$ is amenable if and only if the map j in the short exact sequence for $\mathfrak{A}^*$ has a left inverse which is a bimodule homomorphism (cf. Remark (a)). But before we pursue these aspects, we want to establish how to form new weakly amenable Banach algebras from old ones.

2. New from old. From here on, the term "weakly amenable" will be abbreviated to WA. Whenever $\mathfrak{A}$ is a Banach algebra we define $\mathfrak{A}^2 = \mathrm{span}\,\{a_1 a_2 \mid a_i \in \mathfrak{A},\ i = 1, 2\}$.

PROPOSITION 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be commutative Banach algebras and suppose there is a bounded homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ with dense range. If $\mathfrak{A}$ is WA, then so is $\mathfrak{B}$.*

Proof. Straightforward. ∎

PROPOSITION 2.2. *Let $\mathfrak{A}$ be WA and let I be a closed ideal in $\mathfrak{A}$. Then I is WA if and only if $(I^2)^- = I$.*

Proof. Corollary 1.5 of [6]. ■

Proposition 2.3. *Suppose we have a short exact sequence*

$$0 \to I \xrightarrow{i} \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0$$

of commutative Banach algebras and bounded algebra homomorphisms, i.e. $\mathfrak{A}$ *is an extension of* $\mathfrak{B}$ *by* I. *If* I *and* $\mathfrak{B}$ *are WA, then so is* $\mathfrak{A}$.

Proof. Let X be a commutative Banach $\mathfrak{A}$-module and let D be a bounded X-derivation from $\mathfrak{A}$. We may assume that I is a closed ideal in $\mathfrak{A}$. Since I is WA we have $D(\mathfrak{A}) \subseteq I^{\perp} = \{x \in I \mid I\cdot x = \{0\}\}$ and $D(I) = \{0\}$. Now $I^{\perp}$ is naturally a commutative Banach $\mathfrak{A}$-module and D drops to a map $\tilde{D}$: $\mathfrak{B} \to I^{\perp}$ which is easily seen to be a bounded derivation. Since $\mathfrak{B}$ is WA, $\tilde{D} \equiv 0$. It follows that $D \equiv 0$. ■

Proposition 2.4. *Let* $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ *be WA Banach algebras. Then* $\mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$ *is WA.*

Proof. Induction, using Proposition 2.3. ■

Proposition 2.5. *Let* $(\mathfrak{A}_i, \varphi_{ij})$ *be a directed system of commutative Banach algebras and bounded homomorphisms, with direct limit* $\mathfrak{A}$. *If all* $\mathfrak{A}_i$'*s are WA then so is* $\mathfrak{A}$.

Proof. By Proposition 2.1 it is enough to prove the statement for $\mathfrak{A}$ which is a direct limit of subalgebras, directed by inclusion, and for such $\mathfrak{A}$ the statement is obvious. ■

Before we proceed, let us note the following, also to be used later. If $\mathfrak{A}$ and $\mathfrak{B}$ are (not necessarily commutative) Banach algebras the projective tensor product $\mathfrak{A}\hat{\otimes}\mathfrak{B}$ is naturally a Banach algebra (see e.g. p. 236 of [2]). Since the embedding $\mathfrak{A} \to \mathfrak{A}^{\#}$ has a left inverse of norm 1 the embeddings $\mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}^{\#}$ are isometries.

Proposition 2.6. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be WA. Then* $\mathfrak{A}\hat{\otimes}\mathfrak{B}$ *is WA.*

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ are WA then $(\mathfrak{A}^2)^- = \mathfrak{A}$ and $(\mathfrak{B}^2)^- = \mathfrak{B}$ so $((\mathfrak{A}\hat{\otimes}\mathfrak{B})^2)^- = \mathfrak{A}\hat{\otimes}\mathfrak{B}$. Since $\mathfrak{A}\hat{\otimes}\mathfrak{B}$ is a closed ideal in $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{B}^{\#}$ the result follows from Proposition 2.2, once we have proved that $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{B}^{\#}$ is WA. But $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{B}^{\#} = ((\mathfrak{A}^{\#}\otimes\mathbf{1}_{\mathfrak{B}})(\mathbf{1}_{\mathfrak{A}}\otimes\mathfrak{B}^{\#}))^-$, and any bounded derivation from $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{B}^{\#}$ into a commutative Banach $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{B}^{\#}$-module vanishes by assumption on $\mathfrak{A}^{\#}\otimes\mathbf{1}_{\mathfrak{B}}$ and $\mathbf{1}_{\mathfrak{A}}\otimes\mathfrak{B}^{\#}$. ■

Proposition 2.7. *Let* $(\mathfrak{A}_i)_{i\in I}$ *be a family of Banach subalgebras of a commutative Banach algebra* $\mathfrak{A}$ *such that the algebra generated by the* $\mathfrak{A}_i$'*s is dense in* $\mathfrak{A}$. *If each* $\mathfrak{A}_i$, $i \in I$, *is WA then so is* $\mathfrak{A}$.

