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On semigroups generated by left-invariant positive differential operators on nilpotent Lie groups

by

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Abstract. For a nilpotent Lie group G let

$$\mathcal{L} = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j},$$

where the X_j belong to the Lie algebra of G. Then $-\mathcal{L}$ is the infinitesimal generator of a holomorphic semigroup $T_z f = f * p_z$, $|\operatorname{Arg} z| < \theta$, on $L^1(G)$. We show that for every left-invariant differential operator ∂ on G, $|\partial p_z(x)| \leqslant A_N e^{-N\tau(x)}$ for every constant N and $A_N = A(\theta, z, N)$, where τ is a riemannian distance on G.

Let G be a Lie group and let τ be a subadditive function on G (i.e. $\tau(xy) \leqslant \tau(x) + \tau(y)$), e.g. τ is a riemannian distance from x to the unit element e of G. For elements $X_1, \ldots, X_k$ in the Lie algebra $\mathfrak{g}$ of G which generate $\mathfrak{g}$ as a Lie algebra we consider the left-invariant differential operator

$$\mathcal{L} = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j}. \tag{0.1}$$

In [2] G. B. Folland and E. M. Stein proved that if G is a homogeneous group and $\mathcal{L}$ is homogeneous, then the closure of $-\mathcal{L}$ is the infinitesimal generator of a one-parameter semigroup $\{T_t\}_{t>0}$ of operators on $L^2(G)$ such that

$$T_t f = f * p_t,$$

where p_t is C^∞ and for every ∂ in the enveloping algebra $\mathcal{G}$ of G and every N we have

$$\|\partial p_t (1+\tau)^N\|_{L^\infty} < \infty. \tag{0.2}$$

On the other hand, if $\mathcal{L}$ is of the second order, i.e., if all the n_j's are equal to one, then it is well known and goes back to Nelson and Gårding

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[3], [9] (cf. also [1]) that for an arbitrary Lie group G, every ∂ in $\mathscr{G}$ and an arbitrary N we have

(0.3) $$\|\partial p_t e^{\tau N}\|_{L^\infty} < \infty .$$

The aim of this paper is to prove (0.3) for the operators of the form (0.1) on nilpotent Lie groups. We believe that (0.3) holds for operators of the form (0.1) on arbitrary Lie groups and a good part of our arguments will be done in this generality; however, we do not know whether the following subelliptic estimate proved by B. Helffer and J. Nourrigat for nilpotent groups [4] holds in general:

(0.4) *For every ∂ in $\mathscr{G}$ there are σ and c such that*

$$\|\partial f\|_{L^2} \leqslant c\,\|(1+\mathscr{L})^\sigma f\|_{L^2} \quad \text{for all } f \text{ in } C_c^\infty(G).$$

It is easy to see that the operator $\mathscr{L}$ is positive on $L^2(G)$, and consequently it has a selfadjoint extension. As a matter of fact, the closure of $\mathscr{L}|C_c^\infty$ is selfadjoint; we shall sketch a proof of this in Section 4.

Let $E(\lambda)$ be the spectral resolution of $\mathscr{L}$. Then

$$T_z = \int_0^\infty e^{-z\lambda}\,dE(\lambda), \qquad \operatorname{Re} z > 0,$$

is a holomorphic extension of T_t.

In the case when G is nilpotent and $X_1, \ldots, X_k$ generate $\mathfrak{g}$ we prove that for every N there is a θ such that if $|\operatorname{Arg} z| < \theta$, then for every ∂ in $\mathscr{G}$ we have

$$\|\partial p_z e^{N\tau}\|_{L^\infty} < \infty .$$

This has the following

Corollary. *For every strongly continuous Banach space representation U of G the operator $U(-\mathscr{L})$ is the infinitesimal generator of a holomorphic semigroup of operators $U(T_z)$ for $|\operatorname{Arg} z| < \theta_U$.*

It is perhaps interesting to remark that if

$$\|U(x)\| \leqslant (1+\tau(x))^N, \qquad x \in G,$$

then $\theta_U = \pi/2$.

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1. Preliminaries. Our main tool is the following theorem essentially due to J.-L. Lions [8]. Since its proof is a combination of a number of propositions from [8] and [11] we include it here.

Let $\mathscr{H}$ be a Hilbert space and let V be a dense subspace of $\mathscr{H}$ such that V is a Hilbert space with the inner product $(\cdot,\cdot)_V$ and the norm $\|\cdot\|_V$ and for a constant c we have

$$\|x\|_{\mathscr{H}} \leqslant c\,\|x\|_V \quad \text{for } x \in V.$$

Let $a(x, y)$ be a bounded sesquilinear form on V. It defines an operator $A\colon \mathscr{D}(A) \to \mathscr{H}$ as follows:

$$\mathscr{D}(A) = \{u \in V\colon |a(u, v)| \leqslant C_u \|v\|_{\mathscr{H}},\ v \in V\}, \qquad (Au, v)_{\mathscr{H}} = a(u, v).$$

