STUDIA MATHEMATICA, T. XCIV (1989)

Power-dominated elements in a Banach algebra

by

G. R. ALLAN (Cambridge) and T. J. RANSFORD (Leeds)

Abstract. Let $(A, \|\cdot\|)$ be a Banach algebra. Using only elementary estimates and a simple re-norming technique, we give short new proofs of theorems of Gelfand and of Katznelson and Tzafriri on power-bounded elements of A. Combining these ideas with a version of the theorem of support leads to the following quantitative generalization, valid for a wide range of functions f: if $x \in A$ and $\{\mu(n)\}_{n\geqslant 0}$ is a sequence such that $\|x^n\| \leqslant \mu(n)$ $(n \geqslant 0)$ and $\lim_{n\to\infty} \mu(n+1)/\mu(n) = 1$, then $\limsup_{n\to\infty} \|f(x)x^n\|/\mu(n)$ is majorized by a bound depending only on f and the spectrum of x (not on A). Some examples are then considered and we finish by proving a partial converse.

§ 1. Power-bounded elements and the theorems of Gelfand and Katznelson–Tzafriri. Let A be a (complex, unital) Banach algebra. An element x of A which satisfies $\sup_{n\geqslant 0}\|x^n\| < \infty$ is called *power-bounded*. If moreover x is invertible in A, and if $\sup_{n\in \mathbf{Z}}\|x^n\| < \infty$, then x is called *doubly power-bounded*. From the spectral radius formula it follows that if x is power-bounded then its spectrum satisfies $\operatorname{Sp}(x) \subset \bar{\Delta}$; and if x is doubly power-bounded, then $\operatorname{Sp}(x) \subset \Gamma$. Here, and throughout the paper, we have written $\bar{\Delta} = \{z \in \mathbf{C}\colon |z| \leqslant 1\}$ and $\Gamma = \{z \in \mathbf{C}\colon |z| = 1\}$. In this first section we present short new proofs of a theorem of Gelfand [8] on doubly power-bounded elements, and of another of Katznelson and Tzafriri [12] on power-bounded ones.

Gelfand's result can be proved a number of different ways (see e.g. [8], [10, Theorem 4.10.1], [13, Corollary 4.2]), but the demonstration given below seems particularly brief and elementary. It is based on the method of Bonsall and Crabb [4] in their proof of Sinclair's theorem on the spectral radius of an hermitian element.

THEOREM 1.1 ([8]). *Let A be a Banach algebra and let x be a doubly power-bounded element of A. If $\operatorname{Sp}(x) = \{1\}$ then $x = 1$.*

Proof. Let $h = -i\log(x) = i\sum_{k\geqslant 1}(1-x)^k/k$. Then $\operatorname{Sp}(h) = \{0\}$ and $x = e^{ih}$. Thus, given any integer m, we have $\operatorname{Sp}(\sin mh) = \{\sin 0\} = \{0\}$ and

$$\|(\sin mh)^k\| = \left\|\left(\frac{x^m - x^{-m}}{2i}\right)^k\right\| \leqslant \sup_{n\in\mathbf{Z}}\|x^n\| \quad (k \geqslant 0).$$

If $\sum_{k\geqslant 0} c_k z^k$ is the Taylor expansion of the principal value of $\arcsin(z)$ about $z=0$, then it is elementary that $c_k \geqslant 0$ for all k, and that $\sum_{k\geqslant 0} c_k$ converges to $\arcsin(1)=\pi/2$. Hence

$$\|mh\| = \|\arcsin(\sin mh)\| \leqslant \sum_{k\geqslant 0} |c_k|\,\|(\sin mh)^k\| \leqslant (\pi/2)\sup_{n\in \mathbf{Z}}\|x^n\|.$$

As this holds for any integer m, it follows that $h=0$, and therefore that $x=1$. ∎

Recently, Katznelson and Tzafriri obtained the following interesting generalization of Gelfand's theorem. Their result ([12, Theorem 1], see also [1]) was stated in the language of bounded operators on a Banach space; we have reformulated it in Banach-algebra terms since that fits more naturally with our proof.

THEOREM 1.2 ([12]). *Let A be a Banach algebra and let x be a power-bounded element of A. Then $\|x^{n+1}-x^n\|\to 0$ as $n\to\infty$ if (and only if) $\mathrm{Sp}(x)\cap\Gamma\subset\{1\}$.*

Gelfand's theorem is an immediate consequence of this, because

$$\|x-1\| \leqslant (\sup_{n\geqslant 0}\|x^{-n}\|)(\limsup_{n\to\infty}\|x^{n+1}-x^n\|).$$

On the other hand, we shall show that Theorem 1.2 can also be deduced easily from the weaker Theorem 1.1. The key idea is contained in the following "re-norming" lemma, by which certain problems on power-bounded elements (and power-dominated elements—see § 2) may be reduced to questions about doubly power-bounded ones.

LEMMA 1.3. *Let $(A, \|\cdot\|)$ be a commutative Banach algebra, let $x\in A$, and let $\{\mu(n)\}_{n\geqslant 0}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\mu(n+1)/\mu(n)=1$. Suppose that*

$$0<\limsup_{n\to\infty}\|x^n\|/\mu(n)\leqslant 1.$$

Then there exist a commutative Banach algebra $(B, \|\cdot\|_B)$ and a continuous homomorphism $\pi\colon A\to B$ such that:

(i) *$\pi(x)$ is invertible in B with $\|\pi(x)\|_B=\|\pi(x)^{-1}\|_B=1$.*
(ii) *$\limsup_{n\to\infty}\|ax^n\|/\mu(n)\leqslant\|\pi(a)\|_B\leqslant\|a\|$ $(a\in A)$.*

Proof. Define $p\colon A\to\mathbf{R}$ by

$$p(a)=\limsup_{n\to\infty}\|ax^n\|/\mu(n)\qquad(a\in A).$$

Then p is a seminorm on A which satisfies $p(aa')\leqslant\|a\|\,p(a')$ $(a, a'\in A)$, and by hypothesis $0<p(1)\leqslant 1$. Thus $p^{-1}(0)$ is a proper ideal in A. Define B_0 to be the (unital) quotient algebra $A/p^{-1}(0)$, and let $\pi\colon A\to B_0$ be the quotient homomorphism. Then

$$\|\pi(a)\|_{B_0} := \sup_{y\in p^{-1}(1)} p(ay)\qquad(a\in A)$$

defines an algebra norm $\|\cdot\|_{B_0}$, because it is just the operator norm on B_0, regarded as acting on (B_0, p) by multiplication. Now, since $\lim_{n\to\infty}\mu(n+1)/\mu(n)=1$, it follows that $p(ax)=p(a)$ $(a\in A)$, and hence that

$$\|\pi(a)\,\pi(x)\|_{B_0}=\|\pi(a)\|_{B_0}\qquad(a\in A). \tag{1}$$

By a theorem of Arens [2], (1) implies that $(B_0, \|\cdot\|_{B_0})$ can be embedded isometrically as a subalgebra of a commutative Banach algebra $(B, \|\cdot\|_B)$ in which $\pi(x)$ is invertible, and $\|\pi(x)\|_B=\|\pi(x)^{-1}\|_B=1$. This proves (i), and (ii) now follows from the inequalities

$$\|\pi(a)\|_B=\|\pi(a)\|_{B_0}\leqslant\sup_{y\in p^{-1}(1)}\|a\|\,p(y)=\|a\|\qquad(a\in A)$$

(which also guarantees that $\pi\colon A\to B$ is continuous) and

$$\|\pi(a)\|_B=\|\pi(a)\|_{B_0}\geqslant p(a1/p(1))\geqslant p(a)\qquad(a\in A).\ ∎$$

Remarks. (a) If $\mathrm{Sp}(x)$ is a proper subset of $\bar{\Delta}$ (as is the case in the proof below of Theorem 1.2), then $\pi(x)$ is already invertible in the completion of $(B_0, \|\cdot\|_{B_0})$, so that B may be taken just to be this completion, and there is no need to invoke Arens' theorem. To see this, observe that by equation (1), $\pi(x)-\lambda\pi(1)$ cannot be a topological divisor of zero in B_0, nor therefore in the completion B, unless $|\lambda|=1$. This implies that $\partial\mathrm{Sp}_B(\pi(x))\subset\Gamma$, so that either $\mathrm{Sp}_B(\pi(x))=\bar{\Delta}$ (impossible, because $\mathrm{Sp}_B(\pi(x))\subset\mathrm{Sp}_A(x)\subsetneqq\bar{\Delta}$), or $\mathrm{Sp}_B(\pi(x))\subset\Gamma$.