Proof. Let $\mathscr{F} = \{i_1, \ldots, i_n\}$ be a finite subset of I. Define $\mathfrak{A}_{\mathscr{F}}$ to be the closure of the canonical image of $\mathfrak{A}_{i_1}\hat{\otimes}\ldots\hat{\otimes}\mathfrak{A}_{i_n}$. If the finite subsets of I are directed by inclusion we have $\mathfrak{A} = \lim_{\to} \mathfrak{A}_{\mathscr{F}}$. Use Propositions 2.5 and 2.6. ■

Corollary 2.8. *Let* S *be a commutative semigroup which is a union of groups. Then* $l^1(S)$ *is WA.*

It is worthwhile to compare the above results with hereditary properties of amenability. Propositions 2.1, 2.3, 2.4 and 2.6 hold when "amenable" is inserted instead of "WA". In Proposition 2.2 we have an instance of the correspondence between conditions for amenability and WA mentioned in the introduction. The statement "$(I^2)^- = I$" would be replaced by "I has a bounded approximate identity" to obtain the corresponding result for amenability. Propositions 2.5 and 2.7 do not in general hold when "WA" is replaced by "amenable", the reason being that if D: $\mathfrak{A} \to X^*$ is a bounded derivation into a dual $\mathfrak{A}$-bimodule, then the elements defining the inner derivations $D|_{\mathfrak{A}_i}$ may grow in norm. This is perhaps most strikingly illustrated by a theorem of J. Duncan and I. Namioka [4], which states that if S is an infinite abelian idempotent semigroup then $l^1(S)$ is never amenable (cf. Corollary 2.8).

Corollary 2.9. *Let* $(S, +)$ *be an abelian semigroup satisfying the rule* $s+s = t+t = s+t \Rightarrow s = t$ *for all* $s, t \in S$. *Then* $l^1(S)$ *can be embedded in a WA Banach algebra.*

Proof. Any semigroup satisfying this rule can be embedded in a semigroup which is a (disjoint) union of groups [7]. The double conjugate of this semigroup embedding defines the embedding of $l^1(S)$. ■

3. Characterization of weak amenability. We now return to the short exact sequence (S) and shall suppose that the module is a dual module (p. 50 of [2]). Thus, we replace X by X^* and are hence considering bounded linear maps from a Banach algebra $\mathfrak{A}$ into X^*, where X is a Banach $\mathfrak{A}$-bimodule. It is well known that $B(\mathfrak{A}^{\#}, X^*)$ is isometrically isomorphic to $(\mathfrak{A}^{\#}\hat{\otimes}X)^*$, the isometry being given by $\langle a\otimes x, S\rangle = \langle x, Sa\rangle$ for all $a \in \mathfrak{A}^{\#}$, $x \in X$, $S \in B(\mathfrak{A}^{\#}, X^*)$. We shall freely make use of this identification and hence not distinguish notationally between the $B(\mathfrak{A}^{\#}, X^*)$ point of view and the linear functional point of view.

With this in mind, one readily establishes that the short exact sequence (S), with the module being X^*, is (isomorphic to) the dual sequence of

$$\Sigma(X):\quad 0 \to K \xrightarrow{i} \mathfrak{A}^{\#}\hat{\otimes}X \xrightarrow{\pi} X \to 0,$$

where π is given by $\pi(a\otimes x) = a\cdot x$ for all $a \in \mathfrak{A}^{\#}$, $x \in X$, and i is the injection of the kernel K of π. We change the notation accordingly, so that (S) is replaced by

$$\Sigma^*(X):\quad 0 \to X^* \xrightarrow{\pi^*} B(\mathfrak{A}^{\#}, X^*) \xrightarrow{i^*} K^* \to 0.$$

The previously defined module action of $\mathfrak{A}$ on $B(\mathfrak{A}^{\#}, X^*)$ is the dual action of the natural action of $\mathfrak{A}$ on $\mathfrak{A}^{\#}\otimes X$ given by $a\cdot(b\otimes x) = (ab)\otimes x$ and $(b\otimes x)\cdot a = b\otimes(x\cdot a)$ for all $a\in\mathfrak{A}$, $b\in\mathfrak{A}^{\#}$, $x\in X$. By means of Proposition 1.1 we get

PROPOSITION 3.1. *Let $S\in B(\mathfrak{A}, X^*)$. Then S is a derivation if and only if $S^{\#}\perp[K;\mathfrak{A}]$, where $[K;\mathfrak{A}] = \operatorname{span}\{u\cdot a - a\cdot u \mid u\in K, a\in\mathfrak{A}\}$. In particular, there are no nonzero bounded derivations into X^* if and only if $[K;\mathfrak{A}]^- = K$.*

Proof. S is a derivation

$$\Leftrightarrow\ i^*\delta_{S^{\#}} = 0$$
$$\Leftrightarrow\ \delta_{S^{\#}}(a)|_K = 0 \text{ for all } a\in\mathfrak{A}$$
$$\Leftrightarrow\ \langle u, S^{\#}\cdot a - a\cdot S^{\#}\rangle = 0 \text{ for all } u\in K,\ a\in\mathfrak{A}$$
$$\Leftrightarrow\ \langle u\cdot a - a\cdot u, S^{\#}\rangle = 0 \text{ for all } u\in K,\ a\in\mathfrak{A}.$$

