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W R O C Ł A W S K A D R U K A R N I A N A U K O W A




# Abstract sliding hump technique and characterization of barrelled spaces

by


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**Abstract.** We present an abstract sliding hump technique in a purely functional-analytic setting, leading to a characterization of certain barrelled spaces. We obtain applications in summability theory and in the theory of function spaces.


**Introduction.** In his paper [S], the second author has answered in the affirmative the question posed by Wilansky whether $c_0$ is the only FK-sequence space $E$ densely containing $\Phi$ whose $\beta$-dual $E^\beta$ is $l^1$. Independently, the same result has been obtained by Bennett in [$B_2$]. The result in [S], and also the proof given in [$B_2$], are essentially based on a characterization of certain barrelled sequence spaces by means of their $\beta$-duals. More precisely, a normed AK-sequence space $E$ whose dual $E'$ is also a normed AK-sequence space is barrelled if and only if its $\beta$-dual $E^\beta$ and its dual $E'$ coincide.

The technique of proof leading to the result in [S] (and similar in [$B_2$]) is based on a sliding hump argument. A detailed analysis, presented in this paper, shows that this method carries over—only under slight modifications—to various related situations. For instance, it permits us to provide new characterizations of dense barrelled subspaces e.g. of spaces $\mathfrak{L}^p$ ($p > 1$) or spaces $C_0(T)$ of continuous functions on a locally compact space $T$ vanishing at infinity. In either case, an appropriate concept of sections and sectional convergence has to be developed and a substitute for the $\beta$-dual has to be introduced in an appropriate way.

This pushes forward the question as to whether a more general method can be afforded, providing as special cases the examples mentioned above and the sequence space version given in [S] and [$B_2$]. Here we present such a method and call it an abstract sliding hump technique. We need to define, in an abstract setting, the notions of sections, sectional convergence and $\beta$-dual, familiar in the context of sequence spaces. It turns out that this is actually possible in a satisfactory way, at least in the light of our applications mentioned above. In these cases the abstract sections and the abstract $\beta$-dual of the space under consideration are just what they are expected to be from the concrete point of view. But once these abstract prerequisites are met, the sliding hump argument carries over in a satisfactory way.

The structure of the paper is as follows. In Section 1 we introduce the abstract concept of a system of sections on a dual pairing $\langle E, F\rangle$, providing motivating examples.

In Section 2 we define the concept of sectional convergence (*Abschnittskonvergenz*) with respect to a given system of sections. We indicate that it provides a natural generalization of the classical sequence space notion known under this name. We end Section 2 with an abstract definition of the $\beta$-dual.

Section 3 contains our main results. Theorem 1 provides a purely functional-analytic version of [S, Satz 3.1] and [$B_2$, Lemma 1]. It relates barrelledness to the concept of $\beta$-dual. Theorem 2 presents a somewhat more general version of Theorem 1 which applies to systems of sections on $\langle E'', E'\rangle$ rather than on $\langle E, E'\rangle$. This gain of generality is compensated by an additional condition on $E$.

In Section 4 we present various applications of our main results in the theory of sequence spaces. We prove once more the fact already obtained in [$B_2$] that $c$ and $cs$ have the Wilansky property (in the sense of [$B_2$]). Our proof, however, is more direct since it uses Theorem 2 instead of the sequence space version of Theorem 1. We provide characterizations of dense barrelled subspaces of various summability domains in terms of the $\beta$-dual. The section closes with two results on scarce copies. For instance, we prove that an AK-sequence space $E \subset m$ having $E^\beta \subset cs$ does not contain any barrelled scarce copy.

In the final Section 5 we present two classes of examples of nonstandard systems of sections. We characterize dense barrelled subspaces of spaces $\mathfrak{L}^p$ $(p > 1)$ and $C_0(T)$ by defining appropriate systems of sections which permit an application of the main results in the concrete case.

For all notions concerning functional analysis we refer the reader to the book [$W_1$]. Summability theory is convered by [ZB] and [$W_2$]. A locally convex vector space is always understood to be separated.

**1. Systems of sections.** Let $\langle E, F\rangle$ be a pair of (real or complex) vector spaces placed in duality by a bilinear mapping $\langle\ ,\ \rangle$: $E\times F\to \boldsymbol{K}$. For every $n\in \boldsymbol{N}$ let $x\to x^{[n]}$ and $y\to y^{[n]}$ be linear mappings on $E$ resp. $F$, continuous for $\sigma(E, F)$ resp. $\sigma(F, E)$, and suppose the following axioms are satisfied:

(S1) $\langle x, y^{[n]}\rangle = \langle x^{[n]}, y^{[n]}\rangle = \langle x^{[n]}, y\rangle$ for every $n\in\boldsymbol{N}$ and all $x\in E$, $y\in F$;

(S2) $(x^{[n]})^{[m]} = x^{[n\wedge m]}$ whenever $x\in E$, $n, m\in\boldsymbol{N}$; $n\wedge m$ denotes $\min(n, m)$.

Then we shall refer to this construct as a *system of sections* on $\langle E, F\rangle$.

Notice that any system of sections on $\langle E, F\rangle$ is uniquely determined by



the mappings $x\to x^{[n]}$, $n\in\boldsymbol{N}$, continuous for $\sigma(E, F)$ and satisfying axiom (S2). Indeed, we may define, for every $n\in\boldsymbol{N}$, a $\sigma(E, F)$-continuous linear functional on $E$ by $x\to\langle x^{[n]}, y\rangle$ for every fixed $y\in F$, and this provides a unique element $y^{[n]}$ of $F$ satisfying $\langle x^{[n]}, y\rangle = \langle x, y^{[n]}\rangle$ for every $x\in E$. Using (S2), one concludes that (S1) is then true. Dually, the system of sections might have been reconstructed from the mappings $y\to y^{[n]}$ satisfying (S2), too.

Let us introduce two more notions. We denote by $E^{[n]}$, $F^{[n]}$, $n\in\boldsymbol{N}$, the space of vectors $x^{[n]}$, $x\in E$, and $y^{[n]}$, $y\in F$, respectively. We refer to $x^{[n]}$, $y^{[n]}$ as the *sections* of $x$, $y$ respectively.

The following lemma will be of repeated use to us during the examination to follow.

LEMMA 1. *Let a system of sections be fixed on $\langle E, F\rangle$. Let $\tilde{E}$ be the completion of $E$ with respect to the Mackey topology $\tau(E, F)$. There exists a unique system of sections on $\langle\tilde{E}, F\rangle$ extending the given one.*

Proof. By the above observation we may construct a system of sections on $\langle\tilde{E}, F\rangle$ from the restriction mappings $y\to y^{[n]}$ on $F$ alone. It is therefore sufficient to prove that every restriction $\varrho$: $y\to y^{[n]}$ is $\sigma(F, \tilde{E})$-continuous. $\varrho$ being $\sigma(F, E)$-continuous, its transpose $\varrho'$ maps $E$ to $E$ and is $\sigma(E, F)$-continuous, hence by the Hellinger–Toeplitz theorem it is $\tau(E, F)$-continuous (see [$W_1$, p. 169]), and consequently extends to a continuous linear mapping $\tilde{\varrho}'$: $\tilde{E}\to\tilde{E}$. Clearly its transpose $(\tilde{\varrho}')' = \varrho$ is $\sigma(F, \tilde{E})$-continuous. This proves the result. ■

It is now time to consider a motivating example.

EXAMPLE 1. Let $E$, $F$ be sequence spaces (i.e. subspaces of $\boldsymbol{K}^{\boldsymbol{N}}$), placed in duality by $\langle x, y\rangle = \sum_n x_n y_n$. Suppose that both $E$ and $F$ contain the space $\Phi$ of finite sequences. Then we may define restriction operators by setting

$$x^{[n]} = \sum_{i=1}^{n} x_i e^{(i)}, \qquad y^{[n]} = \sum_{i=1}^{n} y_i e^{(i)},$$

where $e^{(i)}$ denotes the $i$th unit vector in $\boldsymbol{K}^{\boldsymbol{N}}$. This gives rise to a system of sections in the sense introduced above.