(b) The idea of defining the seminorm p was influenced by a proof of Esterle [7, Theorem 9.1].

Proof of Theorem 1.2. The "only if" part of the theorem is a simple consequence of the spectral mapping theorem, so we concentrate on the non-trivial "if" part. There is no loss of generality in assuming that A is commutative.

If $\mathrm{Sp}(x)\cap\Gamma=\emptyset$, then the spectral radius satisfies $r(x)<1$, so that by the spectral radius formula $\lim_{n\to\infty}\|x^n\|=0$, and thus certainly $\lim_{n\to\infty}\|x^{n+1}-x^n\|=0$.

Now suppose that $\mathrm{Sp}(x)\cap\Gamma\neq\emptyset$. This implies that $1\leqslant\|x^n\|\leqslant M$ $(n\geqslant 0)$, where $M=\sup_{n\geqslant 0}\|x^n\|$. By Lemma 1.3, there exist a (commutative) Banach algebra $(B, \|\cdot\|_B)$ and a (continuous) homomorphism $\pi\colon A\to B$ satisfying the conclusions (i) and (ii) of that lemma with $\mu(n)=M$ $(n\geqslant 0)$. By (i), $\pi(x)$ is doubly power-bounded. This implies that $\mathrm{Sp}_B(\pi(x))\subset\Gamma$, and since also $\mathrm{Sp}_B(\pi(x))\subset\mathrm{Sp}_A(x)$, it follows that $\mathrm{Sp}_B(\pi(x))=\{1\}$. By Theorem 1.1,

$\pi(x) = \pi(1)$. Therefore, with $a = x-1$ in Lemma 1.3 (ii) we have

$$\limsup_{n\to\infty} \|x^{n+1}-x^n\|/M \leqslant \|\pi(x-1)\|_B = 0. \ \blacksquare$$

§ 2. **Power-dominated elements.** In this section we prove a substantial generalization of Theorem 1.2. The theorem is extended in three ways: firstly, instead of $\|x^{n+1}-x^n\|$, we consider $\|f(x)x^n\|$ for a large class of functions f for which $f(x)$ makes sense; secondly, the spectrum of x is allowed to lie anywhere within the closed unit disc; and thirdly, the notion of "power-bounded" is replaced by the much wider concept of "power-dominated", which we now describe.

Let x be an element of a Banach algebra A, and let $\{\mu(n)\}_{n\geqslant 0}$ be a sequence of non-negative numbers. We say that x is *power-dominated by* $\{\mu(n)\}$ if $\|x^n\| \leqslant \mu(n)$ $(n \geqslant 0)$. In what follows it will usually be assumed that $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$; it is therefore of interest to know when x can be power-dominated by such a sequence.

Proposition 2.1. *Let x be an element of a Banach algebra A. Then it can be power-dominated by a sequence $\{\mu(n)\}$ with $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$ if and only if its spectral radius satisfies $r(x) \leqslant 1$.*

Proof. Suppose that such a sequence $\{\mu(n)\}$ exists. Then $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$ implies that $\lim_{n\to\infty}\mu(n)^{1/n} = 1$, so by the spectral radius formula we have

$$r(x) = \lim_{n\to\infty}\|x^n\|^{1/n} \leqslant \lim_{n\to\infty}\mu(n)^{1/n} = 1.$$

Conversely, suppose that $r(x) \leqslant 1$. If x is power-bounded, then of course we may take $\mu(n) = \sup_{k\geqslant 0}\|x^k\|$ $(n \geqslant 0)$. If x is not power-bounded, then necessarily $r(x) = 1$, so the sequence $\varrho_n = \log\|x^n\|$ is positive and unbounded, and satisfies $\lim_{n\to\infty}\varrho_n/n = 0$. This implies that $\{\varrho_n\}_{n\geqslant 0}$ has a least concave majorant $\{\sigma_n\}_{n\geqslant 0}$, and that $\lim_{n\to\infty}(\sigma_{n+1}-\sigma_n) = 0$. The sequence $\mu(n) = \exp(\sigma_n)$ $(n \geqslant 0)$ then fulfils the requirements. $\blacksquare$

Remark. If $r(x) = 1$ and x is power-dominated by a sequence $\{\mu(n)\}$ with $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$, then for each $k \in \mathbf{Z}$

$$\limsup_{n\to\infty}\frac{\mu(n)}{\|x^n\|} \geqslant \limsup_{n\to\infty}\frac{\mu(n)}{\mu(n+k)}\,\frac{\|x^{n+k}\|}{\|x^n\|} = \limsup_{n\to\infty}\frac{\|x^{n+k}\|}{\|x^n\|}.$$

Thus, the more "irregular" the sequence $\{\|x^n\|_{n\geqslant 0}\}$, the "further" from it $\{\mu(n)\}_{n\geqslant 0}$ has to be.

Before stating the main theorem, a little more notation is necessary. Let $A(\Gamma)$ be the space of all functions $f\colon \Gamma \to \mathbf{C}$ of the form $f(z) = \sum_{k\in\mathbf{Z}} c_k z^k$ such that $\|f\|_{A(\Gamma)} := \sum_{k\in\mathbf{Z}}|c_k|$ is finite. Under pointwise multiplication $(A(\Gamma), \|\cdot\|_{A(\Gamma)})$ becomes a commutative Banach algebra. Given a closed subset E of Γ, we define

$$I(E) = \{f \in A(\Gamma)\colon f = 0 \text{ on } E\},$$

$$J(E) = \{f \in A(\Gamma)\colon f = 0 \text{ on a neighbourhood of } E \text{ in } \Gamma\}.$$

If $E \neq \emptyset$, then $I(E)$ (respectively $J(E)$) is a proper, closed ideal (respectively proper ideal) in $A(\Gamma)$, and the quotient Banach algebra $(A(\Gamma)/I(E), \|\cdot\|_E)$ (respectively $(A(\Gamma)/\overline{J(E)}, \|\cdot\|_{(E)})$) can be regarded as consisting of functions on (respectively germs of functions near) the set E, and as having character space equal to E (in both cases). If $E = \emptyset$, then $I(E)$ and $J(E)$ are both the whole of $A(\Gamma)$ and we adopt the convention that $\|\cdot\|_E$ and $\|\cdot\|_{(E)}$ are identically zero.

Theorem 2.2. *Let $(A, \|\cdot\|)$ be a Banach algebra, let $\{\mu(n)\}_{n\geqslant 0}$ be a sequence such that $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$, and let x be an element of A which is power-dominated by $\{\mu(n)\}$. If $f(z)$ is any function holomorphic on a neighbourhood of $\mathrm{Sp}(x)$, or if $f(z) = \sum_{k\geqslant 0} c_k z^k$ where $\sum_{k\geqslant 0}|c_k|\,\mu(k) < \infty$, then*

$$(2) \qquad \limsup_{n\to\infty}\|f(x)x^n\|/\mu(n) \leqslant \|f\|_{(E)},$$

where $E = \mathrm{Sp}(x) \cap \Gamma$.