To prove the second claim of the proposition, note that $\mathfrak{A}^{\#}\hat{\otimes}X = K\oplus(\mathbf{1}\otimes X)$, where the sum is topologically direct since π has the right inverse $x\to\mathbf{1}\otimes x$. The canonical decomposition is given by $u = (u - \mathbf{1}\otimes\pi(u)) + \mathbf{1}\otimes\pi(u)$ for all $u\in\mathfrak{A}^{\#}\hat{\otimes}X$. Since $S^{\#}(\mathbf{1}) = 0$, i.e. $S^{\#}\perp\mathbf{1}\otimes X$, it follows that the space of bounded derivations is isomorphic as a Banach space to $(K/[K;\mathfrak{A}]^-)^*$, from which the assertion follows. ∎

We shall now focus on the special module $\mathfrak{A}^*$, i.e. we shall be looking at the algebra $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}$. As mentioned, the embeddings $\mathfrak{A}\hat{\otimes}\mathfrak{A}\to\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}\to\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}^{\#}$ are isometries. Let $K_{\#}$ be the kernel of the map $a\otimes b\to ab$: $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}^{\#}\to\mathfrak{A}^{\#}$ and put

$$K = K_{\#}\cap(\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}),\qquad K_0 = K_{\#}\cap(\mathfrak{A}\hat{\otimes}\mathfrak{A}).$$

When $\mathfrak{A}$ is commutative $K_{\#}$ is a closed two-sided ideal of $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}^{\#}$.

THEOREM 3.2. *Let $\mathfrak{A}$ be a commutative Banach algebra. Then the following are equivalent:*

(i) *$\mathfrak{A}$ is WA.*
(ii) $[K;\mathfrak{A}]^- = K$.
(iii) $(K_{\#}^2)^- = K_{\#}$.
(iv) $(K^2)^- = K$.
(v) $(\mathfrak{A}^2)^- = \mathfrak{A}$ *and* $(K_0^2)^- = [(\mathfrak{A}\hat{\otimes}\mathfrak{A})\cdot K_{\#}]^-$.

If $\mathfrak{A}$ has a bounded approximate identity these are equivalent to

(vi) $(K_0^2)^- = K_0$.

Proof. (i)$\Rightarrow$(ii) follows from Proposition 3.1.

(iv)$\Rightarrow$(ii). As observed

$$K = \{u - \mathbf{1}\otimes\pi(u) \mid u\in\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}\} = (\operatorname{span}\{a\otimes b - \mathbf{1}\otimes ab \mid a, b\in\mathfrak{A}\})^-.$$

Here we use $(\lambda\mathbf{1}+a)\otimes\mathbf{1} - \mathbf{1}\otimes(\lambda\mathbf{1}+a) = a\otimes\mathbf{1} - \mathbf{1}\otimes a$ for all $a\in\mathfrak{A}$ and all $\lambda\in C$ Hence

$$(K^2)^- = (\operatorname{span}\{(a\otimes b - \mathbf{1}\otimes ab)(c\otimes d - \mathbf{1}\otimes cd) \mid a, b, c, d\in\mathfrak{A}\})^-.$$

Now

$$(a\otimes b - \mathbf{1}\otimes ab)(c\otimes d - \mathbf{1}\otimes cd) = (a\otimes\mathbf{1} - \mathbf{1}\otimes a)\,\mathbf{1}\otimes b(c\otimes d - \mathbf{1}\otimes cd)$$
$$= a\cdot u - u\cdot a$$

where $u = \mathbf{1}\otimes b(c\otimes d - \mathbf{1}\otimes cd)\in K$. So $(K^2)^- \subseteq [K;\mathfrak{A}]^-$ from which the claim follows.

(i)$\Rightarrow$(iv). If $\mathfrak{A}$ is WA then $(\mathfrak{A}^2)^- = \mathfrak{A}$. Hence

$$\operatorname{span}\{a\otimes bc - \mathbf{1}\otimes abc \mid a, b, c\in\mathfrak{A}\}$$

is dense in K. Let $a, b, c, d\in\mathfrak{A}$ be given. Put $u = a\otimes bc - \mathbf{1}\otimes abc$. Then

$$u\cdot d - d\cdot u = (\mathbf{1}\otimes d - d\otimes\mathbf{1})\,u = (\mathbf{1}\otimes d - d\otimes\mathbf{1})\,\mathbf{1}\otimes c(a\otimes b - \mathbf{1}\otimes ab)$$
$$= (\mathbf{1}\otimes dc - d\otimes c)(a\otimes b - \mathbf{1}\otimes ab)\in K^2.$$

It follows that $(K^2)^- = [K;\mathfrak{A}]^-$ and hence if $\mathfrak{A}$ is WA that $(K^2)^- = K$.