The following lemma gives some more information about the structure of a system of sections.

LEMMA 2. *Let a system of sections be fixed on $\langle E, F\rangle$. For every $n\in\boldsymbol{N}$, the topology induced by $\sigma(E, F)$ on $E^{[n]}$ is $\sigma(E^{[n]}, F^{[n]})$. Dually, the topology induced by $\sigma(F, E)$ on $F^{[n]}$ is $\sigma(F^{[n]}, E^{[n]})$.*

Proof. We prove the first statement. Clearly $\sigma(E^{[n]}, F^{[n]})$ is coarser than $\sigma(E, F)$ on $E^{[n]}$. On the other hand, for fixed $y\in F$, the linear functional $\langle\ , y\rangle$ on $E^{[n]}$ equals $\langle\ , y^{[n]}\rangle$ in view of axiom (S1). This proves the claim. ■

**2. Sectional convergence.** In this section we introduce the notion of sectional convergence in our abstract setting of a system of sections.

Let a system of sections be fixed on $\langle E, F\rangle$. Let $\tau$ be any admissible locally convex topology on $E$. Then $E$ is said to be an *AK-space* or to *have sectional convergence* with respect to $\tau$ if, for every $x \in E$, $x^{[n]}$ converges to $x$ $(n \to \infty)$ with respect to $\tau$. Sectional convergence with respect to an admissible topology on $F$ is of course defined analogously.

Looking back to Example 1, we see that our abstract notion of sectional convergence coincides with the classical one known under this name in the setting of sequence spaces. Let us consider one more instructive example.

EXAMPLE 2. Let $E$ be a locally convex vector space with dual $F$. Suppose that with respect to some coarser locally convex topology, $E$ has a Schauder basis $(x_n)_{n=1}^{\infty}$. Then we obtain a system of sections on $\langle E, F\rangle$ by setting

$$x^{[n]} = \sum_{i=1}^{n} \xi_i x_i,$$

where $x = \sum_i \xi_i x_i$ is the representation of $x$. Now it is easy to see that $E$ is an AK-space in its original topology with respect to this system of sections if and only if $(x_n)$ is a Schauder basis with respect to the original topology.

Clearly, sectional convergence with respect to some admissible topology always implies sectional convergence with respect to the weak topology. The converse is true for barrelled spaces.

PROPOSITION 1. *Let a system of sections on $\langle E, F\rangle$ be fixed and suppose $E$ is an AK-space with respect to the weak topology. If $E$ is barrelled, then it is also an AK-space with respect to the Mackey topology.*

Proof. The sequence $(\varrho_n)$ of restriction mappings $E \to E$ converges pointwise to id with respect to the weak topology and therefore is pointwise bounded. $E$ being barrelled, it is equicontinuous. But note that $(\varrho_n)$ converges to id with respect to the Mackey topology on the dense subset $E^{[1]} \cup E^{[2]} \cup \ldots$ of $E$ and, consequently, converges to id on all of $E$ pointwise with respect to $\tau(E, F)$. ∎

Proposition 1 above is well known in the setting of sequence spaces (Example 1). Nevertheless, the following consequence of Proposition 1 seems to have escaped K-space specialists so far, perhaps because it makes use of Lemma 1.

PROPOSITION 2. *Let $E$ be a metrizable locally convex vector space with dual $F$ and let a system of sections be fixed on $\langle E, F\rangle$. Suppose $F$ is an AK-space with respect to the strong topology $\beta(F, E)$. Then $E$ is an AK-space in its metrizable topology.*



Proof. By Lemma 1, the system of sections on $\langle E, F\rangle$ extends to $\langle \tilde{E}, F\rangle$, where $\tilde{E}$ denotes the completion of $E$. But note that $\tilde{E}$ is a barrelled space, hence the result follows from Proposition 1 as soon as we shall have proved that $F$ is an AK-space with respect to $\sigma(F, \tilde{E})$. But this follows from $\sigma(F, \tilde{E}) \subset \beta(F, E)$, which is a consequence of the fact that given $\tilde{x} \in \tilde{E}$ and a sequence $(x_n)$ in $E$ converging to $\tilde{x}$, we have $\{x_n\colon n \in N\}^\circ \subset \{\tilde{x}\}^\circ$. This ends the proof. ∎

In the context of sequence spaces, Proposition 2 tells that if $E$ is a K-space containing $\Phi$ whose dual $F$ may be considered a K-sequence space via the natural bilinear mapping $\langle x, y\rangle = \sum_n x_n y_n$ and if $F$ is a strong AK-space, then $E$ is an AK-space, at least when it is metrizable.

We shall now conclude this section by giving an abstract definition of the $\beta$-dual, a notion which is familiar in sequence space theory.

Let a system of sections be fixed on $\langle E, F\rangle$. We denote by $F^\omega$ the vector space consisting of all families $(y_n)_{n=1}^{\infty}$ of vectors having $y_n \in F^{[n]}$ and $y_n^{[m]} = y_m$ whenever $m < n$. We may consider $F$ a subspace of $F^\omega$ via the correspondence

$$y \to (y^{[n]})_{n=1}^{\infty},$$

in case $E$ (or equivalently $F$) is a weak AK-space. Now we define, and denote by $E^{\langle\beta\rangle}$, the *$\beta$-dual* of $E$ to consist of all elements $(y_n)$ of $F^\omega$ such that $\lim_{n\to\infty} \langle x, y_n\rangle$ exists for every $x \in E$.

Specializing to Example 1, we see that $F^\omega$, in this case, may be identified with $\omega = K^N$. Indeed, every sequence $(y_n)$ in $F^\omega$ gives rise to a unique element $y \in \omega$, and conversely, in view of $\Phi \subset F$, every element of $\omega$ may be considered an element of $F^\omega$ via the natural correspondence above. But now it is clear that, in the context of sequence spaces, our abstract notion of $\beta$-dual is just the same as the classical one, i.e. $E^{\langle\beta\rangle} = E^\beta$.

**3. Characterization of barrelledness.** In this section we shall present our main result. We start with the following proposition, which is known in the context of sequence spaces (see [Sw, Thm. 3.1]).

PROPOSITION 3. *Let $E$ be a barrelled locally convex vector space with dual $F$ and let a system of sections be fixed on $\langle E, F\rangle$ such that $E$ is a (weak) AK-space. Then $E^{\langle\beta\rangle} = F$.*

Proof. $E$ being weakly AK, we have $F \subset F^\omega$, so the claimed statement makes sense. Let $(y_n) \in E^{\langle\beta\rangle}$ and define $\varphi \in E^*$ by setting

$$\varphi(x) = \lim_{n\to\infty} \langle x, y_n\rangle.$$

Then $\varphi$ is continuous by the Banach–Steinhaus theorem, hence we have $\varphi = \langle\ , y\rangle$ for some $y \in F$. Inserting the vectors $x^{[m]}$ in the above equality, we find that $y^{[m]} = y_m$, hence $(y_n) \in F$. The converse clearly follows from the fact that $E$ is (weakly) AK. ∎

We shall now prove a converse to Proposition 3, i.e. we show that, in certain situations, the equality $E^{\langle\beta\rangle} = F$ for a system of sections implies that $E$ is barrelled. In the context of K-spaces, our result has been obtained by the second author (see [S]) and, independently, by Bennett in [B$_2$].