Remark. (a) The notation in (2) requires a little explanation. If f is holomorphic on a neighbourhood of $\mathrm{Sp}(x)$, then $f(x)$ is defined via the usual functional calculus. If $f(z) = \sum_{k\geqslant 0} c_k z^k$ where $\sum_{k\geqslant 0}|c_k|\,\mu(k) < \infty$, then $f(x)$ is defined simply to be the sum of the norm-convergent series $\sum_{k\geqslant 0} c_k x^k$. In either case, when E is non-empty, the function $f(z)$ is defined on a neighbourhood of E in Γ (note that $E \neq \emptyset$ forces $\mu(n) \geqslant 1$ for all n), and the germ of f near E belongs to the quotient $A(\Gamma)/\overline{J(E)}$; the expression $\|f\|_{(E)}$ is then to be interpreted as the norm of this germ in the quotient. If E is empty, then according to our convention $\|f\|_{(E)} = 0$, whatever f may be.

(b) The appearance of the norm $\|\cdot\|_{(E)}$ in (2) may seem a little strange, but in fact it is best possible in the following sense: given any non-empty closed subset E of Γ, there exists a Banach algebra A, containing an element x with $\|x^n\| = 1$ $(n \geqslant 0)$ and $\mathrm{Sp}(x) = E$, such that

$$\|f(x)x^n\| = \|f\|_{(E)}$$

for all $n \geqslant 0$, all f holomorphic near $\mathrm{Sp}(x)$, and all f of the form $\sum_{k\geqslant 0} c_k z^k$ with $\sum_{k\geqslant 0}|c_k| < \infty$. To see this, just take A to be $(A(\Gamma)/\overline{J(E)}, \|\cdot\|_{(E)})$, and x to be (the germ near E of) the function $u(z) = z$.

(c) Given A, $\{\mu(n)\}$, x, f and E as in the theorem, with $E \neq \emptyset$, there is also a simple inequality in the other direction (corresponding to the "only if" part of Theorem 1.2), namely

$$\inf_{n\geqslant 0}\|f(x)x^n\| \geqslant |f|_E,$$

where $|\cdot|_E$ denotes the sup-norm relative to E. For if $\zeta \in E$, then there is a character χ on the (commutative) closed subalgebra of A generated by x such that $\chi(x) = \zeta$, and so since characters have norm one,

$$\inf_{n\geqslant 0} \|f(x)x^n\| \geqslant \inf_{n\geqslant 0} |\chi(f(x))\chi(x)^n| = \inf_{n\geqslant 0} |f(\zeta)\zeta^n| = |f(\zeta)|.$$

The proof of Theorem 2.2 has two main ingredients: one is the renorming Lemma 1.3, and the other is the following result, which is a variation upon a well-known theme, sometimes called the theorem of support.

Lemma 2.3. *Let $(B, \|\cdot\|_B)$ be a commutative Banach algebra, let ξ be an invertible element of B such that $\|\xi\|_B = \|\xi^{-1}\|_B = 1$, and let E be a closed subset of Γ which contains $\mathrm{Sp}(\xi)$. Then there exists a norm-decreasing homomorphism ϑ_E: $A(\Gamma)/\overline{J(E)} \to B$ such that $\vartheta_E(u) = \xi$, where u denotes the function $u(z) = z$.*

Proof. Since $\|\xi\|_B = \|\xi^{-1}\|_B = 1$, the map ϑ: $A(\Gamma) \to B$ given by $\vartheta\left(\sum_{k\in \mathbf{Z}} c_k z^k\right) = \sum_{k\in\mathbf{Z}} c_k \xi^k$ is a well-defined, norm-decreasing homomorphism such that $\vartheta(u) = \xi$. To finish the proof it is enough to show that $\vartheta(g) = 0$ for all $g \in J(E)$, for then by continuity ϑ vanishes on $\overline{J(E)}$, and so induces a norm-decreasing homomorphism ϑ_E: $A(\Gamma)/\overline{J(E)} \to B$ which fulfils the requirements.

Let $g \in J(E)$. Then the (closed) support of g is disjoint from E, so as $A(\Gamma)$ is a normal algebra, we can find $h \in A(\Gamma)$ such that $h = 1$ on E and $h = 0$ on the support of g. In particular, we have

(3) $$\vartheta(g)\vartheta(h) = \vartheta(gh) = \vartheta(0) = 0.$$

Now let χ be any character on B. As $\chi \circ \vartheta$ is a character on $A(\Gamma)$, it corresponds to evaluation at some point $\zeta \in \Gamma$. Moreover, we have $\zeta = \chi(\vartheta(u)) = \chi(\xi) \in \mathrm{Sp}(\xi)$, so that $\zeta \in E$. Therefore $\chi(\vartheta(h)) = h(\zeta) = 1$, and since χ is an arbitrary character on B, it follows that $\vartheta(h)$ is invertible in B. Thus (3) implies that $\vartheta(g) = 0$, as was desired. ∎

Proof of Theorem 2.2. We may assume, without loss of generality, that A is commutative. Also, if $\lim_{n\to\infty} \|x^n\|/\mu(n) = 0$ then (2) is obvious, so we can suppose that $\limsup_{n\to\infty} \|x^n\|/\mu(n) > 0$. Then by Lemma 1.3, there exist a commutative Banach algebra $(B, \|\cdot\|_B)$ and a continuous homomorphism π: $A \to B$ satisfying the conclusions (i) and (ii) of that lemma. Now $\mathrm{Sp}_B(\pi(x)) \subset E$ (in particular E must be non-empty), so by Lemma 2.3 applied with $\xi = \pi(x)$, there exists a norm-decreasing homomorphism ϑ_E: $A(\Gamma)/\overline{J(E)} \to B$ such that $\vartheta_E(u) = \pi(x)$. For any function f as in the statement of the theorem, the operation $f(\cdot)$ commutes with continuous homomorphisms, and so

$$\pi(f(x)) = f(\pi(x)) = f(\vartheta_E(u)) = \vartheta_E(f(u)) = \vartheta_E(f).$$

Combining this with Lemma 1.3 (ii) and the fact that ϑ_E is norm-decreasing we obtain

$$\limsup_{n\to\infty} \|f(x)x^n\|/\mu(n) \leqslant \|\pi(f(x))\|_B = \|\vartheta_E(f)\|_B \leqslant \|f\|_{(E)},$$

which proves the theorem. ∎

The following corollary is a generalization of [12, Theorem 5], of which Theorem 1.2 is a very special case.

Corollary 2.4. *Assume that A, $\{\mu(n)\}$ and x satisfy the hypotheses of Theorem 2.2, and let $f(z) = \sum_{k\geqslant 0} c_k z^k$ where $\sum_{k\geqslant 0} |c_k|\mu(k) < \infty$. Suppose either that $\mathrm{Sp}(x) \cap \Gamma = \emptyset$, or that f is of spectral synthesis in $A(\Gamma)$ with respect to $\mathrm{Sp}(x) \cap \Gamma$. Then $\|f(x)x^n\| = o(\mu(n))$ as $n \to \infty$.*

Proof. Let $E = \mathrm{Sp}(x) \cap \Gamma$. The hypothesis about spectral synthesis means precisely that $\|f\|_{(E)} = 0$. Thus the result follows immediately from Theorem 2.2. ∎

There is also a multivariable version of Theorem 2.2. In this result, the role of the functional calculus is now played by the several-variable calculus (see e.g. [5, Chapter I, § 4, Theorem 1]), and the norm $\|\cdot\|_{(E)}$ now refers to the quotient norm on $A(\Gamma^m)/\overline{J(E)}$, where E is a closed subset of Γ^m, and $A(\Gamma^m)$ and $J(E)$ are the obvious m-variable analogues of their one-variable counterparts defined earlier.