(i)$\Rightarrow$(iii). This is established either by repeating the arguments for (i)$\Leftrightarrow$(iv) or by noting that all we used was that the left factor in $\mathfrak{A}^{\#}\hat{\otimes}\mathfrak{A}$ has an identity. Since $\mathfrak{A}^{\#}$ is WA if and only if $\mathfrak{A}$ is WA the result follows.

(v)$\Rightarrow$(i). We prove that $\mathfrak{A}^{\#}$ is WA. Let D: $\mathfrak{A}^{\#}\to(\mathfrak{A}^{\#})^*$ be a bounded derivation. Then D vanishes on $(K_{\#}^2)^-$ and therefore D vanishes on $(K_0^2)^- = (K_{\#}(\mathfrak{A}\hat{\otimes}\mathfrak{A}))^-$. In particular, D vanishes on elements of the form $(\mathbf{1}\otimes c - c\otimes\mathbf{1})\,a\otimes b = a\otimes bc - ac\otimes b$ so that for all $a, b, c\in\mathfrak{A}$ we have $\langle bc, D(a)\rangle = \langle b, D(ac)\rangle$. Using the derivation rule on the right-hand side we get $\langle ab, D(c)\rangle = 0$. Since $(\mathfrak{A}^2)^- = \mathfrak{A}$ we see that $D\equiv 0$ on $\mathfrak{A}$ and hence on $\mathfrak{A}^{\#}$.

(i)$\Rightarrow$(v). Suppose that $\mathfrak{A}$ is WA. By what has already been proved, we get $((K_{\#})^2)^- = K_{\#}$ and $(\mathfrak{A}^2)^- = \mathfrak{A}$. From the obvious inclusion $((\mathfrak{A}\otimes\mathfrak{A})K_{\#})^- \subseteq K_0$ we get $((\mathfrak{A}\otimes\mathfrak{A})K_{\#})^- = ((\mathfrak{A}\otimes\mathfrak{A})^2 K_{\#}^2)^- \subseteq (K_0^2)^-$. The reverse inclusion is trivial.

(vi)$\Rightarrow$(v). This is obvious, since $(K_0^2)^- \subseteq (K_{\#}(\mathfrak{A}\otimes\mathfrak{A}))^- \subseteq K_0$ always holds.

(v)$\Rightarrow$(vi). From $((\mathfrak{A}\otimes\mathfrak{A})K_0)^- = K_0$ we get

$$(K_0^2)^- = [(\mathfrak{A}\otimes\mathfrak{A})K_{\#}]^- \supseteq ((\mathfrak{A}\otimes\mathfrak{A})K_0)^- = K_0.\ ∎$$

Comparing this with Theorem II.21 of [9], or Theorem 3.11 of [3], we see again that "I has a b.a.i." is replaced by "I^2 is dense in I", where I is the appropriate ideal. Note that the Banach space of derivations from $\mathfrak{A}$ into a dual module is, in general, (isomorphic to) a dual space and if the module is $\mathfrak{A}^*$ the space of bounded $\mathfrak{A}^*$-derivations is isomorphic to the dual of a Banach algebra, namely $(K/[K;\mathfrak{A}]^-)^*$.

4. Additive maps and derivations. In this section we shall investigate the weighted discrete convolution algebras $l^1(S, \omega)$ by means of the techniques from Section 3.

First we introduce some notation. Let $(S, +)$ be an abelian semigroup, not necessarily cancellative, and let $\omega: S \to \mathbf{R}_+$ be a strictly positive weight, i.e. $\omega(s+t) \leqslant \omega(s)\omega(t)$ for all $s, t \in S$. We define the Banach algebra $l^1(S, \omega)$ as a set of formal power series with exponents in S:

$$l^1(S, \omega) = \{\sum_{s\in S} \lambda_s X^s \,|\, \lambda_s \in \mathbf{C},\ \|\sum_{s\in S} \lambda_s X^s\| = \sum |\lambda_s|\,\omega(s) < \infty\}$$

with scalar multiplication and addition defined coefficientwise and the algebra multiplication defined as formal multiplication of power series:

$$(\sum \lambda_s X^s)(\sum \mu_s X^s) = \sum_{u\in S} (\sum_{s+t=u} \lambda_s \mu_t) X^u.$$

It is straightforward to check that these definitions define a Banach algebra. It is well known that

$$l^1(S_1, \omega_1) \hat{\otimes} l^1(S_2, \omega_2) = l^1(S_1 \times S_2, \omega_1 \times \omega_2)$$

through the identification $X^s \otimes X^t = X^{(s,t)}$ and thus that

$$(l^1(S_1, \omega_1) \hat{\otimes} l^1(S_2, \omega_2))^* = l^\infty(S_1 \times S_2, 1/\omega_1 \times 1/\omega_2).$$

On S we define the preorder $s < t$ by $t \in s+S$. We define the following sets:

$$V(t) = \{s \,|\, s < t\},$$

$$V(t)^* = \{f \in \mathbf{C}^{V(t)} \,|\, f(s_1+s_2) = f(s_1)+f(s_2) \text{ whenever } s_1+s_2 \in V(t)\},$$

$$[t-s] = \{u \in S \,|\, u+s = t\}.$$

It is our aim to describe possible WA of $l^1(S, \omega)$ in terms of sets of the type $V(t)^*$. Without loss of generality, we may assume that S has a neutral element. Thus, we shall be dealing with the short exact sequence

$$\Sigma: \quad 0 \to K_0 \to l^1(S \times S, \omega \times \omega) \xrightarrow{\pi} l^1(S, \omega) \to 0$$

where $\pi(X^{(s,t)}) = X^{s+t}$.