THEOREM 1. *Let $E$ be a normed space with dual $F$ and let a system of sections be fixed on $\langle E, F\rangle$ such that $F$ is an AK-space with respect to the dual norm topology. Then the following statements are equivalent:*

(1) *$E$ is barrelled.*
(2) *Every $E^{[n]}$, $n \in \mathbf{N}$, is barrelled and $E^{\langle\beta\rangle} = F$.*

Proof. Let $E$ be barrelled. By Proposition 1, $E$ is an AK-space, hence $E^{\langle\beta\rangle} = F$ by Proposition 3. That all $E^{[n]}$ are barrelled follows from the fact that the restriction mapping $\varrho_n\colon E \to E^{[n]}$ is quotient and barrelledness is preserved under quotient mappings.

Let us now assume that statement (2) is satisfied. We have to prove that $E$ is barrelled. So let $B$ be a $\sigma(F, E)$-bounded subset of $F$. We have to prove that $B$ is actually norm bounded. Assume the contrary and select a sequence $(z_n)$ in $B$ having $\|z_n\| \geqslant n2^n$. Let $y_n = n^{-1} z_n$; then $(y_n)$ converges to 0 with respect to $\sigma(F, E)$ and satisfies $\|y_n\| \geqslant 2^n$.

(a) We construct sequences $k_1 < k_2 < \ldots$ and $n_1 < n_2 < \ldots$ of integers such that:

(1) $$\|y_i - y_i^{[k]}\| \leqslant 2^{-j} \quad \text{for } i = 1, \ldots, n_{j-1} \text{ and } k \geqslant k_j;$$

(2) $$\|y_i^{[k]}\| \leqslant 2^{-j} \quad \text{for } k = 1, \ldots, k_j \text{ and } i \geqslant n_j.$$

Indeed, suppose $k_1, \ldots, k_j$ and $n_1, \ldots, n_j$ have been constructed in accordance with conditions (1) and (2). Since, by assumption, $F$ is a normed AK-space, $\|y_i - y_i^{[k]}\|$ converges to 0 $(k \to \infty)$ for every fixed $i \in \{1, \ldots, n_j\}$. Clearly this permits us to choose $k_{j+1}$ in accordance with (1).

Next observe that for fixed $k \in \{1, \ldots, k_{j+1}\}$, $\|y_i^{[k]}\|$ converges to 0 $(i \to \infty)$. Indeed, the sequence $(z_n)$ in $B$ being bounded for $\sigma(F, E)$, $(z_n^{[k]})$ is bounded in $(F^{[k]}, \sigma(F^{[k]}, E^{[k]}))$. But note that $E^{[k]}$ is barrelled by assumption. Hence $(z_n^{[k]})$ is norm bounded and consequently, $y_n^{[k]} = n^{-1} z_n^{[k]}$ converges to 0 $(n \to \infty)$ with respect to the norm topology. Clearly this makes possible a choice of $n_{j+1}$ according to (2).

Let $\alpha_j = 1/\|y_{n_j}\| < 2^{-j}$, $j = 1, 2, \ldots$ Then for fixed $k \in \mathbf{N}$ we may define

$$z_k = \sum_{i=1}^{\infty} \alpha_i y_{n_i}^{[k]} \in F^{[k]}$$

since by (2) above, $(y_{n_i}^{[k]})$ is bounded and $F^{[k]}$ is weakly boundedly complete as



a consequence of the fact that $E^{[k]}$ is barrelled. It is easy to see that $z_k^{[r]} = z_r$ whenever $r < k$, so $(z_k)$ is an element of $F^{\omega}$.

(b) We prove that $(z_k) \in E^{\langle\beta\rangle}$.

Let $x \in E$ with $\|x\| \leqslant 1$ be fixed. For $k \in \mathbf{N}$ choose $j = j(k)$ satisfying $k_j \leqslant k < k_{j+1}$. Then we have

$$\begin{aligned}\langle x, z_k\rangle &= \sum_{i=1}^{\infty} \alpha_i \langle x, y_{n_i}^{[k]}\rangle \\ &= \sum_{i=1}^{j-1} \alpha_i \langle x, y_{n_i}^{[k]}\rangle + \alpha_j \langle x, y_{n_j}^{[k]}\rangle + \sum_{i=j+1}^{\infty} \alpha_i \langle x, y_{n_i}^{[k]}\rangle.\end{aligned}$$

Here the first term converges $(k \to \infty)$ in view of

$$\langle x, y_{n_i}^{[k]}\rangle = \langle x, y_{n_i}\rangle + \delta_{ki}(x) \qquad (i \leqslant j-1)$$

for some $|\delta_{ki}(x)| \leqslant 2^{-j(k)}$ (condition (1) of (a) and $\|x\| \leqslant 1$), where we use the fact that the sequence $(\langle x, y_{n_i}\rangle)_{i=1}^{\infty}$ is bounded.

The third term in the above sum converges $(k \to \infty)$ in view of

$$|\langle x, y_{n_i}^{[k]}\rangle| \leqslant 2^{-j(k)} \qquad (i \geqslant j+1)$$

(condition (2) of (a) and $\|x\| \leqslant 1$).

It remains to prove that $\alpha_j \langle x, y_{n_j}^{[k]}\rangle$ converges $(k \to \infty, j = j(k))$. Using axiom (S2), we have

$$\begin{aligned}\alpha_j \langle x, y_{n_j}^{[k]}\rangle &= \alpha_j \langle x, y_{n_j}\rangle + \alpha_j \langle x, y_{n_j}^{[k]} - y_{n_j}\rangle \\ &= \alpha_j \langle x, y_{n_j}\rangle + \langle x^{[k]} - x, \alpha_j y_{n_j}\rangle.\end{aligned}$$

Here the first term on the right-hand side converges to 0 $(k \to \infty)$. The second term, too, converges to 0 as a consequence of the fact that $E$ is a normed AK-space and $\|\alpha_j y_{n_j}\| = 1$, $j = 1, 2, \ldots$ This proves $(z_j) \in E^{\langle\beta\rangle}$.

(c) We end the proof by showing that $(z_k) \notin F$, thus obtaining the desired contradiction. Suppose $(z_k) \in F$, i.e. there exists $z \in F$ having $z^{[k]} = z_k$ for every $k$. Then, $F$ being a normed AK-space, we deduce that $\langle x, z_k\rangle$ converges to $\langle x, z\rangle$ uniformly on $\|x\| \leqslant 1$, and consequently so does $\langle x, z_{k_j}\rangle$ $(j \to \infty)$. Now, by the argument in (b) we have

$$\langle x, z_{k_j}\rangle = \sum_{i=1}^{j-1} \alpha_i \langle x, y_{n_i}\rangle + \sum_{i=1}^{j-1} \alpha_i \delta_{ij}(x) + \sum_{i=j}^{\infty} \alpha_i \eta_{ij}(x)$$

with $|\delta_{ij}(x)| \leqslant 2^{-j}$ and $|\eta_{ij}(x)| \leqslant 2^{-j}$, uniformly for $\|x\| \leqslant 1$. So $\sum_{i=1}^{j-1} \alpha_i \langle x, y_{n_i}\rangle$ is uniformly convergent, a contradiction since $\|\alpha_i y_{n_i}\| = 1$. ∎

Sequence space versions of Theorem 1 have been obtained in [S] and [B$_2$]. In [B$_2$], the result is obtained for the dual pairing $\langle c_0, l\rangle$, but an additional argument (see for instance [S]) shows that the proof in [B$_2$] carries over to the general case, too.

Note that in both papers, the AK-condition on the space $E$ (in our terminology) is redundant in view of Proposition 2.

Limiting examples for Theorem 1 may be found in [$B_2$]. For instance, neither the AK-condition on $F$ nor the assumption that $E$ is a normed space can be weakened considerably.

We shall now prove a slightly generalized version of Theorem 1 which applies to systems of sections on $\langle E'', E'\rangle$ rather than on $\langle E, E'\rangle$. This generalization is compensated by an additional condition on $E$ resp. $E''$.

