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On the differentiation of integrals of functions from Orlicz classes

by

A. M. STOKOLOS (Odessa)

Dedicated to Professor Władysław Orlicz on the occasion of his 85th birthday

Abstract. This paper is a direct continuation of [4]. The study of conditions on a basis which ensure that it differentiate $\Phi(L)(\boldsymbol{R}^N)$ is a central problem of the theory of differentiation of integrals. There are many papers (see [1]) devoted to the study of connections between the differentiation of various function classes on the one hand, and the halo and covering properties and estimates for the maximal operators on the other. However, the problem whether there exist at all bases which differentiate various classes $\Phi(L)(\boldsymbol{R}^N)$ has not been much discussed. In the present paper we prove the existence of bases which differentiate precisely a given class $\Phi(L)(\boldsymbol{R}^N)$, with natural restrictions on Φ. The analogous question for bases of rectangles is also considered.

1. Introduction. A *differentiation basis at a point* $x \in \boldsymbol{R}^N$ is a collection $B(x)$ of bounded open simply connected subsets of $\boldsymbol{R}^N$ containing x such that there is a sequence $\{R_k\} \subset B(x)$ with $\operatorname{diam} R_k \to 0$ as $k \to \infty$. The family $B = \{R\colon R \in B(x),\ x \in \boldsymbol{R}^N\}$ is then called a *differentiation basis in* $\boldsymbol{R}^N$. A differentiation basis is called *translation invariant* (briefly: a *TI-basis*) if it contains all translates of any of its elements.

If a basis B has the property that for each R in B, if $x \in R$ then $R \in B(x)$, then B is called a *BF-basis*. Unless otherwise stated, we shall consider only BF-bases.

We define the *upper* and *lower derivatives* of the integral of a locally integrable function f at a point x with respect to a basis B by

$$\bar{D}_B(f, x) := \sup_{\substack{R_k \in B(x) \\ \operatorname{diam} R_k \to 0}} \limsup_{k \to \infty} |R_k|^{-1} \int_{R_k} f(y)\, dy,$$

$$\underline{D}_B(f, x) := \inf_{\substack{R_k \in B(x) \\ \operatorname{diam} R_k \to 0}} \liminf_{k \to \infty} |R_k|^{-1} \int_{R_k} f(y)\, dy.$$

We say that a basis B *differentiates the integral of* f if $\bar{D}_B(f, x) = \underline{D}_B(f, x) = f(x)$ a.e.

Let Φ be an Orlicz function (Orlicz function = N-function in [2]) and $\Phi(L)(\boldsymbol{R}^N)$ the Orlicz class. If B differentiates the integral of every function from $\Phi(L)(\boldsymbol{R}^N)$, then we say that B *differentiates* $\Phi(L)(\boldsymbol{R}^N)$; if for any class $\Psi(L)(\boldsymbol{R}^N)$ such that $\Psi(t) = o(\Phi(t))$ as $t \to \infty$, there exists $f \in \Psi(L)(\boldsymbol{R}^N)$ with $\bar{D}_B(f, x) = +\infty$ a.e. then we say that B *does not differentiate* $o(\Phi(L))(\boldsymbol{R}^N)$. Finally, B *differentiates precisely* $\Phi(L)(\boldsymbol{R}^N)$ (written $B \in D(\Phi(L))$) if it differentiates $\Phi(L)(\boldsymbol{R}^N)$ and does not differentiate $o(\Phi(L))(\boldsymbol{R}^N)$.

We denote by $\boldsymbol{B}_s(\boldsymbol{R}^N)$, $s = 1, \ldots, N$, the TI-basis consisting of all N-dimensional intervals (i.e. rectangular parallelepipeds with sides parallel to the axes) R, where

$$R = \{(x_1, \ldots, x_N):\ a_i < x_i < a_i + b_i \text{ for } i = 1, \ldots, N,$$
$$\text{with } b_i = b \text{ for } i = 1, \ldots, s\},$$

while $\boldsymbol{B}(\boldsymbol{R}^N)$ denotes the basis of all rectangular parallelepideds (with arbitrary orientation).

Let us introduce sequences $\Theta = \{\theta_k\}_{k=1}^{\infty}$, $\theta_k \uparrow \infty$ as $k \to \infty$, and $\Theta_0 = \{2^k\}_{k=1}^{\infty}$. The basis of rectangles in the plane with one side at angle $1/\theta_k$ with the positive x-direction (for some $k \in N$) will be denoted by $\boldsymbol{B}_{\Theta}$.