LEMMA 4.1.

(i) $K_0 = \{\sum_{s,t\in S} \lambda_{s,t} Y^{s,t} \,|\, \sum |\lambda_{s,t}|\,\omega(s)\,\omega(t) < \infty\}$

where $Y^{s,t} = X^{(s,t)} - X^{(0,s+t)}$.

(ii) $$[K_0;\ l^1(S, \omega)]^- = (K_0^2)^- = (\text{span}\,\{Y^{s+t,u} - Y^{s,t+u} - Y^{t,s+u} \,|\, s, t, u \in S\})^-.$$

Proof. The description (i) of K_0 follows directly from the decomposition $u = u - X^0 \otimes \pi(u) + X^0 \otimes \pi(u)$ used in the proof of Proposition 3.1. Hence a dense subset of $[K_0; l^1(S, \omega)]$ is spanned by the elements $(X^{(s,0)} - X^{(0,s)})\, Y^{t,u} = Y^{s,0}\, Y^{t,u}$, $s, t, u \in S$. Thus $(K_0^2)^- = [K_0; l^1(S, \omega)]^-$. Calculating the product $Y^{s,0} \cdot Y^{t,u}$ we obtain (ii). ∎

Note that K_0 is isomorphic to $l^1(S\backslash(0) \times S, \omega \times \omega)$ as a Banach space. If we define $\mathcal{D} = (K_0^2)^-$ it follows that $m \in l^\infty(S \times S, 1/\omega \times 1/\omega)$ corresponds to a derivation if and only if $m \perp (\mathcal{D} + X^0 \otimes l^1(S, \omega))$.

The next proposition shows that additive maps always give rise to derivations.

PROPOSITION 4.2. *Suppose there exist* $t \in S$ *and* $f \in V(t)^* \backslash (0)$ *such that*

$$\sup\left\{\frac{|f(s)|}{\omega(s)\,\omega(u)} \,\middle|\, s+u = t\right\} = \alpha < \infty.$$

Then $l^1(S, \omega)$ *is not WA.*

Proof. One checks easily by means of Proposition 3.1 that $m \in l^\infty(S \times S, 1/\omega \times 1/\omega)$ defines a derivation if and only if for all $s, u, v \in S$

$$m(s+u, v) = m(s, u+v) + m(u, s+v).$$

Now let $t \in S$ and $f \in V(t)^*$ be as described in the statement of the proposition and define

$$m(s, u) = f(s)\,\chi_{\{t\}}(s+u),$$

where $\chi_{\{t\}}$ is the characteristic function of the set $\{t\}$. Since $\chi_{\{t\}}(s+u) = 0$ where $f(s)$ is not defined, this makes sense. We get for $s, u, v \in S$

$$m(s+u, v) = f(s+u)\,\chi_{\{t\}}(s+u+v) = \begin{cases} 0 & \text{if } s+u+v \neq t, \\ f(s)+f(u) & \text{if } s+u+v = t \end{cases}$$

$$= f(s)\,\chi_{\{t\}}(s+u+v) + f(u)\,\chi_{\{t\}}(s+u+v)$$

$$= m(s, u+v) + m(u, s+v).$$

Clearly $m \in l^\infty(S \times S, 1/\omega \times 1/\omega)$ with $\|m\| = \alpha$. ∎

We seek a converse to this proposition.

PROPOSITION 4.3. *Suppose that* D *is a bounded derivation from* $l^1(S, \omega)$ *into* $l^\infty(S, 1/\omega)$ *such that for some* $t \in S$ *the map*

$$(s, u) \to X^u \cdot D(X^s)$$

for each fixed $s \in V(t)$ *is independent of* u *and not identically zero on the set*

$\{(s, u) \mid s+u = t\}$. *Then there exists* $f \in V(t)^* \setminus (0)$ *such that*

$$\sup\left\{\frac{|f(s)|}{\omega(s)\,\omega(u)}\,\middle|\, s+u = t\right\} < \infty .$$