Let a system of sections be fixed on $\langle E'', E'\rangle$ and let $\|\ \|$ be a norm on $E''$. Suppose that for fixed $x \in E''$, $n \in N$, we have

$$\|x\| = \max\{\|x^{[n]}\|, \|x - x^{[n]}\|\}.$$

In this case we call $\|\ \|$ a *sup-norm*.

THEOREM 2. *Let $E$ be a normed space with dual $E'$ and bidual $E''$ and let a system of sections be fixed on $\langle E'', E'\rangle$ such that $E'$ is a normed AK-space. Suppose $E''$ admits an equivalent sup-norm. Let $G$ be a linear subspace of $E''$ containing its sections and $E$. Suppose $G^{[n]}$ is barrelled with respect to $\tau(G^{[n]}, E'^{[n]})$ for every $n \in N$. Then $G^{\langle\beta\rangle} = E'$ implies that every $\sigma(E', G)$-bounded subset of $E'$ is in fact norm bounded.*

Proof. Our proof follows the pattern of Theorem 1. We start with a sequence $(y_n)$ in $E'$ which converges to 0 with respect to $\sigma(E', G)$ and satisfies $\|y_n\| \geq 2^n$. Again we seek for a contradiction.

As in the proof of Theorem 1 we select sequences $k_1 < k_2 < \ldots$ and $n_1 < n_2 < \ldots$ of integers satisfying conditions (1) and (2). Denoting by $z^{[i,j]}$ the difference vector $z^{[j]} - z^{[i]}$, we define

$$\beta_i = 1/(i\,\|y_{n_i}^{[k_i,k_i+1]}\|), \qquad z_k = \sum_{i=1}^{\infty} \beta_i (y_{n_i}^{[k_i,k_i+1]})^{[k]},$$

$k = 1, 2, \ldots$ Clearly we have $z_k \in E'^{[k]}$, $z_k^{[r]} = z_r$ for $r < k$, hence $(z_k)$ is an element of $E'^{\omega}$. We prove that $(z_k) \in G^{\langle\beta\rangle}$.

Let $x \in G$, $\|x\| \leq 1$, be fixed. Let $k \in N$ and choose $j = j(k)$ having $k_j < k \leq k_{j+1}$. Then we have

$$\langle x, z_k\rangle = \sum_{i=1}^{j-1} \beta_i \langle x, y_{n_i}^{[k_i,k_i+1]}\rangle + \beta_j \langle x, y_{n_j}^{[k_j,k]}\rangle.$$

For $i \leq j-1$ we note that

$$\begin{aligned}\beta_i \langle x, y_{n_i}^{[k_i,k_i+1]}\rangle &= \beta_i \langle x, y_{n_i}\rangle - \beta_i \langle x, y_{n_i}^{[k_i]}\rangle - \beta_i \langle x, y_{n_i} - y_{n_i}^{[k_i+1]}\rangle \\ &= \beta_i \langle x, y_{n_i}\rangle + \beta_i \delta_i(x) + \beta_i \eta_i(x)\end{aligned}$$

for certain $|\delta_i(x)| \leq 2^{-i}$, $|\eta_i(x)| \leq 2^{-i}$ in view of conditions (1) and (2). Hence the first summand in the above representation of $\langle x, z_k\rangle$ converges in view of $\langle x, y_{n_i}\rangle \to 0$ $(i \to \infty)$. But the second term converges to 0 $(k \to \infty)$ in view of

$$\begin{aligned}|\beta_j \langle x, y_{n_j}^{[k_j,k]}\rangle| &= |\langle x^{[k]} - x^{[k_j]}, \beta_j y_{n_j}^{[k_j,k_j+1]}\rangle| \\ &\leq \|x^{[k]} - x^{[k_j]}\|/j \\ &\leq \|x\|/j \to 0 \quad (k \to \infty,\ j = j(k)).\end{aligned}$$

This proves $(z_k) \in G^{\langle\beta\rangle}$.

As in the proof of Theorem 1, we achieve a contradiction by showing that $(z_k)$ is not represented by an element of $E'$. Indeed, otherwise the sequence $\langle x, z_{k_j}\rangle$ would have to be uniformly convergent $(j \to \infty)$ on $\|x\| \leq 1$. But

$$\langle x, z_{k_j}\rangle = \sum_{i=1}^{j-1} \beta_i \langle x, y_{n_i}^{[k_i,k_i+1]}\rangle = \sum_{i=1}^{j-1} \langle x, \beta_i y_{n_i}^{[k_i,k_i+1]}\rangle$$

is not uniformly convergent on $\|x\| \leq 1$. Indeed, for $i \leq j-1$ there exist certain $x_i$ having $\|x_i\| \leq 1$ and

$$|\langle x_i, \beta_i y_{n_i}^{[k_i,k_i+1]}\rangle - 1/i| \leq 2^{-i}.$$

Observing that $\langle x_i, \beta_i y_{n_i}^{[k_i,k_i+1]}\rangle = \langle x_i^{[k_i,k_i+1]}, \beta_i y_{n_i}^{[k_i,k_i+1]}\rangle$, we define

$$x^{(j)} = \sum_{i=1}^{j-1} x_i^{[k_i,k_i+1]},$$

which provides a vector having $\|x^{(j)}\| \leq 1$ in view of the sup-property of the norm $\|\ \|$. We obtain

$$\sum_{i=1}^{j-1} \langle x^{(j)}, \beta_i y_{n_i}^{[k_i,k_i+1]}\rangle = \sum_{i=1}^{j-1} \langle x_i, \beta_i y_{n_i}^{[k_i,k_i+1]}\rangle = \sum_{i=1}^{j-1} 1/i + \sum_{i=1}^{j-1} \delta_i,$$

where $|\delta_i| \leq 2^{-i}$. This provides the desired contradiction. ∎

Theorem 2 has the following consequence which we shall need in the next section.

COROLLARY 1. *Let $E$ be a normed space and let a system of sections be fixed on $\langle E'', E'\rangle$ such that $E'$ is a normed AK-space. Suppose $E''$ admits an equivalent sup-norm. Let $G$ be a subspace of $E''$ containing its sections and suppose $E$ is a subspace of $G$ of finite codimension. Let $G^{[n]}$ be barrelled with respect to $\tau(G^{[n]}, E'^{[n]})$ for every $n$ and let $G^{\langle\beta\rangle} = E'$. Then $G$ is barrelled.*

Proof. For simplicity we treat the case $G = E + \mathrm{lin}\{e\}$ for some $e \notin E$. Let $U$ be a barrel in $E$. There are two possibilities. Either $\bar{U}$, the closure of $U$ in $(G, \sigma(G, E'))$, is absorbing in $G$, or it is not. In the first case, the $\langle G, E'\rangle$-polar $U^{\circ}$ of $U$ is $\sigma(E', G)$-bounded, hence is norm bounded by Theorem 2, and thus $U^{\circ\circ}$ is a norm neighbourhood of 0 in $G$. But then $U = U^{\circ\circ} \cap E$ is a norm neighbourhood of 0 in $E$.

Let us consider the case where $\bar{U}$ is not absorbing in $G$. This implies $e \notin \operatorname{lin} \bar{U}$. Let

$$V = U + \{\varrho e\colon |\varrho| \leq 1\}.$$

Then $V$ is circled, convex and absorbing in $G$. So $V^\circ$ is $\sigma(E', G)$-bounded, hence norm bounded, hence $V^{\circ\circ}$ is a norm neighbourhood of 0 in $G$ and so $V^{\circ\circ} \cap E$ is a norm neighbourhood of 0 in $E$. It remains to prove that $V^{\circ\circ} \cap E \subset U$. Let $y \in V^{\circ\circ} \cap E$. There exists a net $(y_\gamma)$ in $V$ converging to $y$ with respect to $\sigma(G, E')$. Let $y_\gamma = z_\gamma + \varrho_\gamma e$, $z_\gamma \in U$, $|\varrho_\gamma| \leq 1$. The latter condition permits us to find a cofinal subnet (also denoted by $(y_\gamma)$) such that $(\varrho_\gamma)$ converges to some $\varrho$. So $(z_\gamma)$ converges to $y - \varrho e$, and this gives $\varrho e \in E + \bar{U}$, hence $\varrho = 0$ and so $y \in \bar{U} \cap E = U$. This proves that $E$ is barrelled. Since $E$ has finite codimension in $G$, the result follows. ■

Instead of postulating that $G^{[n]}$ is barrelled in Theorem 2 and Corollary 1 we might have imposed the weaker (but more complicated) condition that every $\sigma(E'^{[n]}, G^{[n]})$-bounded set in $E'^{[n]}$ is norm bounded.