(1.1) Proposition. *Suppose that for some $\alpha > 0$ and $\lambda_0 \in \mathbf{R}$ we have*

$$\alpha\,\|u\|_V^2 \leqslant -\operatorname{Re} a(u, u) + \lambda_0 \|u\|_{\mathscr{H}}^2.$$

Then A is the infinitesimal generator of a strongly continuous semigroup of operators on $\mathscr{H}$ which is holomorphic in a sector

$$S_\varepsilon = \{z\colon |\operatorname{Arg} z| < \varepsilon\}.$$

Proof. For $\lambda > \lambda_0$ we let

$$a_\lambda(u, v) = a(u, v) - \lambda(u, v)_{\mathscr{H}}.$$

Then, by assumption,

(1.2) $$-\operatorname{Re} a_\lambda(u, u) \geqslant \alpha\,\|u\|_V^2 .$$

By [8], Proposition 1.2, p. 11, (1.2) implies that the corresponding operator $A_\lambda = A - \lambda I$ is an isomorphism of $\mathscr{D}(A) = \mathscr{D}(A_\lambda)$ onto $\mathscr{H}$. A_λ is dissipative, i.e. $\operatorname{Re}(A_\lambda u, u)_{\mathscr{H}} \leqslant 0$ for u in $\mathscr{D}(A_\lambda)$, and, by [8], Remarque 1.1, p. 12, $\mathscr{D}(A) = \mathscr{D}(A_\lambda)$ is dense in $\mathscr{H}$. Now applying the Lumer–Phillips theorem (cf. e.g. [11], p. 14), A_λ is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathscr{H}$. Of course,

(1.3) 0 *belongs to the resolvent set $\varrho(A_\lambda)$ of A_λ.*

Also there is a constant c such that $|\operatorname{Im}(A_\lambda u, u)_{\mathscr{H}}| \leqslant c\,\|u\|_V^2$. Consequently, by (1.2), we have

$$NR(-A_\lambda) = \{(-A_\lambda u, u)_{\mathscr{H}}\colon u \in \mathscr{D}(A_\lambda),\ \|u\|_{\mathscr{H}} = 1\} \subset \bar{S}_\theta = \{z\colon |\operatorname{Arg} z| \leqslant \theta\}$$

where $\theta = \arctan(c/\alpha) < \pi/2$. Thus for a δ such that $\theta < \delta < \pi/2$ there is a constant C_δ such that

$$C_\delta |\lambda_1| \leqslant d(\lambda_1, S_\theta) \leqslant d(\lambda_1, \text{closure } NR(-A_\lambda))$$

for

$$\lambda_1 \in \Sigma_\delta = \{z\colon |\operatorname{Arg} z| > \delta\},$$

where $d(z, A) = \inf\{|z-w|\colon w \in A\}$. Now, by [11], Theorem 3.9, since we have

$(-\infty, 0) \subset \Sigma_\delta \cap \varrho(-A_\lambda)$, it follows that

$$-\Sigma_\delta \subset \varrho(A_\lambda), \tag{1.4}$$

$$\|R(\lambda_1, -A_\lambda)\| \leqslant d(\lambda_1, NR(-A_\lambda))^{-1} \leqslant C_\delta^{-1}|\lambda_1|^{-1} \quad \text{for } \lambda_1 \in \Sigma_\delta. \tag{1.5}$$

But (1.3)–(1.5) imply (cf. e.g. [11]) that our semigroup is holomorphic in a sector S_ε, $\varepsilon > 0$. It follows that also the semigroup generated by $A = A_\lambda + \lambda I$ is holomorphic in S_ε.

Now let G be an arbitrary connected Lie group. By $\mathfrak{g}$ we denote its left-invariant Lie algebra and by $\mathscr{G}$ the enveloping algebra. A fixed right-invariant Haar measure is denoted by dx.

If

$$(f, g) = \int fg\, dx,$$

then for $X \in \mathfrak{g}$ we have

$$(Xf, g) = -(f, Xg), \quad f, g \in C_c^\infty(G).$$

Let $d(x, y)$ be a fixed left-invariant riemannian distance on G and let

$$\tau(x) = d(x, e).$$

For a fixed C^∞ nonnegative function f such that $\operatorname{supp} f \subset \{x: \tau(x) < 1\}$ and $\int f\, dx = 1$ we write

$$\varphi(x) = e^{\tau * f(x)}.$$

(1.6) Proposition. *For every submultiplicative function ϱ on G there exist positive numbers m and C such that*

$$\varrho(x) \leqslant C\varphi^m(x) \quad \textit{for all } x \in G.$$

Proof. Cf. e.g. [5].

From now on we shall fix a function η of the form

$$\eta(x) = \varphi^m(x). \tag{1.7}$$

(1.8) Proposition. *For every $\partial \in \mathscr{G}$ there exists a constant $c = c(\partial, \eta)$ such that*

$$|\partial\eta(x)| \leqslant c\eta(x), \quad x \in G.$$

Proof. Cf. [5].