The well-known theorems due to Lebesgue, Jessen–Marcinkiewicz–Zygmund, Saks, and Zygmund give

$$\boldsymbol{B}_s(\boldsymbol{R}^N) \in D\big(L(\log^+ L)^{N-s}\big)(\boldsymbol{R}^N) \quad \text{for } s = 1, \ldots, N,$$

while $\boldsymbol{B}(\boldsymbol{R}^N)$ does not differentiate $L^{\infty}(\boldsymbol{R}^N)$ (for references see [1, 4]).

In the last ten years many new interesting results in the theory of differentiation of integrals have been obtained. In particular, A. Nagel, E. M. Stein and S. Wainger [3] proved that $\boldsymbol{B}_{\Theta}$ differentiates $L^p(\boldsymbol{R}^2)$, $p > 1$, if $\liminf_{k\to\infty} \theta_{k+1}/\theta_k > 1$. On the other hand, in [5] it was shown that if $\liminf_{k\to\infty} \theta_{k+1}/\theta_k = 1$ and $\Delta_2 \theta_k \geqslant 0$ (where $\Delta_2 \theta_k$ is the second difference) then $\boldsymbol{B}_{\Theta}$ does not differentiate $L^{\infty}(\boldsymbol{R}^2)$. Also, J.-O. Strömberg [6] noticed that $\boldsymbol{B}_{\Theta_0}$ does not differentiate $o(L\log^2 L)(\boldsymbol{R}^2)$.

This shows that for $\Phi(L)(\boldsymbol{R}^N) = \big(L(\log^+ L)^k\big)(\boldsymbol{R}^N)$, $k = 0, \ldots, N-1$, there are bases which differentiate precisely these classes. On the other hand, multiplying each function from $\Phi(L)(\boldsymbol{R}^N)$ by a positive constant does not change the differentiation properties of the class, and hence $B \in D(\Phi(L))(\boldsymbol{R}^N) \Leftrightarrow B \in D(\Phi(cL))(\boldsymbol{R}^N)$. This is possible only if $\Phi(cL)(\boldsymbol{R}^N)$ is independent of $c > 0$, which in turn is equivalent to Φ satisfying the Δ_2-condition (for the Δ_2-condition, see [2]).

A natural problem arises: For a given Orlicz class $\Phi(L)(\boldsymbol{R}^N)$ with Φ satisfying the Δ_2-condition, construct a basis which differentiates precisely $\Phi(L)(\boldsymbol{R}^N)$.

Partially, this problem was considered in [4], where the following alternative, rather unexpected from our point of view, was established:

If B is a TI-basis of intervals then either B differentiates $L(\boldsymbol{R}^N)$ or B does not differentiate $o(L\log^+ L)(\boldsymbol{R}^N)$.

On the other hand, if we omit the assumption of translation invariance then there do exist bases of intervals which differentiate classes $\Phi(L)(\boldsymbol{R}^N)$ intermediate between $L(\boldsymbol{R}^N)$ and $L\log^+ L(\boldsymbol{R}^N)$.

In the present paper we give a solution of the above problem and consider analogous questions for bases of rectangles.

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2. Differentiation of $\Phi(L)(\boldsymbol{R}^N)$ by bases of general type

THEOREM 1. *Let Φ be an Orlicz function satisfying the Δ_2-condition. Then there exists a basis B such that $B \in D(\Phi(L))(\boldsymbol{R}^N)$.*

The proof of Theorem 1 is based on the following lemma, which gives a general method of constructing a basis with the given differential properties provided certain covering properties of a simple collection of sets are known.

In the sequel we denote by c, c_i, c', etc. various positive constants which may vary from line to line.

LEMMA A (see [4]). *Let Φ and Φ^* be Young conjugate functions with Φ satisfying the Δ_2-condition. Let*

$$\sigma = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \ldots,$$

where $\mathscr{R}_n$, $n = 1, 2, \ldots$, is a finite collection of bounded open subsets of $\boldsymbol{R}^N$. Suppose $\lambda_n \uparrow \infty$ as $n \to \infty$ and $\{E_n\}_{n=1}^{\infty}$ is a family of bounded open subsets of $\boldsymbol{R}^N$ such that there exist constants c_i, $i = 1, \ldots, 6$, satisfying the following conditions for all $n \in N$:

(i) *All members of $\mathscr{R}_n$ have equal measure.*
(ii) *For any subcollection $\mathscr{S} \subset \mathscr{R}_n$,*

$$\int_W \Phi^*\Big(c_1\Big(\sum_{R\in\mathscr{S}} \chi_R(x) - c_2\Big)\Big)\,dx \leqslant c_3 \sum_{R\in\mathscr{S}} |R|$$

where $W = \{x\colon \sum_{R\in\mathscr{S}} \chi_R(x) \geqslant c_4\}$.
(iii) *$|R \cap E_n|/|R| \geqslant c_5/\lambda_n$ for $R \in \mathscr{R}_n$.*
(iv) *$\big|\bigcup_{R\in\mathscr{R}_n} R\big| \geqslant c_6\,\Phi(\lambda_n)\,|E_n|$.*

Then there is a basis $B(\boldsymbol{R}^N)$ whose every element is a dilation of some member of σ and such that $B(\boldsymbol{R}^N)$ differentiates precisely $\Phi(L)(\boldsymbol{R}^N)$.

Proof of Theorem 1. Let $n_0 = \max\{n\colon \Phi^*(n) < n\}$. Define $\mathscr{R}_n$, $n > n_0$, to be any family of n bounded open simply connected sets such that

$$(1) \qquad R \cap R' = [0, 1]^N, \quad |R| = \Phi^*(n)/n \quad \text{for } R, R' \in \mathscr{R}_n.$$

Let $\lambda_n := \Phi^*(n)/n$, $E_n := [0, 1]^N$, $n > n_0$. Let us now check the assumption of Lemma A.

For any $\mathscr{S} \subset \mathscr{R}_n$ with card $\mathscr{S} = s \leqslant n$ we have

$$\int \Phi^*\Big(\sum_{R\in\mathscr{R}_n} \chi_R\Big) \leqslant \Phi^*(s) + \sum_{R\in\mathscr{S}} |R| \leqslant s\Phi^*(n)/n + \sum_{R\in\mathscr{S}} |R| = 2\sum_{R\in\mathscr{S}} |R|.$$

Moreover,

$$|R \cap E_n|/|R| = 1/|R| = 1/\lambda_n \quad \text{for } R \in \mathscr{R}_n.$$

It remains to prove (iv). Denote by φ and φ^* the inverse functions of Φ and Φ^* respectively. Then, as is well known (see e.g. [2]),

$$(2)\qquad v \leqslant \varphi(v)\,\varphi^*(v) \leqslant 2v, \quad v \geqslant 0.$$

Putting $v = \Phi^*(u)$ yields $\Phi^*(u) \leqslant \varphi(\Phi^*(u))u \leqslant 2\Phi^*(u)$, hence

$$(3)\qquad \Phi(\Phi^*(u)/u) \leqslant \Phi^*(u) \leqslant \Phi(2\Phi^*(u)/u)$$

or, equivalently,

$$(4)\qquad \Phi^*(\Phi(u)/u) \leqslant \Phi(u) \leqslant \Phi^*(2\Phi(u)/u).$$

Further, from (1) it follows that for some $c \in (0, 1)$

$$(5)\qquad \Big|R \setminus \bigcup_{R'\in\mathscr{R}_n, R'\neq R} R'\Big| \geqslant c|R| \quad \text{for } R \in \mathscr{R}_n,$$

and from (3) and (5) we obtain

$$\Big|\bigcup_{R\in\mathscr{R}_n} R\Big| \geqslant c' \sum_{R\in\mathscr{R}_n} |R| = c'\Phi^*(n) \geqslant c'\Phi(\Phi^*(n)/n) = c'\Phi(\lambda_n)|E_n|.$$

Hence (iv) is verified, which completes the proof.

Theorem 1 admits various generalizations and more precise versions. Without going into details, we present some of them.

THEOREM 1′. *Let Φ be an Orlicz function which satisfies the Δ_2-condition. Then there exists a basis $B \in D(\Phi(L))(\mathbf{R}^N)$ such that for any Orlicz function Ψ which satisfies the Δ_2-condition and $\Psi(t) = o(\Phi(t))$ as $t \to \infty$, there exists a subbasis $\tilde{B}$ of B with $\tilde{B} \in D(\Psi(L))(\mathbf{R}^N)$.*

THEOREM 1″. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition such that $\Psi(t) = o(\Phi(t))$ as $t \to \infty$. Then there exists a basis $B \in D(\Phi(L))(\mathbf{R}^N)$ such that no subbasis $\tilde{B}$ of B differentiates $\Psi(L)(\mathbf{R}^N)$.*

3. Differentiation of $\Phi(L)(\mathbf{R}^2)$ by bases of rectangles. The bases which have been constructed in Section 2 are generated by sets which, in general, have rather complicated geometry.