**4. Applications to sequence spaces.** In this section we will present various applications of the main results to the study of sequence spaces. Besides some nice consequences to be obtained in the standard duality $\langle x, y\rangle = \sum_i x_i y_i$ combined with the usual notion of sections, we shall apply Theorem 1 and Corollary 1 to certain nonstandard dual pairings resp. nonstandard notions of sections. Let us, however, start with some applications in the standard constellation.

THEOREM 3 (cf. [S], [B$_2$]). *Let $E$ be a normed K-space whose dual (via $\langle x, y\rangle = \sum_i x_i y_i$) is a BK-AK-sequence space. Then $E$ has the Wilansky property (in the sense of* [B$_2$]), *i.e. any subspace $F$ of $E$ satisfying $F^\beta = E^\beta = E'$ is necessarily barrelled.*

Proof. We apply Theorem 1 to the dual pairing $\langle F + \Phi, E'\rangle$, where $\Phi$ denotes the space of finite sequences. We define sections as in Example 1. Then the barrelledness of $F + \Phi$ is equivalent to $(F + \Phi)^\beta = E'$. But $F^\beta = (F + \Phi)^\beta$, so $F^\beta = E'$ implies that $F + \Phi$ is barrelled, and by the Levin–Saxon–Valdivia theorem (see [W$_1$, p. 250]), this implies the barrelledness of $F$. ■

Using his counterpart to Theorem 3, Bennett ([B$_2$]) was able to characterize dense barrelled subspaces of the spaces $c$ and $cs$. As both these spaces do not satisfy the conditions of Theorem 3, his proof requires additional arguments. Here we show that both results may be derived directly from Corollary 1.



THEOREM 4. *A dense subspace $E$ of $c$ is barrelled if and only if $E^\beta = l$, i.e. $c$ has the Wilansky property. More generally, every subspace $G$ of $m$ containing $c_0$ as a subspace of finite codimension has the Wilansky property.*

Proof. We prove the first statement. Clearly every dense barrelled subspace $E$ of $c$ satisfies $E^\beta = l$. The converse follows from Corollary 1. Indeed, we may assume that $E$ contains the space $\Phi + \operatorname{lin}\{e\}$, where $e = (1, 1, \ldots)$. Now $E$ contains its sections in $\langle m, l\rangle$, has $\beta$-dual $l$ and contains $E \cap c_0$ as a subspace of codimension 1. Thus Corollary 1 applies and yields the result. ■

Theorem 4 is no longer valid in the case where $G$ is a subspace of $m$ containing $c_0$ as a subspace of countable codimension since a space $G = c_0 + H$ with $\dim(H)$ countable need not be barrelled.

EXAMPLE 3. Let $E$ be a Banach space and let $x_1, x_2, \ldots$ be linearly independent vectors in $E''$ having $x_i \notin E$. Then $G = E + \operatorname{lin}\{x_n\colon n \in N\}$ is not barrelled, for we may define a sequence $(f_n)$ of continuous linear functionals such that $f_n \equiv 0$ on $E + \operatorname{lin}\{x_1, \ldots, x_{n-1}\}$, $f_n(x_n) \neq 0$, $\|f_n\| \to \infty$. Then $(f_n)$ converges to 0 pointwise but is unbounded. This shows that a linear subspace $G$ of $m$ containing $c_0$ does not have the Wilansky property unless it contains $c_0$ as a subspace of finite codimension.

We shall now turn towards the problem of characterizing dense barrelled subspaces of summability domains. Recall that, given a normed K-space $E$ and an infinite matrix $A$, the *summability domain* $E_A$ consists of all $x \in \omega$ such that $Ax$ exists and lies in $E$. Now $E_A$ is a K-space when endowed with the topology coming from the seminorms $x \to \|Ax\|$, $x \to |x_i|$, $i = 1, 2, \ldots$, and $q_i(x) = \sup_n |\sum_{j=1}^n a_{ij} x_j|$, $i = 1, 2, \ldots$

PROPOSITION 4. *Let $E$ be a normed K-space whose dual $E'$ (via $\langle x, y\rangle = \sum_i x_i y_i$) is a BK-AK-space. Let $A$ be an infinite matrix such that $\Phi$ is contained in $A(E_A)$. Suppose the K-topology on $E_A$ is generated by the seminorms $x \to \|Ax\|$ and finitely many $x \to |x_i|$, $i = 1, \ldots, N$. Let $D$ be a dense subspace of $E_A$. Then $D$ is barrelled if and only if $A(D)^\beta = E'$.*

Proof. Since the topology of $E_A$ is generated by $x \to \|Ax\|$ and the $x \to |x_i|$, $i = 1, \ldots, N$, the dual of $E_A$ (and hence of $D$) may be identified with $E' + K^N$ via the bilinear mapping

$$\langle x, y + \eta\rangle = \sum_{i=1}^{\infty} (Ax)_i y_i + \sum_{i=1}^{N} x_i \eta_i.$$

Suppose now we have $\Phi \subset A(D)$. We define a system of sections on $\langle D, E' + K^N\rangle$ by setting $(y + \eta)^{[n]} = y^{[n]} + \eta$, where $y^{[n]}$ denotes the usual section in $E'$. We note that $y + \eta \to (y + \eta)^{[n]}$ is continuous with respect to

$\sigma(E'+K^N, D)$. Indeed, for fixed $x\in D$ we have

$$\langle x, (y+\eta)^{[n]}\rangle = \sum_{i=1}^{\infty} (Ax)_i (y^{[n]})_i + \sum_{i=1}^{N} x_i \eta_i$$
$$= \sum_{i=1}^{\infty} ((Ax)^{[n]})_i y_i + \sum_{i=1}^{N} x_i \eta_i,$$

proving that $y+\eta \to (y+\eta)^{[n]}$ is weakly continuous in view of $\Phi \subset A(D)$. We next claim that $E'+K^N$ has sectional convergence with respect to the norm topology. Let $y+\eta \in E'+K^N$. Then we have

$$\langle x, (y+\eta)-(y+\eta)^{[n]}\rangle = \sum_{i=n+1}^{\infty} (Ax)_i y_i,$$

which tends to 0 ($n\to\infty$) uniformly on $\|Ax\| \leqslant 1$, $E'$ being an AK-space by assumption.

Applying Theorem 1 to the dual pairing $\langle D, E'+K^N\rangle$, we deduce that $D$ is barrelled if and only if $D^{\langle\beta\rangle} = E'+K^N$, since the section spaces $D^{[n]}$ are finite-dimensional. But note that $D^{\langle\beta\rangle}$ is just the subspace of $\omega+K^N$ consisting of all vectors $y+\eta$ such that $\sum_i (Ax)_i y_i$ converges for all $x\in D$, hence the equality $D^{\langle\beta\rangle} = E'+K^N$ amounts to $A(D)^\beta = E'$. This proves the result in the case $\Phi\subset A(D)$.

The general case is easily deduced if we consider a linear subspace $\Psi$ of $E_A$ of countable dimension having $A(\Psi) = \Phi$, for then the barrelledness of $D$ is equivalent to the barrelledness of $D+\Psi$. ∎

Various situations are known in which a summability domain $E_A$ is normable and its topology is generated by $\|Ax\|$ and finitely many $x\to|x_i|$ alone.