Let

$$\mathscr{H} = \mathscr{H}_\eta = L^2(\eta) = \{f: \int |f(x)|^2 \eta(x)\, dx = \|f\|_\eta^2 < \infty\}.$$

To simplify our notation we shall also write $(\cdot, \cdot)_{\mathscr{H}_\eta} = (\cdot, \cdot)_\eta$.

Let U be a symmetric relatively compact neighbourhood of the identity of G. We write

$$\tau_U(x) = \min\{n: x \in U^n\}. \tag{1.9}$$

Of course, τ_U is subadditive.

We shall need the following simple

(1.10) Lemma. *There are a subset Λ of G and a constant c such that*

$$G = \bigcup_{a \in \Lambda} aU, \quad \sum_{a \in \Lambda} e^{-c\tau_U(a)} < \infty. \tag{1.11}$$

Proof. Let F be a finite subset of G such that $FU \supset U^2$. Then

$$F^{n-1}U \supset U^n. \tag{1.12}$$

Let

$$F_n = \{a \in F^{n-1}: aU \cap (U^{n-1})^c \neq 0\}.$$

By (1.12), $F_n U \supset U^n - U^{n-1}$. Consequently, if $\Lambda = \bigcup_{n=1}^\infty F_n$, then the equality in (1.11) holds. On the other hand, if $a \in F_n$, then $\tau_U(a) \geqslant n-2$, whence, for $c > \log|F|$,

$$\sum_{a \in \Lambda} e^{-c\tau_U(a)} \leqslant \sum_n \sum_{a \in F_n} e^{-c\tau_U(a)} \leqslant \sum_n |F|^n e^{-c(n-2)} < \infty.$$

For a submultiplicative function ϱ on G we write

$$L^1(\varrho) = \{f \in L^1(dx): \int |f(x)|\varrho(x)\, dx < \infty\}.$$

We have

$$\|f * g\|_{L^1(\varrho)} \leqslant \|f\|_{L^1(\varrho)} \|g\|_{L^1(\varrho)}. \tag{1.13}$$

Finally, let G be unimodular and let N be a unimodular normal subgroup of G. Let dx, $d\xi$, $d\dot{x}$ be Haar measures on G, N and G/N respectively such that

$$\int f(x)\, dx = \int_{G/N} \int_N f(\xi y)\, d\xi\, d\dot{y}. \tag{1.14}$$

Let π be the natural homomorphism $\pi: G \to G/N$. For a submultiplicative function ϱ on G/N the function $\varrho \circ \pi = \dot{\varrho}$ is submultiplicative on G. We write

$$\pi f(\dot{x}) = \int_N f(\xi x)\, d\xi.$$

Then

$$\|\pi f\|_{L^1(\varrho)} \leqslant \|f\|_{L^1(\dot{\varrho})}.$$

It is well known, and also follows from (1.6), (1.10), that for sufficiently large k

$$\int \varphi(x)^{-k}\, dx < \infty.$$

2. Some lemmas

(2.1) LEMMA. *Let $X \in \mathfrak{g}$. Then for every natural number n there is a constant c such that*

$$\|X^n f\|_\eta^2 \leqslant c\,\{|(X^{2n}f, f)_\eta| + \sum_{k=0}^{n-1} \|X^k f\|_\eta^2\}.$$

Proof. Since $\|X^n f\|_\eta^2 = |(X^n f, X^n f)_\eta|$, we have

$$\|X^n f\|_\eta^2 \leqslant |(X^n f, X^n(f\eta))| + \sum_{k=0}^{n-1} \binom{n}{k} |(X^n f, X^k f X^{n-k}\eta)|$$

and by Proposition (1.8) for an arbitrary $\varepsilon > 0$ there is c_0 such that

$$\|X^n f\|_\eta^2 \leqslant |(X^{2n}f, f\eta)| + \sum_{k=0}^{n-1} (\varepsilon \|X^n f\|_\eta^2 + c_0 \|X^k f\|_\eta^2)$$

$$= |(X^{2n}f, f\eta)| + n\varepsilon \|X^n f\|_\eta^2 + c_0 \sum_{k=0}^{n-1} \|X^k f\|_\eta^2.$$

Now for ε small enough we take $n\varepsilon \|X^n f\|_\eta^2$ to the other side, and the proof is complete.