Proof. Let m be the functional corresponding to D. We shall use the notation $t-s$ for any $u \in [t-s]$. The conditions on D and t are precisely that the map $(s, v) \to m(s, t-s+v)$ is well defined and not identically zero on $V(t) \times S$. Fix $v \in S$. Then for $s_1+s_2 \in V(t)$ we have

$$\begin{aligned} m(s_1+s_2,\, t-(s_1+s_2)+v) &= m(s_1,\, t-(s_1+s_2)+s_2+v)+m(s_2,\, t-(s_1+s_2)+s_1+v) \\ &= m(s_1,\, t-s_1+v)+m(s_2,\, t-s_2+v), \end{aligned}$$

where the last step is valid because the maps considered are well defined. Furthermore,

$$\sup\left\{\frac{|m(s,\, u+v)|}{\omega(s)\,\omega(u)}\,\middle|\, s+u = t\right\} \leqslant \omega(v) \sup\left\{\frac{|m(s,\, u+v)|}{\omega(s)\,\omega(u+v)}\,\middle|\, s+u = t\right\} \leqslant \omega(v)\,\|m\| .$$

If we define $f(s) = m(s, t-s+v)$ for appropriate v we get the conclusion. ∎

We shall now concentrate on giving conditions which ensure that "$\varphi_t(s) = X^{t-s} \cdot D(X^s)$" can be defined and nonzero on $V(t)$ for some $t \in S$. We phrase the question by regarding bounded derivations as bounded linear functionals on the ideal K_0 in $l^1(S \times S, \omega \times \omega)$. That $m \in K_0^*$ corresponds to a derivation means

(i): $\quad m \perp \mathscr{D}$

and that for some $t \in S$ the map φ_t, given by the derivation corresponding to m, is well defined, means that for some $t \in S$

Finally, the nonvanishing of the map φ_t is equivalent to

(ii)$_t$: $\quad m \perp \{Y^{s,v+u} - Y^{s,v+u'} \mid v \in S;\ s \in V(t);\ u, u' \in [t-s]\}$.

Put

$$W_t = \operatorname{span}\{Y^{s,v+u} - Y^{s,v+u'} \mid v \in S;\ s \in V(t);\ u, u' \in [t-s]\},$$

$$Z_t = \operatorname{span}\{Y^{s,v+u} \mid s \in V(t);\ u \in [t-s];\ v \in S\}.$$

Suppose now that we have "failure of additivity", i.e. if D is a bounded derivation such that the map φ_t is well defined for some $t \in S$, then φ_t is necessarily equal to the zero map. Using the descriptions (i), (ii)$_t$, and (iii)$_t$ this means that if $m \perp (\mathscr{D} + W_t)$ then $m \perp Z_t$.

We arrive at the condition

$$\text{(F-A):} \qquad \forall\, t \in S: \quad (\mathscr{D} + W_t)^- \supseteq Z_t .$$

Fix t and v in S. If (F-A) holds, we get $Y^{t,v} \in (\mathscr{D} + W_t)^-$, since clearly $Y^{t,v} \in Z_t$. Let $(d_n + w_n)_{n \in N}$, $d_n \in \mathscr{D}$, $w_n \in W_t$, be a sequence tending to $Y^{t,v}$. Then $(d_n + w_n)\, X^{(0,t)} \to Y^{t,t+v}$ as $n \to \infty$. Since $\mathscr{D}$ is a closed ideal and $W_t X^{(0,t)} = \{0\}$ it follows that $Y^{t,t+v} \in \mathscr{D}$ and hence, since t and v were arbitrary, that $Y^{u,v} \in \mathscr{D}$ whenever $u < v$. We collect our observations in

Lemma 4.5. *Suppose* (F-A) *holds. Then*

(a) $Y^{u,v} \in \mathscr{D}$ *for all* $u, v \in S$ *such that* $u < v$.
(b) $Y^{nu,v} \in \mathscr{D}$ *for all* $u, v \in S$, $n \in N$, $n \geqslant 2$.
(c) *If* $V(u) = \{u, 0\}$ *then* $\mathscr{D} \supseteq Z_u$.

Proof. (a) has already been established. From $Y^{nu,v} - Y^{(n-1)u,u+v} - Y^{u,(n-1)u+v} \in \mathscr{D}$ and (a), it follows that

$$Y^{nu,v} \equiv Y^{(n-1)u,u+v} \pmod{\mathscr{D}}$$

and hence by induction that

$$Y^{nu,v} \equiv Y^{u,(n-1)u+v} \equiv 0 \pmod{\mathscr{D}} \qquad \text{for } n \geqslant 2.$$

To prove (c) just note that if $V(u) = \{u, 0\}$ then $W_u = (0)$. ∎

Lemma 4.6. *Let* $u_1, \ldots, u_n$, $v \in S$ *and suppose that* $Y^{u_i,v} \in \mathscr{D}$ *for* $i = 1, \ldots, n$. *Then* $Y^{u_1+\ldots+u_n,v} \in \mathscr{D}$.

Proof. Put $\sigma = u_1 + \ldots + u_n$ and $\sigma_i = u_1 + \ldots + u_{i-1} + u_{i+1} + \ldots + u_n$. Then a straightforward induction on n shows that

$$Y^{\sigma,v} \equiv \sum_{i=1}^{n} Y^{u_i,\sigma_i+v} \pmod{\mathscr{D}}.$$

Since $\mathscr{D}$ is an ideal in $l^1(S \times S, \omega \times \omega)$ and $Y^{u_i,\sigma_i+v} = Y^{u_i,v} X^{(0,\sigma_i)}$ the claim follows. ∎

A combination of the results of this section yields the desired connection between derivations and additive maps for a large class of semigroups.