If $A$ is row-finite, then it is well known that the seminorms $q_i$ may be discarded, so that if $E_A$ is normable, finitely many $x\to|x_i|$ and $\|Ax\|$ suffice. The latter is the case e.g. when $A$ is a lower triangular matrix with only finitely many diagonal entries $\neq 0$.

If $A$ is an invertible matrix, then the topology of $E_A$ is induced by $\|Ax\|$ alone.

An interesting case is the following one. If $E_A$ is a BK-space containing $\Phi$ and having bounded sections (in the usual sense), then the seminorms $q_i$ may again be discarded, as executed in [K, Thm. 2].

Proposition 4 being based on Theorem 1, we now obtain a similar version based on Corollary 1. We state it in a particularly interesting case.

PROPOSITION 5. *Let $A$ be an infinite matrix such that $c_A$ is normable and the topology is generated by the seminorms $x\to\|Ax\|_\infty$ and finitely many $x\to|x_i|$, $i=1,\ldots,N$. Suppose $A(c_A)$ contains $\Phi+\operatorname{lin}\{e\}$. Then a dense subspace $D$ of $c_A$ is barrelled if and only if $A(D)^\beta = l$.*



Proof. We may again assume that $A(D)$ contains $\Phi+\operatorname{lin}\{e\}$. Now apply Corollary 1 to the pairing $\langle D, l+K^N\rangle$, where $\langle x, y+\eta\rangle$ is defined as in the proof of Proposition 4. Notice that $D$ may be given an equivalent norm having the sup-property, for $l+K^N \cong l$ has an additive norm if we introduce sections as in the proof of Proposition 4. ∎

As a consequence we obtain

THEOREM 5. *A dense subspace $E$ of $cs$ is barrelled if and only if $E^\beta = bv$.*

Proof. Let $S=(s_{ij})$ be the summation matrix, i.e. $s_{ij}=1$ if $i\geqslant j$, $s_{ij}=0$ if $i<j$. Then $cs = c_S$ and is normed with $\|Sx\|_\infty$ alone. Hence by Proposition 5, a dense subspace $E$ of $cs$ is barrelled if and only if $S(E)^\beta = l$. So it remains to prove that $S(E)^\beta = l$ is equivalent to $E^\beta = bv$, at least when $\Phi\subset E$.

Suppose $E^\beta = bv$. We have to prove $S(E)^\beta \subset l$. Let $y\in S(E)^\beta$, and choose $z\in\omega$ such that $y_i = z_i - z_{i+1}$. It suffices to prove $z\in bv$. So let $x\in E$. We have to prove $xz\in cs$. Now for fixed $n\in N$,

$$\sum_{i=1}^{n} (Sx)_i (z_i - z_{i+1}) = \sum_{i=1}^{n} x_i z_i - (Sx)_n z_{n+1},$$

so $y\in S(E)^\beta$ shows that the right-hand side converges ($n\to\infty$) for every $x\in E$. Inserting the unit vector $(1,0,0,\ldots)\in E$ proves $z\in c$, hence, in view of $Sx\in c$ for every $x\in E$, the second term on the right side converges for every $x\in E$, and consequently, so does the first. This proves $z\in E^\beta$, hence $y\in l$.

Conversely, suppose $S(E)^\beta = l$. Then $S(E)$ is a barrelled subspace of $c$, hence $E$ is barrelled in $cs$, and hence $E^\beta = bv$ follows. ∎

COROLLARY 2. *A dense subspace $E$ of $(c_0)_S$ is barrelled if and only if $E^\beta = bv$.* ∎

We conclude this section with two applications of Theorem 1 to the study of scarce copies. For definitions we refer to [$B_1$]. In this paper it is stated that most classical sequence spaces (prominent exceptions $\omega$ and $l$) do not have any barrelled scarce copy. Here we obtain the somewhat more general

PROPOSITION 6. *Let $E$ be any AK-sequence space having $E\subset m$ and $E^\beta \subset cs$. Then no scarce copy of $E$ is barrelled.*

Proof. Let $\Sigma(E, r)$ be a scarce copy of $E$ and suppose it is barrelled. $E$ being an AK-space, this implies $E' = E^\beta = \Sigma(E, r)^\beta$. We claim that $\Sigma(c_0, r)$ is now a barrelled scarce copy of $c_0$, and this will provide a contradiction, since $c_0$ does not have any barrelled scarce copy.

Let $y\notin l$. Choose $x\in c_0$ having $xy\notin cs$. So, by assumption, $xy\notin E^\beta = \Sigma(E, r)^\beta$. Choose $z\in\Sigma(E, r)$ such that $zxy\notin cs$. By the definition of $\Sigma(E, r)$, there exist $z_1,\ldots,z_n\in\sigma(E, r)$ such that $z = z_1+\ldots+z_n$. This implies

$z_i xy \notin cs$ for some $i$. But note that $x \in c_0$ and $z_i \in E \subset m$ give $z_i x \in c_0$, while $c_m(z_i x) \leqslant c_m(z_i) \leqslant r_m$ for all $m$ implies $z_i x \in \sigma(c_0, r) \subset \Sigma(c_0, r)$. Hence we have proved that $\Sigma(c_0, r)^\beta = l$ and hence $\Sigma(c_0, r)$ is barrelled. ■

In [B$_1$], Bennett proves that $\Sigma(E)$, the linear hull of all scarce copies of a space $E$, is a barrelled subspace of $E$ in case $E$ is a solid FK-space. Here we prove that the same result holds for monotone (i.e. $m_0 E \subset E$) BK-spaces whose dual is a BK-AK-space.

PROPOSITION 7. *Let $E$ be a monotone BK-AK-space whose dual is a BK-AK-space. Then $\Sigma(E)$ is a barrelled subspace of $E$.*

Proof. By Theorem 3 it suffices to prove $\Sigma(E)^\beta = E^\beta$. Now recall that $\Sigma(E)$ is precisely the linear hull of all vectors of the form $\lambda x$, where $x \in E$, $\lambda = (\lambda_n)$ a sequence of 0 or 1 entries only and satisfying $\lim_{C_1} \lambda = 0$, where $C_1$ refers to Cesàro means. It will therefore be sufficient to prove the following (classical) lemma in order to complete the proof.

LEMMA. *Let $(x_n)$ be a positive sequence having $\sum_n x_n = \infty$. Then there exists a sequence $\lambda = (\lambda_n)$ having $\lambda_n \in \{0, 1\}$ and $\lim_{C_1} \lambda = 0$ such that still $\sum_n \lambda_n x_n = \infty$.*

Proof. We give a sketch of the proof since we do not know of a reference.

We define a binary tree of countable infinite height whose nodes are infinite subsets of $N$. Start with the root $N$. Then, given any node $N$ in the tree already defined, define two successors $N'$, $N''$ of $N$ as follows. Let $N = \{n_1, n_2, \ldots\}$, where $n_i < n_{i+1}$. Then set $N' = \{n_1, n_3, n_5, \ldots\}$ and $N'' = \{n_2, n_4, n_6, \ldots\}$.

Let us now select a cofinal branch in the tree. Let $N' = \{1, 3, \ldots\}$ and $N'' = \{2, 4, \ldots\}$ be the successors of $N$. Then either $\sum_{n \in N'} x_n$ or $\sum_{n \in N''} x_n$ is $\infty$. Accordingly, let $N_1 = N'$ or $N_1 = N''$. Choose $n_1$ such that $\sum_{i=1, i \in N_1}^{n_1} x_i > 1$. Now let $N'$, $N''$ be the successors of $N_1$. Then either $\sum_{i > n_1, i \in N'} x_i$ or $\sum_{i > n_1, i \in N''} x_i$ is infinite. Accordingly choose $N_2 = N'$ or $N_2 = N''$. Then choose $n_2 > 2n_1$ such that $\sum_{i=n_1+1, i \in N_2}^{n_2} x_i > 2$, etc.