The aim of the present section is the construction of bases generated by "elementary" sets, i.e. rectangles in the plane. However, we have not been able to obtain an exact analogue of Theorem 1 in this case, even for $\mathbf{R}^2$. Thus the problem of existence of bases of rectangles (or other convex sets) which differentiate precisely an arbitrary class $\Phi(L)$ (with Φ satisfying the Δ_2-condition) remains open.

THEOREM 2. *Let Φ be an Orlicz function such that for some $\varepsilon > 0$*

$$(6)\qquad \Phi(t)/t^{2-\varepsilon} \downarrow 0 \quad \text{as } t \to +\infty,\ t \geqslant t_0.$$

Then there exists a basis $B \subset \mathbf{B}(\mathbf{R}^2)$ such that $B \in D(\Phi(L))(\mathbf{R}^2)$.

Proof. Let Φ^* be the conjugate function to Φ and let φ^* and φ be their respective inverse functions.

By (6), Φ satisfies the Δ_2-condition and

$$(7)\qquad \varphi(t)\,t^{1/(\varepsilon-2)} = \varphi(t)\,t^{-1/2-\varepsilon'} \uparrow \infty \quad \text{as } t \to \infty.$$

Rewrite (2) as

$$1 \leqslant (\varphi(t)\,t^{-1/2-\varepsilon'})(\varphi^*(t)\,t^{-1/2+\varepsilon'}) \leqslant 2.$$

Then

$$t^{1/(2-\varepsilon)}/\varphi(t) \leqslant \varphi^*(t)\,t^{-1/(2+\varepsilon'')} \leqslant 2t^{1/(2-\varepsilon)}/\varphi(t).$$

Finally, using (7) we obtain

$$(8)\qquad \Phi^*(t)/t^{2+\varepsilon''} \leqslant c\Phi^*(\tau)/\tau^{2+\varepsilon''}$$

where $t < \tau$ and the constant c depends on ε.

We now deduce from (8) several inequalities which will be used in the sequel. First,

$$(9)\qquad (\Phi^*(s)/s)(\Phi(n)/n) \leqslant csn, \quad 1 \leqslant s \leqslant \Phi(n)/n.$$

Indeed, by (8) and (4),

$$\Phi^*(s)/s^2 \leqslant c\Phi^*(\Phi(n)/n)\,n^2/\Phi^2(n) \leqslant cn^2/\Phi(n)$$

and (9) is proved.

Secondly,

$$(10)\qquad \sum_{k=1}^{s} \Phi^*(k)\,k^{-2} = \sum_{k=1}^{s} \Phi^*(k)\,k^{-2-\varepsilon''}\,k^{\varepsilon''} \leqslant c\Phi^*(s)\,s^{-2-\varepsilon''} \sum_{k=1}^{s} k^{\varepsilon''} \leqslant c\Phi^*(s)/s,$$

$$(11)\qquad \sum_{k=1}^{s} \Phi^*(k)\,k^{-3} = \sum_{k=1}^{s} \Phi^*(k)\,k^{-2-\varepsilon''}\,k^{\varepsilon''-1} \leqslant c\Phi^*(s)\,s^{-2-\varepsilon''} \sum_{k=1}^{s} k^{\varepsilon''-1} \leqslant c\Phi^*(s)/s^2.$$

Let us start the construction of a basis. Let $m_n = [\Phi(n)/n]+1$, $m_n \in N$. Let $\mathscr{R}_n$ consist of $R^0 = [0, n]\times[0, 1]$ together with the rectangles obtained from R^0 by rotation through the angles $kn/\Phi(n)$, $k = 1, \ldots, m_n$, about the origin. Then

(12) $$|R| = n \quad \text{for } R \in \boldsymbol{R}_n,$$

and clearly for some $n_0 \in N$

$$\Big|R \setminus \bigcup_{R' \in \mathscr{R}_n, R' \neq R} R'\Big| \geqslant \tfrac{1}{2}|R| \quad \text{for } R \in \mathscr{R}_n,\ n \geqslant n_0,$$

whence

(13) $$\Big|\bigcup_{R \in \mathscr{R}_n} R\Big| \geqslant c \sum_{R \in \mathscr{R}_n} |R| \geqslant c\Phi(n).$$

Define $E = \{x\colon |x| < 1\}$. Obviously (see Fig. 1)