(2.2) LEMMA. *Let $X \in \mathfrak{g}$. For every $b > 0$ there is B such that for every natural number j we have*

$$\|X^j f\|_\eta^2 \leqslant b\,\|X^{j+1} f\|_\eta^2 + B\,\|X^{j-1} f\|_\eta^2, \quad f \in C_c^\infty(G).$$

Proof. By Proposition (1.8), for a constant c we have

$$\|X^j f\|_\eta^2 = |(X^j f, X^j f\eta)| = |(X^{j-1} f, X[X^j f\eta])|$$
$$\leqslant |(X^{j-1} f, X^{j+1} f\eta)| + |(X^{j-1} f, X^j f X\eta)|$$
$$\leqslant |(X^{j-1} f, X^{j+1} f\eta)| + c\,|(X^{j-1} f, X^j f\eta)|.$$

Hence, by the Schwarz inequality for $\varepsilon > 0$ there is c_ε such that

$$\|X^j f\|_\eta^2 \leqslant c_\varepsilon \|X^{j-1} f\|_\eta^2 + \varepsilon \|X^{j+1} f\|_\eta^2 + cc_\varepsilon \|X^{j-1} f\|_\eta^2 + \varepsilon c \|X^j f\|_\eta^2.$$

Since c depends only on η and X, for ε sufficiently small we may take $\varepsilon c \|X^j f\|_\eta^2$ to the other side thus completing the proof.

(2.3) LEMMA. *Let $X \in \mathfrak{g}$. For every integer $n \geqslant 2$ and every $b > 0$ there is $B > 0$ such that*

$$\|X^j f\|_\eta^2 \leqslant b\,\|X^n f\|_\eta^2 + B\,\|f\|_\eta^2, \quad f \in C_c^\infty(G),\ 1 \leqslant j < n. \tag{2.4}$$

Proof. We proceed by induction on n. The case $n = 2$, $j = 1$ is an immediate consequence of Lemma (2.2). Suppose (2.4) holds for $n \leqslant m$, i.e.

$$\|X^j f\|_\eta^2 \leqslant \varepsilon \|X^m f\|_\eta^2 + C_\varepsilon \|f\|_\eta^2, \quad 1 \leqslant j < m, \tag{2.5}$$

where ε can be taken arbitrarily small. By Lemma (2.2), for an arbitrary $b_1 > 0$ and some B_1

$$\|X^m f\|_\eta^2 \leqslant b_1 \|X^{m+1} f\|_\eta^2 + B_1 \|X^{m-1} f\|_\eta^2. \tag{2.6}$$

Hence,

$$\|X^m f\|_\eta^2 \leqslant b_1 \|X^{m+1} f\|_\eta^2 + B_1 \|X^{m-1} f\|_\eta^2$$
$$\leqslant b_1 \|X^{m+1} f\|_\eta^2 + B_1 (\varepsilon \|X^m f\|_\eta^2 + C_\varepsilon \|f\|_\eta^2)$$
$$= b_1 \|X^{m+1} f\|_\eta^2 + B_1 \varepsilon \|X^m f\|_\eta^2 + B_1 C_\varepsilon \|f\|_\eta^2,$$

so taking ε small enough, the conclusion follows.

A similar argument yields

(2.7) LEMMA. *Let $X \in \mathfrak{g}$. Then for every positive integer n and every $\varepsilon > 0$ there is a constant C_ε such that*

$$|(X^k f, X^l f\eta)| \leqslant \varepsilon \|X^n f\|_\eta^2 + C_\varepsilon \|f\|_\eta^2$$

for all k, $l \leqslant n-1$.

Now let $X_1, \ldots, X_k$ be some fixed elements from $\mathfrak{g}$ and $n_1, \ldots, n_k$ some fixed positive integers. We introduce a Hilbert space norm on $C_c^\infty(G)$ by

$$\|f\|_{V_\eta}^2 = \|f\|_\eta^2 + \sum_{j=1}^{k} \sum_{s=1}^{n_j} \|X_j^s f\|_\eta^2.$$

By Lemma (2.3), this norm is equivalent to

$$\|f\|_\eta^2 + \sum_{j=1}^{k} \|X_j^{n_j} f\|_\eta^2,$$

and we shall use the same notation for both.

Let V_η be the completion of $C_c^\infty(G)$ in the norm $\|\cdot\|_{V_\eta}$. The following lemma has a standard proof (cf. e.g. [1]).

(2.8) LEMMA. *$f \in V_\eta$ iff $f \in \mathscr{H}_\eta$ and $X_j^s f \in \mathscr{H}_\eta$ for all $1 \leqslant s \leqslant n_j$, $j = 1, \ldots, k$, where $X_j^s f$ is understood in the sense of distributions.*

3. A semigroup of operators on weighted Hilbert spaces. We are still considering a general Lie group G. For elements $X_1, \ldots, X_k$ in the Lie algebra $\mathfrak{g}$ of G we consider the operator

$$\mathscr{L} = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j} \tag{3.1}$$

defined on $C_c^\infty(G)$. As in the previous sections, we write $\eta = \varphi^m$ and we define a sesquilinear form a_η by

$$a_\eta(f, g) = -\sum_{j=1}^{k} \int_G X_j^{n_j} f(x) X_j^{n_j} [g(x)\eta(x)]^- \, dx.$$

For $\eta = 1$ we write $a_\eta(f, g) = a(f, g)$.