Theorem 2.5. *Let $(A, \|\cdot\|)$ be a commutative Banach algebra, let $\{\mu(n)\}_{n\geqslant 0}$ be a sequence such that $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$, and let $x_1, \ldots, x_m$ be elements of A which satisfy*

$$\|x_1^{k_1}\ldots x_m^{k_m}\| \leqslant \mu(k_1 + \ldots + k_m) \quad (k_1, \ldots, k_m \geqslant 0).$$

If $f(z_1, \ldots, z_m)$ is holomorphic on a neighbourhood of the joint spectrum $\mathrm{Sp}(x_1, \ldots, x_m)$, or if

$$f(z_1, \ldots, z_m) = \sum_{k_1,\ldots,k_m\geqslant 0} c_{k_1,\ldots,k_m} z_1^{k_1}\ldots z_m^{k_m}$$

where

$$\sum_{k_1,\ldots,k_m\geqslant 0} |c_{k_1,\ldots,k_m}|\mu(k_1 + \ldots + k_m) < \infty,$$

then

$$\lim_{n\to\infty}\Big(\sup_{k_1,\ldots,k_m\geqslant n} \|f(x_1, \ldots, x_m)x_1^{k_1}\ldots x_m^{k_m}\|/\mu(k_1 + \ldots + k_m)\Big) \leqslant \|f\|_{(E)},$$

where $E = \mathrm{Sp}(x_1, \ldots, x_m) \cap \Gamma^m$.

Proof. Define p: $A \to \mathbf{R}$ by

$$p(a) = \lim_{n\to\infty}\Big(\sup_{k_1,\ldots,k_m\geqslant n} \|ax_1^{k_1}\ldots x_m^{k_m}\|/\mu(k_1 + \ldots + k_m)\Big) \quad (a \in A).$$

If $p(1) = 0$ then the result is clear, so we may suppose that $p(1) > 0$. The hypothesis $\lim_{n\to\infty} \mu(n+1)/\mu(n) = 1$ ensures that $p(ax_j) = p(a)$ $(a \in A,\ j = 1, \ldots, m)$, so as in Lemma 1.3 we can construct a commutative Banach algebra $(B, \|\cdot\|_B)$ and a continuous homomorphism π: $A \to B$, such that $\pi(x_j)$ is invertible in B with $\|\pi(x_j)\|_B = \|\pi(x_j)^{-1}\|_B = 1$ $(j = 1, \ldots, m)$, and such that $p(a) \leqslant \|\pi(a)\|_B \leqslant \|a\|$ $(a \in A)$. By the obvious analogue of Lemma 2.3, there exists a norm-decreasing homomorphism ϑ_E: $A(\Gamma^m)/\overline{J(E)} \to B$ such that $\vartheta_E(u_j) = \pi(x_j)$ $(j = 1, \ldots, m)$, where u_j denotes the function $u_j(z_1, \ldots, z_m) = z_j$ $(j = 1, \ldots, m)$. The rest of the proof goes through as for Theorem 2.2. ∎

As before, we single out the special case corresponding to $\|f\|_{(E)} = 0$. This time it is a generalization of [12, Theorem 8], even in the case when $x_1, \ldots, x_m$ are all power-bounded.

COROLLARY 2.6. *Assume that* A, $\{\mu(n)\}$ *and* $x_1, \ldots, x_m$ *satisfy the hypotheses of Theorem* 2.5, *and let*

$$f(z_1, \ldots, z_m) = \sum_{k_1,\ldots,k_m \geqslant 0} c_{k_1,\ldots,k_m} z_1^{k_1} \ldots z_m^{k_m}$$

where

$$\sum_{k_1,\ldots,k_m \geqslant 0} |c_{k_1,\ldots,k_m}|\, \mu(k_1 + \ldots + k_m) < \infty.$$

Suppose that either $\mathrm{Sp}(x_1, \ldots, x_m) \cap \Gamma^m = \emptyset$, *or that f is of spectral synthesis in* $A(\Gamma^m)$ *with respect to* $\mathrm{Sp}(x_1, \ldots, x_m) \cap \Gamma^m$. *Then*

$$\|f(x_1, \ldots, x_m) x_1^{k_1} \ldots x_m^{k_m}\| = o\big(\mu(k_1 + \ldots + k_m)\big) \quad \text{as } \min(k_1, \ldots, k_m) \to \infty. \ ∎$$

§ 3. **Some examples.** To use Theorem 2.2 as a quantitative result, we need to be able to estimate $\|f\|_{(E)}$. This is the problem addressed in this section.

We shall consider sets of the form

$$E_\tau = \{e^{i\vartheta} \in \Gamma\colon |\vartheta| \leqslant \tau\},$$

where $0 \leqslant \tau < \pi$. Any proper closed subset E of Γ can be rotated so as to lie inside some such E_τ, so upper bounds for $\|\cdot\|_{(E_\tau)}$ are also good enough to cover $\|\cdot\|_{(E)}$, albeit at the cost of some loss of sharpness. By standard harmonic analysis (see e.g. [11, Chapter VIII, § 7.3]), it can be shown that for these sets E_τ we have $\overline{J(E_\tau)} = I(E_\tau)$; thus it suffices to consider $\|\cdot\|_{E_\tau}$ (the quotient norm on $A(\Gamma)/I(E_\tau)$). The following theorem estimates $\|f\|_{E_\tau}$ for a class of functions f wide enough to include a number of examples of interest.

THEOREM 3.1. *Let* $0 \leqslant \tau < \pi$, *and let* f: $E_\tau \to \boldsymbol{C}$ *be a function of the form* $f(e^{it}) = e^{i\gamma t} g(t)$ $(t \in [-\tau, \tau])$, *where* $\gamma \in \boldsymbol{R}$ *and* g: $[-\tau, \tau] \to \boldsymbol{R}$ *is an odd, increasing C^2-function which (if $\tau > 0$) satisfies*

$$g''(t) \geqslant -\left(\frac{\pi g'(0)}{2\tau}\right)\sin\left(\frac{\pi t}{2\tau}\right) \quad (t \in [0, \tau]). \tag{4}$$

(i) *The norm* $\|f\|_{E_\tau}$ *obeys the general inequalities*

$$g(\tau) \leqslant \|f\|_{E_\tau} \leqslant g(\tau)\sec(\tau/2).$$

(ii) *If moreover* $\tau = \pi/p$ *and* $\gamma = q/2$, *where* $p, q \in \boldsymbol{Z}$ *with* $p \geqslant 2$, *then*

$$\|f\|_{E_\tau} = \begin{cases} g(\tau), & \text{if } p+q \text{ is even}, \\ g(\tau)\sec(\tau/2), & \text{if } p+q \text{ is odd}. \end{cases}$$

Remarks. (a) The right-hand side of (4) is negative for all $t \in [0, \tau]$. Thus (4) holds automatically whenever g is convex on $[0, \tau]$.

(b) The inspiration for Theorem 3.1 was the paper of Crabb and Duncan [6]. In particular, the proof of part (ii) is modelled on similar ones in [6]. The following lemma, crucial to Theorem 3.1, is also implicit in [6]. The proof of it given below is a little more direct, but contains no essentially new ideas, and is included mainly to keep this exposition as self-contained as possible.