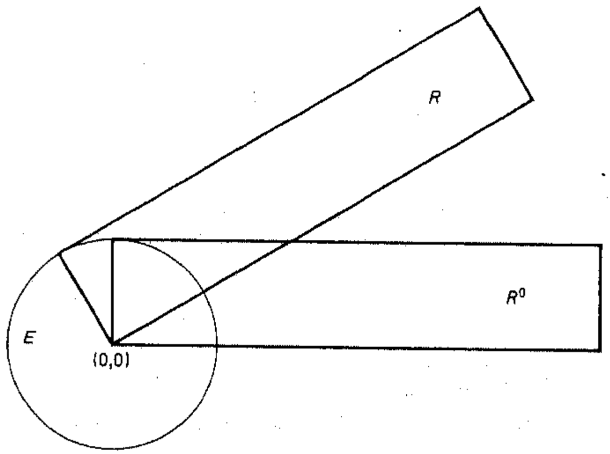

Fig. 1

(14) $$|R \cap E|/|R| = \pi/(4n) \quad \text{for } R \in \mathscr{R}_n.$$

Let $\sigma = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \ldots$, $E_n = E$, $\lambda_n = n$, $n \in N$, $n \geqslant n_0$. Then (12)–(14) imply (i), (iii) and (iv) of Lemma A.

We now show (ii). Let $\mathscr{S} = \{R_1, \ldots, R_s\} \subset \mathscr{R}_n$ with the respective rotation angles $\theta_1 < \ldots < \theta_s$. Without loss of generality we may assume that $s \geqslant 4$. Set

$$F_i := \{x\colon \operatorname{card}\{m\colon x \in R_m\} \geqslant i\}, \quad G_i := \{x\colon \operatorname{card}\{m\colon x \in R_m\} = i\},$$

$$I := \int \Phi^*\Big(\sum_{k=1}^{s} \chi_{R_k}(x)\Big)\,dx.$$

We aim at obtaining the inequality

$$I \leqslant c \sum_{k=1}^{s} |R_k|.$$

We have

$$I = \sum_{i=2}^{s-1} \Delta\Phi^*(i)|F_i| + \Phi^*(1)|F_1| + \Phi^*(s)|G_s| - \Phi^*(s-1)|F_s|,$$

where $\Delta\Phi^*(i) = \Phi^*(i) - \Phi^*(i-1)$, and therefore

$$I \leqslant \Phi^*(1) \sum_{k=1}^{s} |R_k| + \Phi^*(s)|G_s| + \sum_{i=2}^{s-1} \Delta\Phi^*(i)|F_i|.$$

Let us estimate $|G_s|$. By Fig. 2, $G_s = R_1 \cap R_s$, and hence

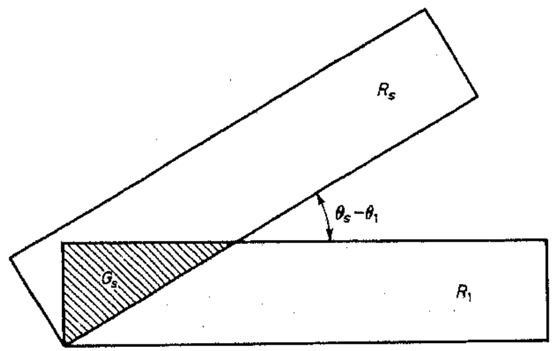

Fig. 2

(15) $$|G_s| = \tfrac{1}{2}\cot(\theta_s - \theta_1) \leqslant 2\Phi(n)/(sn).$$

Let us point out that similar estimates, with details of proof omitted, will appear in several places of this paper.

Further, from (15), (9) and (12) we obtain

$$\Phi^*(s)|G_s| \leqslant 2(\Phi^*(s)/s)(\Phi(n)/n) \leqslant csn = c\sum_{k=1}^{s} |R_k|.$$

It remains to estimate $\sum := \sum_{i=2}^{s-1} \Delta\Phi^*(i)|F_i|$. We have

$$F_i = \bigcup_{k=i}^{s-1} P_k \cup Q,$$

where

$$P_k = \{x \in R_k \setminus \bigcup_{j>k} R_j\colon \text{ card } \{m\colon x \in R_m\} \geqslant i\},$$

$$Q = \{x \in R_s\colon \text{ card } \{m\colon x \in R_m\} \geqslant i\}.$$

Therefore

$$\sum \leqslant \sum_{i=2}^{s-1} \Delta\Phi^*(i) \sum_{k=i}^{s-1} |P_k| + \sum_{i=2}^{s-1} \Delta\Phi^*(i)|Q| =: \sum' + \sum''.$$