We note that in virtue of Proposition (1.8) for some $c > 0$ we have

$$|a_\eta(f, g)| \leqslant c\|f\|_{V_\eta}\|g\|_{V_\eta}$$

and so as in Section 1, a_η defines an operator A^η whose domain is

$$\mathcal{D}(A^\eta) = \{f \in V_\eta\colon\ |a_\eta(f, g)| \leqslant C_f\|g\|_\eta\}$$

and

$$(A^\eta f, g)_\eta = a_\eta(f, g) \quad \text{for } f \in \mathcal{D}(A^\eta),\ g \in V_\eta.$$

Also, $a_\eta(f, g) = -(\mathcal{L}f, g)_\eta$ for $f, g \in C_c^\infty(G)$, i.e. $-\mathcal{L} \subset A^\eta$.

Now, by (2.3), (2.7) and (1.8) we have

(3.2) Theorem. *There exist λ_0 and $\alpha > 0$ such that*

$$-\operatorname{Re} a_\eta(f, f) + \lambda_0\|f\|_\eta^2 \geqslant \alpha\|f\|_{V_\eta}^2.$$

Consequently, by Proposition (1.1),

(3.3) Theorem. *A^η is the infinitesimal generator of a holomorphic semigroup $\{T_t^\eta\}_{t>0}$ of operators on $\mathcal{H}_\eta$.*

Also it requires only a routine verification to prove

(3.4) Theorem. *For $f \in \mathcal{D}(A^\eta)$, $A^\eta f = -\mathcal{L}f$ and $\mathcal{D}(A^\eta) = \{f \in V_\eta\colon \mathcal{L}f \in \mathcal{H}_\eta\}$, where $\mathcal{L}f$ is meant in the sense of distributions.*

Hence the following theorem follows:

(3.5) Theorem. *If $m_1 \leqslant m_2$ and $\eta_1 = \varphi^{m_1}$, $\eta_2 = \varphi^{m_2}$, then $A^{\eta_2} \subset A^{\eta_1}$.*

The next theorem is a consequence of (3.5) but its proof, though routine and easy, seems to require passing from the inclusion (3.5) to the resolvents and using the Hille–Yosida–Phillips theorem.

(3.6) Theorem. *If η_1, η_2 are as above, then for f in $\mathcal{H}_{\eta_2}$ we have*

$$T_t^{\eta_1} f = T_t^{\eta_2} f$$

for all positive t and consequently, since the semigroups are holomorphic, for t in the smaller of the corresponding sectors.

Finally, we note that for η equal identically to 1, the operator $A^\eta = A$ is selfadjoint, being symmetric and the infinitesimal generator of a (holomorphic) semigroup of operators.

4. $\mathcal{L}$ as an operator on $L^2(G)$. In this section we are going to prove that our operator $\mathcal{L}$ as defined by (3.1) is essentially selfadjoint on $C_c^\infty(G)$. This of course follows from an argument by E. Nelson and W. F. Stinespring [10] but requires some care: $C_c^\infty(G)$ is not the Gårding space of the right regular representation (it is only dense in it). Moreover, for the right regular representation the ellipticity or hypoellipticity of $\mathcal{L}$ is not needed: we can regularize from the left-hand side.

For an element ∂ of the left-invariant enveloping algebra $\mathcal{G}$ of G let ∂^+ be the element of $\mathcal{G}$ such that $\partial \to \partial^+$ is the linear anti-automorphism of $\mathcal{G}$ such that $X^+ = -X$ for X in $\mathfrak{g}$.

The following is a lemma of E. Nelson and W. F. Stinespring [10].

(4.1) Lemma. *Let f be a positive-definite smooth function on G. Then for ∂ in $\mathcal{G}$*

$$\partial^+ \partial f(e) \geqslant 0.$$

Let $\mathcal{S}^\infty$ be the Gårding space for the right regular representation:

$$\mathcal{S}^\infty = \operatorname{lin}\{f * \omega\colon f \in L^2(G),\ \omega \in C_c^\infty(G)\}. \tag{4.2}$$

Of course, for every ∂ in $\mathcal{G}$ and $f * \omega \in \mathcal{S}^\infty$ we have

$$\partial(f * \omega) = f * \partial\omega \in \mathcal{S}^\infty \subset L^2(G) \cap C^\infty(G).$$

(4.3) Lemma. *Let $L = \sum_{j=1}^{k} (-1)^{n_j} X_j^{2n_j}$ be a differential operator defined on $\mathcal{S}^\infty$. Then L is essentially selfadjoint.*

Proof. We note that L is of the form

$$L = \sum_{j=1}^{k} \partial_j^+ \partial_j. \tag{4.4}$$

It suffices to prove that $(1 + L)(\mathcal{S}^\infty)$ is dense in $L^2(G)$.

Suppose that for some $g \in L^2(G)$

$$0 = ([1 + L][\xi * \xi * g * \omega], g)$$

for all $\xi, \omega \in C_c^\infty(G)$. Then, by (4.4), since ω is arbitrary,

$$0 = \Big(1 + \sum_{j=1}^{k} \partial_j^+ \partial_j\Big)\big((\xi * g)^\sim * \xi * g\big)(e),$$

i.e., by (4.1), $0 = (\xi * g)^\sim * (\xi * g)(e) = \|\xi * g\|_{L^2}^2$, hence $\|g\|_{L^2} = 0$.