Theorem 4.7. *Suppose that S satisfies one of the following:*

(a) *S is a cancellation semigroup.*
(b) *Every element of S is divisible by some* $n \in N$, $n \geqslant 2$.
(c) *If* $u \in S \setminus (0)$ *then* $\langle V(u) \rangle = S$, *where* $\langle V(u) \rangle$ *is the subsemigroup generated by* $V(u)$.

Then for any weight $\omega\colon S \to \mathbf{R}_+$, $l^1(S, \omega)$ is WA if and only if

$$\left\{f \in V(t)^* \,\middle|\, \sup\left\{\frac{|f(s)|}{\omega(s)\,\omega(u)} \,\middle|\, s+u = t\right\} < \infty\right\} = \{0\}$$

for all $t \in S$.

Proof. The "only if" part follows from Proposition 4.2. Now suppose that the condition holds for all $t \in S$. Then (F-A) holds by Proposition 4.3.

If S is a cancellation semigroup we have $W_t = \{0\}$ and thus $\mathscr{D} \supseteq Z_t$ for all $t \in S$. But this clearly implies $\mathscr{D} = K_0$.

If every element of S is n-divisible for some $n \geqslant 2$, possibly depending on the element, then Lemma 4.5(b) implies that $\mathscr{D} = K_0$.

Finally, assume that S satisfies (c) and let $u, v \in S$. Consider first the case $v \neq 0$. Write $u = u_1 + \ldots + u_n$, where $u_i \in V(v)$, $i = 1, \ldots, n$. By Lemmas 4.6 and 4.5(a) it follows that $Y^{u,v} \in \mathscr{D}$. If $v = 0$ and $V(u) = \{u, 0\}$ we get $Y^{u,v} \in \mathscr{D}$ by Lemma 4.5(c). Hence we are left with the case $u = u_1 + u_2$, where $u_i \in S \setminus (0)$, $i = 1, 2$. But then

$$Y^{u,0} \equiv Y^{u_1,u_2} + Y^{u_2,u_1} \pmod{\mathscr{D}}$$

and again, by the first case, $Y^{u,0} \in \mathscr{D}$. It follows in all the cases (a), (b), and (c) that if (F-A) holds then $\mathscr{D} = K_0$ and therefore $l^1(S, \omega)$ is WA. ∎

Corollary 4.8. *Let G be an abelian group and let $\omega\colon G \to \mathbf{R}_+$ be a weight. The Beurling algebra $l^1(G, \omega)$ is WA if and only if*

$$\sup\left\{\frac{|f(g)|}{\omega(g)\,\omega(-g)} \,\middle|\, g \in G\right\} = +\infty$$

for all $f \in \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{C}) \setminus (0)$.

Proof. First observe that $V(t) = G$ for all $t \in G$, so that $V(t)^* = \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{C})$. Let $f \in \mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{C})$ and put

$$\alpha(t) = \sup\left\{\frac{|f(g)|}{\omega(g)\,\omega(t-g)} \,\middle|\, g \in G\right\}.$$

Then $\omega(-t)^{-1}\alpha(t) \leqslant \alpha(0) \leqslant \omega(t)\alpha(t)$ for all $t \in G$, and so the condition in 4.7 holds for every $t \in G$ if and only if it holds for $t = 0$. Thus the corollary is a special case of the theorem. ∎

Remark. In [1] the authors consider conditions for weak amenability of Beurling algebras on the integers. In [1, Theorem 2.2] they give the sufficient condition

$$\text{A:} \quad \frac{\omega(n)\,\omega(-n)}{n} \to 0 \quad \text{as } n \to \infty,$$

and in [1, Theorem 2.3] they give the necessary condition

$$\text{B:} \quad \sup\left\{\frac{\omega(n+m)}{\omega(n)\,\omega(m)}\,\frac{1+|n|}{1+|m+n|} \,\middle|\, m, n \in \mathbf{Z}\right\} = \infty.$$

Noting that each $f \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{C})$ has the form $n \to f(1)n$, the condition of Corollary 4.8 above is in the case $G = \mathbf{Z}$

$$\text{C:} \quad \sup\left\{\frac{n}{\omega(n)\,\omega(-n)} \,\middle|\, n \in \mathbf{Z}\right\} = \infty.$$

It is straightforward to verify $\mathrm{A} \Rightarrow \mathrm{C} \Rightarrow \mathrm{B}$ and clearly $\mathrm{B} \not\Rightarrow \mathrm{C}$ and $\mathrm{C} \not\Rightarrow \mathrm{A}$, so Corollary 4.8 covers a gap left in [1].