Suppose $N_1, N_2, N_3, \ldots$ have been chosen in this way. Define $\lambda = (\lambda_n)$ by setting $\lambda_n = 1$ if $n_{i-1} < n \leqslant n_i$ and $n \in N_i$, $\lambda_n = 0$ otherwise. Clearly $\sum_n \lambda_n x_n = \infty$. It remains to prove that $\lim_{C_1} \lambda = 0$. For $n \in N$, $n_{r-1} < n \leqslant n_r$, we have

$$\begin{aligned} n^{-1} \sum_{i=1}^{n} \lambda_i &= n^{-1} \sum_{j=1}^{r-1} \sum_{i=n_{j-1}+1}^{n_j} \lambda_i + n^{-1} \sum_{i=n_{r-1}+1}^{n} \lambda_i \\ &\leqslant n^{-1} \sum_{j=1}^{r-1} n_j 2^{-j} + n^{-1}(n - n_{r-1}) 2^{-r} \\ &\leqslant n^{-1}(r-1) n_{r-1} 2^{-r+1} + 2^{-r} \to 0 \quad (n \to \infty,\ r = r(n)). \end{aligned}$$

This proves $\lim_{C_1} \lambda = 0$ as desired. ■



**5. Two special cases.** In this final section we consider two particular cases of systems of sections with infinite-dimensional section spaces.

Let $T$ be a locally compact space and let $C_0(T)$ be the space of scalar-valued continuous functions on $T$ vanishing at infinity. Let $C_0(T)$ be endowed with $\| \ \|_\infty$; then its dual may be identified with $M(T)$, the space of finite Radon measures on $T$. The corresponding bilinear mapping is $\langle f, \mu \rangle = \int_T f \, d\mu$.

Suppose that $T$, in addition, is countable at infinity. Let $(T_n)$ be a sequence of compact sets with union $T$ having $T_n \subset \operatorname{int} T_{n+1}$. We define restriction mappings $\mu \to \mu^{[n]}$ on $M(T)$ by setting

$$\int_T f \, d\mu^{[n]} = \int_{T_n} f \, d\mu,$$

$f \in C_{00}(T)$. This provides a system of sections on $\langle C_0(T)'', M(T) \rangle$, since $\mu \to \mu^{[n]}$ is norm continuous. Notice that, in general, sections do not exist within $C_0(T)$. They do e.g. when the sets $T_n$ are compact and open. With this notation we obtain

PROPOSITION 8. *Let $E$ be a subspace of $C_0(T)$ containing $C_{00}(T)$. Then the following statements are equivalent:*

(1) *$E$ is barrelled.*

(2) *Every Radon measure $\mu$ on $T$ such that $\lim_{n \to \infty} \int_{T_n} f \, d\mu$ exists for every $f \in E$ is necessarily finite.*

Proof. (1)$\Rightarrow$(2). Observe that $\mu^{[n]}$ converges pointwise to the linear functional $\varphi$ defined by

$$\varphi(f) = \lim_{n \to \infty} \int_{T_n} f \, d\mu,$$

hence $\varphi$ is continuous by the Banach–Steinhaus theorem, i.e. there exists $\nu \in M(T)$ having $\varphi(f) = \int f \, d\nu$ for all $f \in E$. Since $C_{00}(T) \subset E$, this implies $\mu = \nu$ and so $\mu$ is finite.

(2)$\Rightarrow$(1). Observe that $M(T)$ is a normed AK-space in view of

$$|\langle f, \mu - \mu^{[n]} \rangle| = \Big| \int_{T \setminus T_n} f \, d\mu \Big| \leqslant \|f\|_\infty |\mu|(T \setminus T_n),$$

which converges to 0 $(n \to \infty)$ uniformly on $\|f\|_\infty \leqslant 1$.

Let $B$ be a barrel in $E$. We may assume that $B$ is contained in the unit ball of $E$. Let $\bar{B}$ denote the closure of $B$ in $(E'', \sigma(E'', M(T)))$. Let $G$ be the linear hull of $\bar{B}$. We claim that $G$ contains its sections.

Let $f \in E$, $\|f\|_\infty \leqslant 1$, be fixed. We prove that $f^{[n]} = f\chi_n \in G$, where $\chi_n$ denotes the characteristic function of $T_n$. Observe that $C_{00}(T) \subset E$ implies that $E_i = \{f \in E\colon \operatorname{supp}(f) \subset T_i\}$ is barrelled, so there exists $\alpha > 0$ such that $\|f\|_\infty \leqslant 1$ and $\operatorname{supp}(f) \subset T_{n+1}$ imply $f \in \alpha B$.

Let $\mu \in M(T)$ and $\varepsilon > 0$ be fixed. Choose an open set $G$ contained in int $T_{n+1}$ and containing $T_n$ such that $|\mu|(G\setminus T_n)<\varepsilon$. Let $\chi\colon T\to[0,1]$ be continuous with $\chi\equiv 1$ on $T_n$ and $\chi\equiv 0$ on $T\setminus G$. Then $\|f\chi\|_\infty\leqslant 1$ and $\operatorname{supp}(f\chi)\subset T_{n+1}$ give $f\chi\in\alpha B$. Moreover,

$$|\langle f^{[n]}-f\chi,\,\mu\rangle|\leqslant\int|1-\chi|\,d|\mu|\leqslant|\mu|(G\setminus T_n)<\varepsilon.$$

This proves $f^{[n]}\in\alpha\bar{B}$. Since $f\in E$, $\|f\|_\infty\leqslant 1$, and $n\in N$ were chosen arbitrarily, we obtain

$$\{f^{[n]}\colon f\in E,\ \|f\|_\infty\leqslant 1,\ n\in N\}\subset\alpha\bar{B}.$$

This implies that $G$ contains its sections, for let $g\in\bar{B}$, $(g_\gamma)$ in $B$, $g_\gamma\to g$ weakly; then $\|g_\gamma\|\leqslant 1$, hence $g_\gamma^{[n]}\in\alpha\bar{B}$, so $g^{[n]}=\lim_\gamma g_\gamma^{[n]}\in\alpha\bar{B}$, too.

Next observe that $M(T)^\omega$ may be identified with the space $\mathfrak{M}(T)$ of all Radon measures on $T$, for every sequence $(\mu_n)$ having $\mu_n^{[m]}=\mu_m$, $m<n$, gives rise to a unique $\mu\in\mathfrak{M}(T)$, defined by $\int f\,d\mu=\int f\,d\mu_n$ in case $\operatorname{supp}(f)\subset T_n$. Dually, every $\mu\in\mathfrak{M}(T)$ defines an element $(\mu_n)$ of $M(T)^\omega$ by $\mu_n=\mu|T_n$. But now $E^{\langle\beta\rangle}$ is the space of all measures $\mu\in\mathfrak{M}(T)$ such that $\int_{T_n}f\,d\mu$ converges for every $f\in E$. So statement (2) above translates into $E^{\langle\beta\rangle}=M(T)$, hence $G^{\langle\beta\rangle}=M(T)$, and hence Theorem 2 yields that every $\sigma(M(T),G)$-bounded set is norm bounded in $M(T)$, since for every $n$, $\sigma(M(T)^{[n]},E)$-bounded sets are norm bounded. Now $\bar{B}$ is absorbing in $G$, hence $B^{\circ\circ}$ is a norm neighbourhood of 0 in $G$, and so is $B=B^{\circ\circ}\cap E$ in $E$. This ends the proof. ∎

The assumption that $T$ is countable at infinity may be avoided in Proposition 8. This yields

THEOREM 6. *Let $E$ be a subspace of $C_0(T)$ containing $C_{00}(T)$. Then the following statements are equivalent*:

(1) *$E$ is barrelled.*

(2) *Whenever $\mu$ is a Radon measure on $T$ supported by a $\sigma$-compact set $\bigcup_{n=1}^\infty T_n$, $T_{n-1}\subset\operatorname{int}T_n$, $T_n$ compact, such that $\lim_{n\to\infty}\int_{T_n}f\,d\mu$ exists for every $f\in E$, then $\mu$ must be finite.*

Proof. For the proof that (1) implies (2) see Proposition 8. We establish the reverse implication. We have to prove that $\sigma(M(T),E)$-bounded sequences $(\mu_n)$ are norm bounded. Let $(\mu_n)$ be such a sequence. For every $n$ let $K_i^{(n)}$ be compact subsets of $T$ having $K_{i-1}^{(n)}\subset\operatorname{int}K_i^{(n)}$ such that

$$\mu_n\Big(T\setminus\bigcup_{i=1}^\infty K_i^{(n)}\Big)=0.$$

Using a diagonal procedure, we may therefore find a sequence $(T_i)$ of compact subsets of $T$ having $T_{i-1}\subset\operatorname{int}T_i$ such that

$$\mu_n\Big(T\setminus\bigcup_{i=1}^\infty T_i\Big)=0$$



for all $n$, i.e. each $\mu_n$ is supported by $\bigcup_i T_i$. Now let $S=\bigcup_i T_i$. Then $S$ is an open, hence locally compact subset of $T$, which is countable at infinity. We may assume the sequence $(\mu_n)$ sited in $M(S)$, hence it suffices to prove that $(\mu_n)$ is norm bounded in $M(S)$. If $S$ is relatively compact in $T$, this is clear in view of $C_{00}(T)\subset E$. So let us assume that $S$ is not relatively compact in $T$.

Let $F=\{f|S\colon f\in E\}$. Then $F$ is a dense subspace of $C_0(S)$ containing $C_{00}(S)$. It suffices to prove that $F$ is barrelled, for clearly $(\mu_n)$ is $\sigma(M(S),F)$-bounded, and this will imply that it is norm bounded. But note that $S$ is countable at infinity, so we may apply Proposition 8. Since condition (2) above implies that (2) in Proposition 8 is satisfied for $M(S)$ and $F$, we deduce that $F$ is in fact barrelled. ∎

We conclude our paper with an application of Theorem 1 to spaces of integrable functions. Let $(\Omega,\mathfrak{A},\mu)$ be a $\sigma$-finite measure space and let $(\Omega_n)$ be an increasing sequence in $\mathfrak{A}$ with union $\Omega$ such that $\mu(\Omega_n)<\infty$ for every $n$. For $p\geqslant 1$ let $\mathfrak{L}^p=\mathfrak{L}^p(\Omega,\mathfrak{A},\mu)$ denote the space of equivalence classes of $p$-integrable functions endowed with the usual $p$-norm. For $p>1$ we obtain the following characterization of dense barrelled subspaces of $\mathfrak{L}^p$.

PROPOSITION 9. *Let $E$ be a dense subspace of $\mathfrak{L}^p$ $(p>1)$ containing all $\mathfrak{L}_n^p=\mathfrak{L}^p(\Omega_n,\mathfrak{A}\cap\Omega_n,\mu|\Omega_n)$, $n\in N$. Then the following statements are equivalent*:

(1) *$E$ is barrelled.*

(2) *Whenever $g\colon\Omega\to\mathbf{K}$ is a measurable function such that $g|\Omega_n\in\mathfrak{L}_n^q$ for every $n$ and $\lim_{n\to\infty}\int_{\Omega_n}fg\,d\mu$ exists for every $f\in E$, then $g\in\mathfrak{L}^q$.*

Proof. Define a system of sections on $\langle E,\mathfrak{L}^q\rangle$ $(1/p+1/q=1)$ by setting $f^{[n]}=f\chi_n$, where $\chi_n$ denotes the characteristic function of $\Omega_n$. Then $\mathfrak{L}^q$ is a normed AK-space in view of

$$|\langle f,\,g-g^{[n]}\rangle|=\Big|\int fg(1-\chi_n)\,d\mu\Big|\leqslant\|f\|_p\|g(1-\chi_n)\|_q\to 0\quad(n\to\infty).$$

The space $(\mathfrak{L}^q)^\omega$ may be identified naturally with the space of all classes of measurable functions $g$ on $\Omega$ having $g|\Omega_n\in\mathfrak{L}_n^q$ for all $n$. But then statement (2) of Theorem 1 translates naturally into statement (2) above. ∎

COROLLARY 3. *Let $E$ be a dense subspace of $\mathfrak{L}^p$ $(p>1)$ containing all $\mathfrak{L}_n^p$ and suppose $f\in E$ and $|g(t)|=1$ for all $t\in\Omega$ imply $fg\in E$ whenever $g$ is measurable. Then the following statements are equivalent*:

(1) *$E$ is barrelled.*

(2) *Whenever $g$ is measurable on $\Omega$ having $fg\in\mathfrak{L}^1$ for all $f\in E$, then $g\in\mathfrak{L}^q$.*

Proof. Statement (2) in Proposition 9 may be replaced by (2) above since for a function $h\geqslant 0$ the existence of $\lim_{n\to\infty}\int_{\Omega_n}h\,d\mu$ may be expressed equivalently by $h\in\mathfrak{L}^1$. ∎

Clearly Proposition 9 and Corollary 3 are both false in the case $p = 1$, $q = \infty$. This may be seen by considering the dual pairing $\langle l^1, m\rangle$ and taking an example as in [B₂]. Notice that Corollary 3 again becomes false if we omit the condition of monotonicity on $E$. Take e.g. the subspace $E$ of $l^p$ $(p > 1)$ consisting of all vectors $x$ having $x_{2n+1} = x_{2n}$ eventually. Then $E^\alpha = l^q$, but $E$ is not barrelled (since the $\beta$-dual is not $l^q$).

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# Weighted integral inequalities for the ergodic maximal operator and other sublinear operators. Convergence of the averages and the ergodic Hilbert transform


by

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**Abstract.** In this paper we study those pairs of weights $(u, w)$ for which the ergodic maximal operator and other sublinear operators, associated to an automorphism of a $\sigma$-finite measure space $(X, \mathcal{M}, \mu)$, are of weak or strong type $(\phi, \phi)$ with respect to the measures $w d\mu$ and $u d\mu$, where $\phi$ is an N-function. As consequences, we get the $\mu$-almost everywhere convergence and the norm convergence of certain weighted averages (which include the Cesàro averages), as well as of the ergodic Hilbert transform, for every function in the Orlicz space $L_\phi(w d\mu)$, where the weight $w$ belongs to a certain class.


**1. Introduction and preliminaries. The $A_\phi(T, \mu)$-condition.** Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space and $T$ an invertible $\mu$-measure-preserving transformation from $X$ into itself. Such a transformation will be called an *automorphism* of $(X, \mathcal{M}, \mu)$. Let $\mathfrak{M} \equiv \mathfrak{M}(X, \mathcal{M}, \mu)$ be the space of $\mathcal{M}$-measurable and $\mu$-almost everywhere ($\mu$-a.e.) finite functions from $X$ to $\boldsymbol{R}$ or to $\boldsymbol{C}$.

For each pair of nonnegative integers, $m$ and $n$, and every $f \in \mathfrak{M}$ we consider the average

$$R_{m,n} f(x) = (m+n+1)^{-1} \sum_{i=-m}^{n} f(T^i x) \tag{1.1}$$

and let $M_T$ be the *ergodic maximal operator*, acting on $\mathfrak{M}$, defined by

$$M_T f = \sup_{m,n \geqslant 0} R_{m,n} |f|. \tag{1.2}$$

The role of the operator $M_T$ in the study of the convergence of the

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This paper contains some of the results of the author's Doctoral Thesis written under the direction of Professor Alberto de la Torre at the University of Málaga.