(4.5) Theorem. *The closure of $\mathcal{L}$, $\bar{\mathcal{L}}$, is selfadjoint, $\mathcal{L}$ being defined on $C_c^\infty(G)$.*

Proof. We are going to show that $\bar{\mathcal{L}} = \bar{L}$. The inclusion $\mathcal{L} \subset \bar{L} = -A$ is an obvious consequence of (3.4), (4.3) and the note at the end of Section 3. To show $\bar{\mathcal{L}} \supset L$ we let $\{\gamma_n\}$ be a sequence of functions in $C_c^\infty(G)$ converging with all derivatives almost uniformly to 1. Let $u \in \mathcal{S}^\infty = \mathcal{D}(L)$. Since $\partial u \in L^2(G)$ for all $\partial \in \mathcal{G}$, by the Leibniz formula,

$$\lim_{n\to\infty} \|\mathcal{L}(\gamma_n u) - Lu\|_{L^2} = 0$$

and also

$$\lim_{n\to\infty} \|\gamma_n u - u\|_{L^2} = 0.$$

Hence $u \in \mathcal{D}(\bar{\mathcal{L}})$, i.e. $L \subset \bar{\mathcal{L}}$.

5. The case of graded groups. We now assume that G is a graded group (cf. [2]) and that the operator $\mathcal{L}$ defined by (3.1) is of degree, say, λ, i.e.

(5.1) $$\mathcal{L}(f \circ \delta_r) = r^\lambda (\mathcal{L}f) \circ \delta_r,$$

where $\{\delta_r\}_{r>0}$ is the family of dilations of G. We also assume that $X_1, \ldots, X_k$ satisfy

(5.2) $$\operatorname{Lie}\{X_1, \ldots, X_k\} = \mathfrak{g}.$$

Our basic tool now is the following subelliptic estimate due to B. Helffer and J. Nourrigat [4].

For every ∂ in $\mathcal{G}$ there is an integer $\sigma(\partial)$ such that

(5.3) $$\|\partial u\|_{L^2(G)} \leqslant c\,\|(1+\bar{\mathcal{L}})^{\sigma(\partial)} u\|_{L^2(G)}$$

for all $u \in \mathcal{D}(\bar{\mathcal{L}}^{\sigma(\partial)})$.

Consequently (cf. [10]), since G is unimodular, there are N and c such that for $u \in \mathcal{D}(\bar{\mathcal{L}}^N)$

(5.4) $$\|u\|_{L^\infty(G)} \leqslant c\,\|(1+\bar{\mathcal{L}})^N u\|_{L^2(G)}.$$

Since $\bar{\mathcal{L}}$ is selfadjoint on $L^2(G)$ and positive, we write

$$T_z f = \int_0^\infty e^{-z\lambda}\, dE(\lambda) f, \quad \operatorname{Re} z > 0,$$

where $dE(\lambda)$ is the spectral measure of $\bar{\mathcal{L}}$. For every z with $\operatorname{Re} z > 0$,

$$T_z\colon L^2(G) \to \bigcap_n \mathcal{D}(\bar{\mathcal{L}}^n),$$

hence, by (5.3) and (5.4), we have

(5.5) $$T_z f \in C^\infty(G) \quad \text{for all } f \text{ in } L^2(G).$$

Moreover, since T_t commutes with left translations, and $T_t = T_t^*$, we verify as e.g. in [7] that

$$T_z f = f * p_z, \quad p_z \in L^2(G),$$

and $p_t^* = p_t$ for t real, whence $p_{\bar{z}}(x) = \overline{p_z(x)}$. Also

$$p_z * p_w = p_{z+w}, \quad \operatorname{Re} z > 0,\ \operatorname{Re} w > 0.$$

Hence, finally, $p_z \in C_0(G)$.

We note that $T_t = T_t^\eta$ for $\eta \equiv 1$, as defined in Section 3.

Now assume that η is an arbitrary function of the form (1.7). Since $\{T_t^\eta\}_{t>0}$ is a holomorphic semigroup on $\mathcal{H}_\eta$, in virtue of (5.5) and (3.6), we have

$$T_t\colon L^2(\eta) \to \bigcap_n \mathcal{D}((A^\eta)^n) \cap C^\infty(G).$$

A routine application of (1.8) and the Leibniz formula yield

(5.7) Lemma. *For every X in $\mathfrak{g}$ there is a constant c which may depend on X, t and η such that*

$$|(XT_t f, XT_t f)_\eta| \leqslant \|X^2 T_t f\|^2_{L^2(G)} + c\,\|T_t f\eta\|^2_{L^2(G)} + c\,\|XT_t f\|^2_{L^2(G)}.$$