Example. Let S be the closed unit interval with binary operation defined by $(s, t) \to \min\{s+t, 1\}$. Then $l^1(S)$ is not WA. We want to determine the canonical form of a derivation, so let $m \in l^\infty(S \times S)$ correspond to a derivation. As in the proof of Proposition 4.3 one sees that for each $t \in S$ the map $s \to m(s, t-s)\colon [0, t] \to \mathbf{C}$ is additive and hence, being bounded, is of the form $s \to \alpha_t s$, where $\alpha_t \in \mathbf{C}$. Since $V(1)^* = \{0\}$ and $V(0) = \{0\}$ we see that α_0 can be chosen arbitrarily, whereas $\alpha_1 = 0$. Put $f(t) = \alpha_t$. Then $\sup\{|tf(t)| \mid t > 0\} \leqslant \|m\|$. In particular, every left translate of f belongs to $l^\infty(S)$. Since $m(s, t) = \langle X^t \mid D(X^s)\rangle$ where D is the derivation under consideration, it follows that

$$(*) \qquad D\Big(\sum_{s \in S} \lambda_s X^s\Big) = \Big(\sum s\lambda_s X^s\Big)\cdot f.$$

Conversely, starting with $f\colon [0, 1] \to \mathbf{C}$ satisfying $f(1) = 0$ and $\sup\{|tf(t)| \mid t > 0\} < \infty$ it is immediate to verify that $(*)$ defines a bounded derivation from $l^1(S)$ into $l^\infty(S)$. Note that since $V(1) = S$ and $V(1)^* = (0)$ the domains of additive maps are essentially restricted to sets of the type $V(s)$, and cannot in general be enlarged to all of S.

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References

[1] W. G. Bade, P. C. Curtis, Jr., and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. 55 (1987) 359–377.

[2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, New York 1973.

[3] P. C. Curtis, Jr., and R. J. Loy, *The structure of amenable Banach algebras*, J. London Math. Soc., to appear.

[4] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Royal Soc. Edinburgh 80A (1978), 309–321.
[5] N. Groenbaek, *Derivations and semigroups, commutative radical Banach algebras*, Ph. D. Thesis, University of California, Los Angeles 1980.
[6] —, *Commutative Banach algebras, module derivations and semigroups*, J. London Math. Soc., to appear.
[7] E. Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. 83 (1956), 70–97.
[8] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).
[9] R. V. Kadison, *Local derivations*, J. Algebra, to appear.
[10] A. Ya. Khelemskii, *Flat Banach modules and amenable algebras*, Trudy Moskov. Mat. Obshch. 47 (1987), 179–218; AMS Translation 1985, 199–244.

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The Fréchet envelopes of vector-valued Smirnov classes

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Abstract. In this paper the Nevanlinna class $N(X)$ and the Smirnov class $N_*(X)$ of analytic functions on the unit disc in the complex plane with values in a Banach space X are studied. Representations of the Fréchet envelope of the Smirnov class by Fréchet spaces of analytic functions are found. The continuous linear functionals on $N_*(X)$ and multipliers of $N_*(X)$ into the Hardy spaces H_p $(0 < p \leqslant \infty)$ are described. The best possible estimates of the mean growth of functions and their Taylor coefficients are obtained. The proofs of our results are mainly soft and simplify earlier ones given in the scalar case by N. Yanagihara [22–24].

0. Introduction. Let $E = (E, \tau)$ be an F-space (i.e. a complete metrizable t.v.s.) with separating dual space and let τ^c be the locally convex topology associated to τ, i.e. the strongest locally convex topology on E which is weaker than τ. Obviously, if $\mathscr{B}$ is a base of neighbourhoods of zero for τ, then the family $\{\operatorname{conv} U\colon\ U \in \mathscr{B}\}$ is a base of neighbourhoods of zero for τ^c. Therefore, the topology τ^c produces the same space E' of continuous linear functionals as τ and has a countable base at zero. The dual space E' of (E, τ^c) separates the points of E, so τ^c is metrizable. The completion $\hat{E}$ of (E, τ^c) is a Fréchet space (i.e. locally convex F-space) which is called the *Fréchet envelope* of E.

Observe that (E, τ^c) is a Mackey space, i.e. τ^c coincides with the Mackey topology of the dual pair (E, E') (cf. [18], IV.3.4). Therefore, the Fréchet envelope of E is the unique, up to isomorphism, Fréchet space containing E as a dense subspace and such that the restriction map $T \to T|_E$ takes $\hat{E}'$ onto E'.

For many classes of concrete (function or sequence) spaces, it is possible to describe $\hat{E}$ as another concrete space. This has been done e.g. for the Hardy spaces H_p $(0 < p < 1)$ [8], the Bergman spaces $B_{p,\alpha}$ $(0 < p < 1,\ \alpha > -1)$ [19], the Orlicz function and sequence spaces [13, 14, 6], and the Lorentz sequence spaces $d(w, p)$ $(0 < p < 1)$ [15]. In 1973, N. Yanagihara [22, 23] identified the dual space and then the Fréchet envelope of the Smirnov class N_* of analytic functions on the unit disc on the complex plane.