LEMMA 3.2. *If* $0 \leqslant \tau < \pi$ *and* $\alpha \in \boldsymbol{R}$, *then* $\|e^{i\alpha t}\|_{E_\tau} \leqslant \sec(\tau/2)$.

Proof. If $\tau = 0$, then $A(\Gamma)/I(E_\tau)$ is one-dimensional and the result is obvious, so we may suppose that $0 < \tau < \pi$. Since $\|e^{i\alpha t}\|_{E_\tau} = \|e^{-i\alpha t}\|_{E_\tau}$ for all $\alpha \in \boldsymbol{R}$, and since $\|e^{int}\|_{E_\tau} = 1$ for all $n \in \boldsymbol{Z}$, a simple compactness argument shows that there exists $\beta \in [0, \frac{1}{2}]$ such that

$$\|e^{i\beta t}\|_{E_\tau} = \sup_{\alpha \in \boldsymbol{R}} \|e^{i\alpha t}\|_{E_\tau}.$$

With this value of β, define F: $\Gamma \to \boldsymbol{C}$ by

$$F(e^{is}) = e^{-i\beta\tau s/\pi}\sin(1-\beta)\tau + e^{i(1-\beta)\tau s/\pi}\sin\beta\tau \quad (s \in [-\pi, \pi]).$$

Then $F \in A(\Gamma)$ and

$$F(e^{is}) = \sum_{k \in \boldsymbol{Z}} a_k e^{iks} \quad (s \in [-\pi, \pi]),$$

where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{is}) e^{-iks}\, ds = \frac{(-1)^k \sin\beta\tau \cdot \sin(1-\beta)\tau}{\tau\{(\beta + k\pi/\tau) - (\beta + k\pi/\tau)^2\}} \quad (k \in \boldsymbol{Z}).$$

In particular, $a_0 \geqslant 0$ and $(-1)^k a_k \leqslant 0$ $(k \in \boldsymbol{Z} \setminus \{0\})$, so setting $s = \pi$ in the two expressions for $F(e^{is})$ above, we obtain

$$|a_0| - \sum_{k \neq 0} |a_k| = F(e^{i\pi}) = \sin\tau. \tag{5}$$

Also, substituting s by $\pi t/\tau$ in these two expressions for $F(e^{is})$ and multiplying by $e^{i\beta t}$ gives

$$\sin(1-\beta)\tau + e^{it}\sin\beta\tau = \sum_{k\in\mathbf{Z}} a_k e^{i(\beta+k\pi/\tau)t} \qquad (t\in[-\tau,\tau]).$$

Thus

$$\begin{aligned}\sin(1-\beta)\tau+\sin\beta\tau &\geq \|\sin(1-\beta)\tau+e^{it}\sin\beta\tau\|_{E_\tau}\\ &= \Big\|\sum_{k\in\mathbf{Z}} a_k e^{i(\beta+k\pi/\tau)t}\Big\|_{E_\tau}\\ &\geq |a_0|\,\|e^{i\beta t}\|_{E_\tau} - \sum_{k\neq 0}|a_k|\sup_{\alpha\in\mathbf{R}}\|e^{i\alpha t}\|_{E_\tau}\\ &= (\sin\tau)\sup_{\alpha\in\mathbf{R}}\|e^{i\alpha t}\|_{E_\tau},\end{aligned}$$

the last equality arising from (5) and the choice of β. Hence

$$\sup_{\alpha\in\mathbf{R}}\|e^{i\alpha t}\|_{E_\tau} \leq \sec(\tau/2)\cos(\tfrac{1}{2}-\beta)\tau \leq \sec(\tau/2). \ \blacksquare$$

Proof of Theorem 3.1. (i) Since "evaluation at $e^{i\tau}$" is a character on $A(\Gamma)/I(E_\tau)$, it has norm one, and therefore

$$\|f\|_{E_\tau} \geq |f(e^{i\tau})| = g(\tau),$$

which gives the desired lower bound.

For the upper bound, we may suppose as in Lemma 3.1 that $0<\tau<\pi$. The key observation is that if $h\colon[-\pi,\pi]\to\mathbf{R}$ is the function given by $h(s)=g(\tau s/\pi)$ $(s\in[-\pi,\pi])$, then

(6) $$(-1)^k\int_0^\pi h(s)\sin[(k+\tfrac{1}{2})s]\,ds \geq 0 \qquad (k\in\mathbf{Z}).$$

To see this, note that from (4) we have

$$h''(s)+\tfrac{1}{2}h'(0)\sin(s/2)\geq 0 \qquad (s\in[0,\pi]),$$

and therefore for each $k\in\mathbf{Z}$

$$\int_0^\pi\big(h''(s)+\tfrac{1}{2}h'(0)\sin(s/2)\big)\big(1-(-1)^k\sin(k+\tfrac{1}{2})s\big)\,ds\geq 0.$$

Integrating twice by parts yields the inequality

$$(-1)^k\int_0^\pi h(s)\sin[(k+\tfrac{1}{2})s]\,ds \geq \tfrac{1}{4}h'(0)(k+\tfrac{1}{2})^2(-1)^k\int_0^\pi(\cos ks-\cos(k+1)s)\,ds,$$

whose right-hand side is non-negative for all $k\in\mathbf{Z}$ (and indeed equals zero unless $k=0$ or -1). Thus (6) is proved.

Now define $H\colon\Gamma\to\mathbf{C}$ by

$$H(e^{is}) = e^{-is/2}h(s) \qquad (s\in[-\pi,\pi]).$$

Then $H\in A(\Gamma)$, and

$$H(e^{is}) = \sum_{k\in\mathbf{Z}} b_k e^{iks} \qquad (s\in[-\pi,\pi])$$

where

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-is/2}h(s)e^{-iks}\,ds = -\frac{i}{\pi}\int_0^\pi h(s)\sin[(k+\tfrac{1}{2})s]\,ds,$$

since g, and therefore h, are odd. From (6) it therefore follows that $|b_k| = i(-1)^k b_k$ $(k\in\mathbf{Z})$, and hence, setting $s=\pi$ in the two expressions for $H(e^{is})$ above, we obtain

(7) $$\sum_{k\in\mathbf{Z}}|b_k| = iH(e^{i\pi}) = h(\pi) = g(\tau).$$

Also, substituting s by $\pi t/\tau$ in these two expressions for $H(e^{is})$ and multiplying by $e^{i(\gamma+\pi/2\tau)t}$ gives

$$e^{i\gamma t}g(t) = \sum_{k\in\mathbf{Z}} b_k e^{i(\gamma+(k+1/2)\pi/\tau)t} \qquad (t\in[-\tau,\tau]).$$

Thus

(8) $$\|f\|_{E_\tau} \leq \sum_{k\in\mathbf{Z}}|b_k|\,\|e^{i(\gamma+(k+1/2)\pi/\tau)t}\|_{E_\tau} \leq g(\tau)\sup_{k\in\mathbf{Z}}\|e^{i(\gamma+(k+1/2)\pi/\tau)t}\|_{E_\tau},$$

the last inequality arising from (7). The result now follows from Lemma 3.1.

(ii) First suppose that $p+q$ is even, say $p+q=2l$. Then from (8) we have

$$\|f\|_{E_\tau} \leq g(\tau)\sup_{k\in\mathbf{Z}}\|e^{i(l+kp)t}\|_{E_\tau} \leq g(\tau)\sup_{n\in\mathbf{Z}}\|e^{int}\|_{E_\tau} = g(\tau).$$

By (i), $\|f\|_{E_\tau}\geq g(\tau)$, so in fact equality holds.