(5.8) Lemma. *For each ∂ in $\mathcal{G}$ there is $c = c(t, \partial)$ such that*

$$\|\partial T_t f\|_{L^2(G)} \leqslant c\,\|f\|_{L^2(G)}.$$

Proof. By (5.3) with $N = \sigma(\partial)$,

$$\|\partial T_t f\|_{L^2(G)} \leqslant C\,\|(1+\bar{\mathcal{L}})^N T_t f\|_{L^2(G)} \leqslant C\,\Big\|\int_0^\infty (1+\lambda)^N e^{-\lambda t}\, dE(\lambda) f\Big\| \leqslant c\,\|f\|_{L^2(G)}.$$

For $f \in L^2(\eta^2)$ we have

$$\|(T_t f)\eta\|^2_{L^2(G)} = \|T_t f\|^2_{L^2(\eta^2)} \leqslant c\,\|f\|^2_{L^2(\eta^2)}$$

and so, by (5.7) and (5.8),

$$(XT_t f, XT_t f)_\eta \leqslant c\,\|f\|^2_{L^2(\eta^2)}.$$

Hence, by an easy induction, we obtain

(5.9) Lemma. *For every ∂ in $\mathcal{G}$ and every $t > 0$ there is k such that*

$$\|\partial T_t f\|_{L^2(\eta)} \leqslant c\,\|f\|_{L^2(\eta^k)}$$

for all $f \in L^2(\eta^k)$. Consequently, for some s

$$\|\partial[(T_t f)\eta]\|_{L^2(G)} \leqslant c\,\|f\|_{L^2(\eta^s)}.$$

This, by (5.4), implies

(5.10) Corollary. *For every ∂ in $\mathcal{G}$, every η and every $t > 0$ there are c and s such that*

$$\|\partial(T_t f)\eta\|_{L^\infty(G)} \leqslant c\,\|f\|_{L^2(\eta^s)}.$$

(5.11) THEOREM. *For every* $t > 0$ *and every* $\eta = \varphi^m$ (cf. (1.7)) *we have*

$$\int p_t(x)^2 \eta(x)\,dx < \infty.$$

Proof. We already know that $p_t \in L^2(G)$. Let U be a symmetric relatively compact neighbourhood of the identity in G, $t > 0$ and $m > 0$. By (1.10), there are k and a countable subset A of G such that

$$G = \bigcup_{a\in A} a^{-1}U, \quad \sum_{a\in A} \eta(a^{-1})^{-k} < \infty.$$

Let

(5.12) $$f_a = {}_{a^{-1}}p_t | U.$$

Since $p_t \in C_0(G)$ we have $|f_a| \leqslant C$ and $\|f_a\|_{L^2(G)} \leqslant M$, for all $a \in A$. Consequently, by (5.10), since $\operatorname{supp} f_a \subset U$,

(5.13) $$\left|\int {}_x f_a(y)\, p_t(y)\,dy\, \eta(x)^{k+1}\right| = |T_t f_a(x)\, \eta(x)^{k+1}| \leqslant C\|f_a\|_{L^2(\eta^{k+s+1})} \leqslant c\|f_a\|_{L^2(G)} \leqslant cM.$$

Since $\eta(xy) \leqslant c\eta(x)\eta(y)$ for some c,

$$\begin{aligned} I = \int p_t(y)^2 \eta(y)\,dy &\leqslant \sum_{a\in A} \int_{a^{-1}U} p_t(y)^2 \eta(a^{-1}ay)\,dy \\ &\leqslant c\sum_{a\in A}\int_{a^{-1}U} p_t(y)^2 \eta(a^{-1})\eta(ay)\,dy \\ &\leqslant cM_0 \sum_{a\in A}\int_{a^{-1}U} p_t(y)^2 \eta(a^{-1})^{k+1}\eta(a^{-1})^{-k}\,dy, \end{aligned}$$

where $M_0 = \sup\{\eta(y)\colon\ y \in U\}$. Consequently, by (5.13),

$$\begin{aligned} I &\leqslant cM_0 \sum_{a\in A}\int_{a^{-1}U} {}_a f_a(y)\, p_t(y)\, \eta(a^{-1})^{k+1}\eta(a^{-1})^{-k}\,dy \\ &\leqslant cM_0 M \sum_{a\in A} \eta(a^{-1})^{-k} < \infty. \end{aligned}$$

(5.14) THEOREM. *For every* η, *every* $\partial \in \mathcal{G}$ *and every* $t > 0$

$$\|\partial p_t \eta\|_{L^\infty} < \infty.$$

Proof. Since $p_{t/2} \in L^2(\eta)$ for all η and $p_t = T_{t/2}\, p_{t/2}$, (5.14) follows immediately from (5.10).

Now we consider p_z with z complex.

(5.15) THEOREM. *For every* η *there is a sector* S_{θ_0}, $\theta_0 > 0$, *such that for all* $\partial \in \mathcal{G}$ *and* $z \in S_{\theta_0}$

$$\|\partial p_z \eta\|_{L^\infty} < \infty.$$

Proof. As we have shown in Section 3 for $z \in S_{\theta_0}$ the operator T_z^η maps $L^2(\eta)$ into $L^2(\eta)$. Hence for t small enough

$$\partial p_z = \partial(p_{z-t} * p_t) = p_{z-t} * \partial p_t \in L^2(\eta),$$

which, by (5.9), proves (5.15).

It follows immediately from (5.15) that for every η there is a sector S_θ, $\theta > 0$, such that for $z \in S_\theta$

$$p_z \in L^1(\eta).$$

Also it is easy to see that the map $z \to p_z \in L^1(\eta)$ is holomorphic for z in S_θ.

(5.16) THEOREM. $\{T_t\}_{t>0}$ *is a holomorphic semigroup on* $L^1(\eta)$ *for every* η.

Proof. To prove (5.16) it suffices to show that

(5.17) $$\|p_z\|_{L^1(\eta)} \leqslant C \quad \text{for } |z| \leqslant 1 \text{ and } z \in S_\theta.$$

Let $|z| = 1$ be in the sector S_θ in which $z \to p_z \in L^1(\eta)$ is holomorphic. Since the infinitesimal generator $\mathcal{L}$ of the semigroup $\{T_t\}_{t>0}$ is homogeneous of degree say λ, if Q is the homogeneous dimension of G (cf. [2]) we easily verify that

$$p_{tz} = t^{-Q/\lambda} p_z \circ \delta_{t^{-1/\lambda}}.$$

Hence for $t < 1$

$$\|p_{tz}\|_{L^1(\eta)} = \int |p_z(x)|\, \eta(\delta_t x)\,dx \leqslant C_{z,\eta}$$

because if τ is a riemannian distance on G and δ_t is an automorphism of G, then $\tau(\delta_t x) \leqslant t^d \tau(x)$ for some d and, on the other hand, $\eta(x) \leqslant e^{C\tau(x)+C}$. This completes the proof of (5.17).

6. Arbitrary nilpotent Lie groups. Now we use the trick of L. Rothschild and E. M. Stein [12] applied by many authors afterwards: We pass to the free nilpotent group on which the operator $\mathcal{L}$ is homogeneous. The details are as follows.

If $\dot{G}$ is an arbitrary nilpotent Lie group, of step p, say, and $\dot{X}_1, \ldots, \dot{X}_k$ are elements of the Lie algebra $\dot{\mathfrak{g}}$ of $\dot{G}$ such that

$$\operatorname{Lie}\{\dot{X}_1, \ldots, \dot{X}_k\} = \dot{\mathfrak{g}}$$

we let G be a free step p nilpotent Lie group whose Lie algebra is freely generated by $X_1, \ldots, X_k$. We introduce dilations on G such that the operator (3.1) is homogeneous. The mapping

$$\pi\colon\ \mathfrak{g} \ni X_j \to \dot{X}_j \in \dot{\mathfrak{g}}$$

defines a homomorphism of $\mathfrak{g}$ onto $\dot{\mathfrak{g}}$ and consequently a homomorphism of G onto $\dot{G}$ and of $\mathcal{G}$ onto $\dot{\mathcal{G}}$. Let N be the kernel of π in G.

Consequently, the operator

$$\dot{\mathscr{L}} = \sum_{j=1}^{k} (-1)^{n_j} \dot{X}_j^{2n_j}$$

is the image by π of $\mathscr{L}$ defined on the free nilpotent group G by (0.1). Also it is easy to verify that for z in a sector S_θ, $\theta > 0$,

$$\dot{p}_z(\dot{x}) = \int_N p_z(\xi x)\, d\xi$$

defines a strongly continuous holomorphic convolution semigroup on $L^1(\dot{G}, \psi)$, where ψ is a submultiplicative function on $\dot{G}$, whose infinitesimal generator is the closure of $-\dot{\mathscr{L}}$. Also

$$\int_{\dot{G}} |\dot{\partial}\dot{p}_z(\dot{x})|\, \psi(\dot{x})\, d\dot{x} = \int_{\dot{G}} \int_N |\partial p_z(\xi)|\, \psi(\pi(\xi x))\, d\xi\, d\dot{x}$$
$$\leqslant \int_G |\partial p_z(x)|\, \psi \circ \pi(x)\, dx.$$

This shows that Theorems (5.15) and (5.16) hold for an arbitrary nilpotent Lie group and arbitrary operator of the form (0.1).

(6.1) Corollary. *For any strongly continuous Banach space representation U of a nilpotent Lie group G, if $\mathscr{L}$ is the operator defined by* (0.1) *the operator $U(-\mathscr{L})$ is the infinitesimal generator of a strongly continuous holomorphic semigroup of operators.*

An easy application of (5.16) and the results of [6] and [7] yield the following

(6.2) Corollary. *If G and $\mathscr{L}$ are as above and U is an isometric strongly continuous representation of G (or more generally, $\|U(x)\|$ grows at most like a polynomial as $x \to \infty$), then the semigroup generated by $U(-\mathscr{L})$ is holomorphic in the half-plane* $\operatorname{Re} z > 0$.

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