Now suppose that $p+q$ is odd, say $q-p=2m+1$. Let $\varphi\colon A(\Gamma)/I(E_\tau)\to\mathbf{C}$ be the linear functional given by

$$\varphi(v) = e^{-i\tau/2}v(e^{i\tau}) + e^{i\tau/2}v(e^{-i\tau}) \qquad (v\in A(\Gamma)/I(E_\tau)).$$

Since $\tau=\pi/p$, it follows that whenever $\sum_{k\in\mathbf{Z}}|c_k|<\infty$ we have

$$\Big|\varphi\Big(\sum_{k\in\mathbf{Z}}c_k z^k\Big)\Big| = \Big|\sum_{k\in\mathbf{Z}}2c_k\cos((k-\tfrac{1}{2})\pi/p)\Big| \leq 2\cos(\pi/2p)\Big(\sum_{k\in\mathbf{Z}}|c_k|\Big),$$

and therefore $\|\varphi\|\leq 2\cos(\tau/2)$. Also

$$\varphi\big(e^{i(1/2+\pi/2\tau)t}g(t)\big) = e^{i\pi/2}g(\tau)+e^{-i\pi/2}g(-\tau) = 2ig(\tau),$$

so as $f(e^{it}) = e^{iqt/2}g(t) = e^{imt}e^{i(1/2+\pi/2\tau)t}g(t)$, we deduce that

$$\|f\|_{E_\tau} = \|e^{i(1/2+\pi/2\tau)t}g(t)\|_{E_\tau} \geq \big|\varphi\big(e^{i(1/2+\pi/2\tau)t}g(t)\big)\big|/\|\varphi\| \geq g(\tau)\sec(\tau/2).$$

By (i), $\|f\|_{E_\tau}\leq g(\tau)\sec(\tau/2)$, so in fact equality holds. $\blacksquare$

We now give some examples of the use of Theorems 2.2 and 3.1. The first of these, a quantitative generalization of Theorem 1.2, was obtained for doubly power-bounded elements by Crabb and Duncan [6]. A related result for power-bounded ones has been given by Bachelis and Saeki [3].

COROLLARY 3.3. *Let A be a Banach algebra, let x be an element of A which is power-dominated by a sequence $\{\mu(n)\}_{n\geqslant 0}$ with $\lim_{n\to\infty}\mu(n+1)/\mu(n) = 1$, and suppose that $\mathrm{Sp}(x)\cap\Gamma\subset E_\tau$, where $0\leqslant\tau<\pi$. Then*

$$\limsup_{n\to\infty}\|x^{n+1}-x^n\|/\mu(n)\leqslant 2\tan(\tau/2),$$

and the right-hand side is sharp when π/τ is an even integer. If π/τ is an odd integer, then the estimate may be improved to

$$\limsup_{n\to\infty}\|x^{n+1}-x^n\|/\mu(n)\leqslant 2\sin(\tau/2),$$

and the right-hand side is sharp.

Proof. Define $f\colon C\to C$ by $f(z)=z-1$. The restriction $f\,|\,E_\tau$ is of the form $f(e^{it})=ie^{i\gamma t}g(t)$, where $\gamma=\frac{1}{2}$ and $g(t)=2\sin(t/2)$ $(t\in[-\tau,\tau])$. Hence by Theorem 3.1 (i)

$$\|f\|_{E_\tau}\leqslant 2\sin(\tau/2)\sec(\tau/2)=2\tan(\tau/2),$$

and if moreover $\tau=\pi/p$ where p is odd, then by Theorem 3.1 (ii)

$$\|f\|_{E_\tau}\leqslant 2\sin(\tau/2).$$

Combining these inequalities with the estimate from Theorem 2.2, namely

$$\limsup_{n\to\infty}\|f(x)x^n\|/\mu(n)\leqslant\|f\|_{(E_\tau)}=\|f\|_{E_\tau},$$

we obtain the desired bounds for $\limsup_{n\to\infty}\|x^{n+1}-x^n\|/\mu(n)$.

To justify the sharpness assertions, take the Banach algebra A to be $(A(\Gamma)/I(E_\tau),\|\cdot\|_{E_\tau})$, the element x to be the function $u(z)=z$ $(z\in E_\tau)$, and the sequence $\{\mu(n)\}$ to be identically 1. Then the hypotheses of the corollary are satisfied, and in this case we have

$$\limsup_{n\to\infty}\|f(x)x^n\|/\mu(n)=\|f\|_{E_\tau}=\begin{cases}2\tan\tau/2, & \text{if } \pi/\tau \text{ is an even integer,}\\ 2\sin\tau/2, & \text{if } \pi/\tau \text{ is an odd integer,}\end{cases}$$

where the second equality comes from Theorem 3.1 (ii). This shows that the stated bounds for $\limsup_{n\to\infty}\|x^{n+1}-x^n\|/\mu(n)$ cannot be improved when π/τ is an integer. ■

COROLLARY 3.4. *Under the hypotheses of Corollary 3.3,*

$$\limsup_{n\to\infty}\|(x+1)^{-1}(x-1)x^n\|/\mu(n)\leqslant\tan(\tau/2)\sec(\tau/2)$$

and the right-hand side is sharp when π/τ is an odd integer. If π/τ is an even integer, then this estimate may be improved to

$$\limsup_{n\to\infty}\|(x+1)^{-1}(x-1)x^n\|/\mu(n)\leqslant\tan(\tau/2),$$

and the right-hand side is sharp.

Proof. Define $f\colon C\setminus\{-1\}\to C$ by $f(z)=(z+1)^{-1}(z-1)$. The restriction $f\,|\,E_\tau$ is of the form $f(e^{it})=ie^{i\gamma t}g(t)$, where $\gamma=0$ and $g(t)=\tan(t/2)$ $(t\in[-\tau,\tau])$, so by Theorem 3.1

$$\|f\|_{E_\tau}\leqslant\tan(\tau/2)\sec(\tau/2),\quad\text{and}$$

$$\|f\|_{E_\tau}=\begin{cases}\tan(\tau/2)\sec(\tau/2), & \text{if } \pi/\tau \text{ is an odd integer,}\\ \tan(\tau/2), & \text{if } \pi/\tau \text{ is an even integer.}\end{cases}$$

The rest of the proof goes through as for Corollary 3.3. ■

COROLLARY 3.5. *In addition to the hypotheses of Corollary 3.3, assume that 0 lies in the unbounded component of $C\setminus(\mathrm{Sp}(x)\cup E_\tau)$. Then*

$$\limsup_{n\to\infty}\|x^n\log x\|\leqslant\tau\sec(\tau/2),$$

and the right-hand side is sharp when π/τ is an odd integer. If π/τ is an even integer, then the estimate may be improved to

$$\limsup_{n\to\infty}\|x^n\log x\|\leqslant\tau,$$

and the right-hand side is sharp.

Proof. As 0 lies in the unbounded component of $C\setminus(\mathrm{Sp}(x)\cup E_\tau)$, there exists a holomorphic branch $f(z)$ of $\log z$ on a neighbourhood of $\mathrm{Sp}(x)\cup E_\tau$ such that $f(e^{it})=it$ $(t\in[-\tau,\tau])$. By Theorem 3.1 applied with $\gamma=0$ and $g(t)=t$ $(t\in[-\tau,\tau])$, we have

$$\|f\|_{E_\tau}\leqslant\tau\sec(\tau/2),\quad\text{and}$$

$$\|f\|_{E_\tau}=\begin{cases}\tau\sec(\tau/2), & \text{if } \pi/\tau \text{ is an odd integer,}\\ \tau, & \text{if } \pi/\tau \text{ is an even integer.}\end{cases}$$

The rest of the proof goes through as for Corollary 3.3. ■

§ 4. **A converse result.** There is another extension of Gelfand's Theorem 1.1, which is not implied by any of the results mentioned so far. This generalization, due to Hille ([9], see also [10, Theorem 4.10.1]), replaces the hypothesis of double power-boundedness by an $o(n)$ condition without weakening the conclusion.

THEOREM 4.1 ([9]). *Let A be a Banach algebra, and let x be an invertible element of A such that $\|x^n\|+\|x^{-n}\| = o(n)$ as $n \to \infty$. If $\mathrm{Sp}(x) = \{1\}$ then $x = 1$.* ∎

(Note that $o(n)$ cannot be replaced by $O(n)$; consider, for example, $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the algebra A of 2×2 matrices). It is natural to ask whether there are corresponding extensions of any of the other results in this paper, starting with Katznelson and Tzafriri's Theorem 1.2. Even for this theorem, we shall see the answer essentially is no.

First, a little more notation is needed. A sequence $\{\omega(n)\}_{n\geq 0}$ of non-negative numbers is called a *weight sequence* if

$$\omega(0) = 1, \quad \omega(m+n) \leqslant \omega(m)\,\omega(n) \quad (m, n \geqslant 0).$$

Also if $\zeta \in C$ and $r > 0$, then we write $\Delta(\zeta, r) = \{z \in C: |z-\zeta| < r\}$ and $\bar{\Delta}(\zeta, r) = \{z \in C: |z-\zeta| \leqslant r\}$ (thus $\bar{\Delta}(0, 1)$ is just $\bar{\Delta}$). We can now state the main result of this section, which may be regarded as a sort of converse of Theorem 1.2.

THEOREM 4.2. *Let $\{\omega(n)\}_{n\geq 0}$ be a weight sequence which satisfies $\omega(n) \leqslant \omega(n+1)$ $(n \geqslant 0)$ and $\liminf_{n\to\infty} \omega(2n)/\omega(n) < \infty$. Then there exists a commutative Banach algebra $(A, \|\cdot\|)$ such that, given any sequence $\{\varepsilon(n)\}_{n\geq 0}$ with $\lim_{n\to\infty} \varepsilon(n) = 0$, we can find an element x of A with the following properties:*

(i) $\mathrm{Sp}(x) \subset \Delta(0, 1) \cup \{1\}$.

(ii) $\|x^n\| \leqslant \omega(n)$ $(n \geqslant 0)$.

(iii) $\|x^{n+1} - x^n\| \geqslant \varepsilon(n)\,\omega(n)$ *for infinitely many n.*

Remarks. (a) If $\alpha > 0$, then by taking $\omega(n) = (1+n)^\alpha$ and $\varepsilon(n) = 1/\log(n+10)$ Theorem 4.2 shows that it is possible to find a Banach algebra A containing x such that $\mathrm{Sp}(x) \subset \Delta(0, 1) \cup \{1\}$, with $\|x^n\| = O(n^\alpha)$ as $n \to \infty$, but $\|x^{n+1} - x^n\| \neq O(n^\beta)$ for any $\beta < \alpha$. This and other similar examples would appear to rule out the possibility of extending the Katznelson–Tzafriri Theorem 1.2 (and *a fortiori*, any of its subsequent generalizations) along the lines of Theorem 4.1.

(b) By Theorem 1.2, if an element x of a Banach algebra satisfies $\mathrm{Sp}(x) \subset \Delta(0, 1) \cup \{1\}$ and $\|x^n\| \leqslant 1$ $(n \geqslant 0)$, then the sequence $\|x^{n+1} - x^n\|$ necessarily converges to zero as $n \to \infty$. However, taking $\omega(n) = 1$ $(n \geqslant 0)$ in Theorem 4.2 shows that it may converge arbitrarily slowly.

The proof of Theorem 4.2 proceeds via two elementary preparatory lemmas. If $(W, \|\cdot\|)$ is a Banach algebra, then we write $l^\infty(W)$ for the Banach algebra of bounded sequences $\{w_j\}_{j\geq 0}$ of elements of W, with coordinatewise multiplication, and norm given by $\|\{w_j\}\|_\infty = \sup_{j\geq 0} \|w_j\|$.

LEMMA 4.3. *Let W be a Banach algebra, and let $\{w_j\}_{j\geq 0}$ be a sequence in W which converges to an element $w \in W$. If $x = \{w_j\}_{j\geq 0}$, then*

$$\mathrm{Sp}_{l^\infty(W)}(x) \subset \bigcup_{j\geq 0} \mathrm{Sp}_W(w_j) \cup \mathrm{Sp}_W(w).$$

Proof. Let $\lambda \in C$ and suppose that $\lambda \notin (\bigcup_{j\geq 0} \mathrm{Sp}_W(w_j) \cup \mathrm{Sp}_W(w))$. Then $w_j - \lambda 1$ and $w - \lambda 1$ are invertible in W for all $j \geqslant 0$. Put $y = \{(w_j - \lambda 1)^{-1}\}_{j\geq 0}$. Since $(w_j - \lambda 1)^{-1} \to (w - \lambda 1)^{-1}$ as $j \to \infty$, we certainly have $y \in l^\infty(W)$. Also $y(x - \lambda 1) = (x - \lambda 1)\,y = 1$ in $l^\infty(W)$, so $\lambda \notin \mathrm{Sp}_{l^\infty(W)}(x)$. ∎

Given a weight sequence $\{\omega(n)\}_{n\geq 0}$ such that $\omega(n) \geqslant 1$ $(n \geqslant 0)$, we denote by $A_+(\Gamma, \omega)$ the space of all functions $f\colon \Gamma \to C$ of the form $f(z) = \sum_{k\geq 0} c_k z^k$ such that $\|f\|_\omega := \sum_{k\geq 0} |c_k|\,\omega(k)$ is finite. Under pointwise multiplication $(A_+(\Gamma, \omega), \|\cdot\|_\omega)$ becomes a commutative Banach algebra.

LEMMA 4.4. *Let $\{\omega(n)\}_{n\geq 0}$ be a weight sequence such that $\omega(n) \geqslant 1$ $(n \geqslant 0)$, let $\vartheta \in R$, and define $v \in A_+(\Gamma, \omega)$ by $v(z) = \frac{1}{2}(z + e^{i\vartheta})$ $(z \in \Gamma)$. Then:*

(i) $\mathrm{Sp}(v) \subset \bar{\Delta}(\frac{1}{2}e^{i\vartheta}, \frac{1}{2}R)$, *where* $R = \lim_{n\to\infty} \omega(n)^{1/n}$.

(ii) $\|v^n\|_\omega = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \omega(k)$ $(n \geqslant 0)$.

(iii) $\|v^{n+1} - v^n\|_\omega \geqslant \|v^n\|_\omega |\sin\vartheta|$ $(n \geqslant 0)$.

Proof. (i) If $\lambda \in \mathrm{Sp}(v)$, then $2\lambda - e^{i\vartheta} \in \mathrm{Sp}(u)$, where u denotes the function $u(z) = z$. By the spectral radius formula,

$$r(u) = \lim_{n\to\infty} \|u^n\|^{1/n} = \lim_{n\to\infty} \omega(n)^{1/n} = R,$$

and hence $|\lambda - \frac{1}{2}e^{i\vartheta}| \leqslant \frac{1}{2}R$.

(ii) For each $n \geqslant 0$

$$\|v^n\|_\omega = \Big\| 2^{-n} \sum_{k=0}^{n} \binom{n}{k} e^{i(n-k)\vartheta} z^k \Big\|_\omega = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \omega(k).$$

(iii) For each $n \geqslant 0$

$$\|v^{n+1} - v^n\|_\omega = \Big\| 2^{-n} \sum_{k=0}^{n} \binom{n}{k} e^{i(n-k)\vartheta} z^k \big(\tfrac{1}{2}(z + e^{i\vartheta}) - 1\big) \Big\|_\omega$$

$$\geqslant 2^{-n} \sum_{k=0}^{n} \Big| \frac{1}{2}\Big\{ \binom{n}{k-1} + \binom{n}{k} \Big\} e^{i\vartheta} - \binom{n}{k} \Big| \omega(k)$$

$$\geqslant 2^{-n} \sum_{k=0}^{n} \binom{n}{k} |\sin\vartheta|\, \omega(k) = \|v^n\|_\omega |\sin\vartheta|. \quad ∎$$

Proof of Theorem 4.2. We take A to be $l^\infty(W)$, where $W = A_+(\Gamma, \omega)$. Let $\{\varepsilon(n)\}_{n\geq 0}$ be a sequence tending to zero: we shall construct an element x of A with properties (i), (ii) and (iii) of the theorem.

Let $C = 1+\liminf_{n\to\infty}\omega(2n)/\omega(n)$. Altering finitely many terms of the sequence $\{\varepsilon(n)\}$ does not disturb the validity of the theorem, so as $\lim_{n\to\infty}\varepsilon(n) = 0$, there is no loss of generality in supposing that $\varepsilon(n) < 1/2C$ $(n \geq 0)$. We may therefore choose a sequence $\{\vartheta_n\}_{n\geq 0}$ such that $\lim_{n\to\infty}\vartheta_n = 0$ and $\sin\vartheta_n > 2C\varepsilon(n)$ $(n\geq 0)$, and then another sequence $\{r_n\}_{n\geq 0}$ such that $0 < r_n < 1$ and

$$2\omega(1)(1-r_n^{n+1}) < \sin\vartheta_n - 2C\varepsilon(n) \quad (n\geq 0). \tag{9}$$

These conditions certainly force $\lim_{n\to\infty} r_n = 1$. Now define v_j, $w_j \in A_+(\Gamma, \omega)$ by

$$v_j(z) = \tfrac{1}{2}(z+e^{i\vartheta_j}), \quad w_j(z) = r_j v_j(z) \quad (z\in\Gamma,\ j\geq 0),$$

and let x be the element of A given by $x = \{w_j\}_{j\geq 0}$. This x has the desired properties, as we now verify.

(i) As $j\to\infty$, we have $w_j \to w$, where $w(z) = \frac{1}{2}(z+1)$ $(z\in\Gamma)$. Therefore by Lemma 4.3

$$\mathrm{Sp}_A(x) \subset \bigcup_{j\geq 0}\mathrm{Sp}_W(w_j)\cup \mathrm{Sp}_W(w).$$

Now the hypotheses on $\{\omega(n)\}$ imply that $\lim_{n\to\infty}\omega(n)^{1/n} = 1$, so Lemma 4.4 (i) gives $\mathrm{Sp}_W(w) \subset \bar{\Delta}(\frac{1}{2}, \frac{1}{2})$ and $\mathrm{Sp}_W(w_j) \subset \bar{\Delta}(\frac{1}{2}r_j e^{i\vartheta_j}, \frac{1}{2}r_j) \subset \Delta(0, 1)$ $(j\geq 0)$. Hence $\mathrm{Sp}_A(x) \subset \Delta(0,1)\cup\{1\}$.

(ii) Using Lemma 4.4 (ii) and the assumption that $\{\omega(k)\}_{k\geq 0}$ is increasing, we have

$$\|x^n\|_\infty \leq \sup_{j\geq 0}\|v_j^n\|_\omega = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\omega(k) \leq \omega(n) \quad (n\geq 0).$$

(iii) Using Lemma 4.4 (iii), for each $n\geq 0$ we have

$$\begin{aligned}\|x^{n+1}-x^n\|_\infty &\geq \|w_n^{n+1}-w_n^n\|_\omega \\ &\geq \|v_n^{n+1}-v_n^n\|_\omega - (1-r_n^{n+1})\|v_n^{n+1}\|_\omega - (1-r_n^n)\|v_n^n\|_\omega \\ &\geq \|v_n^n\|_\omega(\sin\vartheta_n - (1-r_n^{n+1})\|v_n\|_\omega - (1-r_n^n)) \\ &\geq \|v_n^n\|_\omega(\sin\vartheta_n - 2\omega(1)(1-r_n^{n+1})) \geq \|v_n^n\|_\omega\cdot 2C\varepsilon(n),\end{aligned}$$

the last inequality coming from (9). Now by Lemma 4.4 (ii), for each $m\geq 0$

$$\|v_{2m}^{2m}\|_\omega \geq 2^{-2m}\sum_{k=m+1}^{2m}\binom{2m}{k}\omega(k) \geq \tfrac{1}{4}\omega(m),$$

where again we have used the hypothesis that $\{\omega(k)\}_{k\geq 0}$ is an increasing sequence. As $C > \liminf_{m\to\infty}\omega(2m)/\omega(m)$, it follows that there is an infinite

subset S of $\{0, 1, 2, \ldots\}$ such that for all $m\in S$

$$\|v_{2m}^{2m}\|_\omega \geq \omega(2m)/2C.$$

Hence, setting $n = 2m$ above, we deduce that for all $m\in S$

$$\|x^{2m+1}-x^{2m}\|_\infty \geq \varepsilon(2m)\,\omega(2m),$$

which completes the proof of (iii), and of the whole theorem. ∎

References

[1] G. R. Allan, A. G. O'Farrell and T. J. Ransford, *A tauberian theorem arising in operator theory*, Bull. London Math. Soc. 19 (1987), 537–545.

[2] R. Arens, *Inverse-producing extensions of normed algebras*, Trans. Amer. Math. Soc. 88 (1958), 536–548.

[3] G. F. Bachelis and S. Saeki, *Power bounded elements in a Banach algebra with small peripheral spectrum*, to appear.

[4] F. F. Bonsall and M. J. Crabb, *The spectral radius of a hermitian element of a Banach algebra*, Bull. London Math. Soc. 2 (1970), 178–180.

[5] N. Bourbaki, *Théories spectrales*, Hermann, Paris 1967.

[6] M. J. Crabb and J. Duncan, *Some inequalities for norm unitaries in Banach algebras*, Proc. Edinburgh Math. Soc. 21 (1978), 17–23.

[7] J. Esterle, *Quasimultipliers, representations of H^∞, and the closed ideal problem for commutative Banach algebras*, in: Radical Banach Algebras and Automatic Continuity, Lecture Notes in Math. 975, Springer, 1983, 66–162.

[8] I. M. Gelfand, *Zur Theorie der Charaktere der abelschen topologischen Gruppen*, Rec. Math. N. S. (Mat. Sb.) 9 (51) (1941), 49–50.

[9] E. Hille, *On the theory of characters of groups and semi-groups in normed vector rings*, Proc. Nat. Acad. Sci. U.S.A. 30 (1944), 58–60.

[10] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, 1957.

[11] Y. Katznelson, *An Introduction to Harmonic Analysis*, 2nd ed., Dover, New York 1976.

[12] Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. 68 (1986), 313–328.

[13] M. M. Neumann, *Decomposable operators and generalized intertwining linear transformations*, to appear.

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS
16 Mill Lane, Cambridge CB2 1SB England

SCHOOL OF MATHEMATICS
UNIVERSITY OF LEEDS
Leeds LS2 9JT, England

Received October 9, 1987 (2369)