But, as $Q = R_s \cap R_{s-i+1}$ by repeating the proof of (15), we obtain $|Q| = \frac{1}{2}\cot(\theta_s - \theta_{s-i+1}) \leqslant 2\Phi(n)/(in)$, hence

$$\sum'' \leqslant c(\Phi(n)/n) \sum_{i=2}^{s-1} \Delta\Phi^*(i)/i.$$

It follows from (10) that

$$\sum_{i=2}^{s-1} \Delta\Phi^*(i)/i = \sum_{i=2}^{s-1} \Phi^*(i)/i - \sum_{i=2}^{s-1} \Phi^*(i-1)/i$$
$$\leqslant c\Big(\sum_{i=2}^{s-2} \Phi^*(i)/i^2 + \Phi^*(s)/s\Big) \leqslant c\Phi^*(s)/s$$

and by (9)

$$\sum'' \leqslant c(\Phi^*(s)/s)(\Phi(n)/n) \leqslant c \sum_{k=1}^{s} |R_k|.$$

It remains to estimate $\sum'$. From Fig. 3 we have

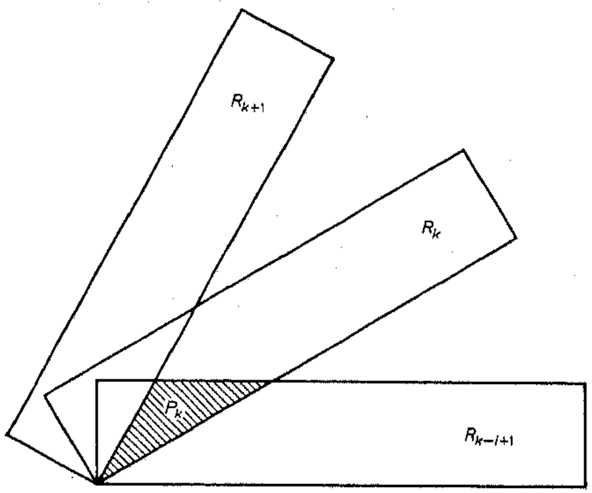

Fig. 3

$$P_k = (R_k \cap R_{k-i+1}) \setminus (R_{k+1} \cap R_{k-i+1})$$

and therefore $|P_k| = \frac{1}{2}\cot(\theta_k - \theta_{k-i+1}) - \frac{1}{2}\cot(\theta_{k+1} - \theta_{k-i+1})$. Hence

$$2\sum' = \sum_{i=2}^{s-1} \sum_{k=i}^{s-1} \Delta\Phi^*(i)\cot(\theta_k - \theta_{k-i+1})$$
$$- \sum_{i=2}^{s-1} \sum_{k=i}^{s-1} \Delta\Phi^*(i)\cot(\theta_{k+1} - \theta_{k-i+1})$$
$$\leqslant \sum_{k=2}^{s-1} \Delta\Phi^*(2)\cot(\theta_k - \theta_{k-1}) + \sum_{i=3}^{s-1} \sum_{k=i}^{s-1} \Delta_2\Phi^*(i)\cot(\theta_k - \theta_{k-i+1})$$
$$=: \sum\nolimits_1' + \sum\nolimits_2'$$

where $\Delta_2\Phi^*(i) = \Delta\Phi^*(i) - \Delta\Phi^*(i-1)$. Since $\cot(\theta_k - \theta_{k-1}) \leqslant 2\Phi(n)/n$ from (12) and (6) we obtain

$$\sum\nolimits_1' \leqslant c\Phi(n)s/n = c\Phi(n)sn/n^2 \leqslant csn = c\sum_{k=1}^{s} |R_k|,$$

and we are left with the main sum $\sum_2'$.

Analogously to (15), $\cot(\theta_k - \theta_{k-i+1}) \leqslant 2\Phi(n)/(in)$, and so

$$\sum\nolimits_2' \leqslant (\Phi(n)/n) \sum_{i=3}^{s-1} \sum_{k=i}^{s-1} \Delta_2\Phi^*(i)/i.$$

Applying twice the Abel transform and reasoning as above gives

$$\sum\nolimits_2' \leqslant c(\Phi(n)/n)\Big(\Phi^*(s)/s + \sum_{i=1}^{s} \sum_{k=i}^{s} \Phi^*(i)/i^3\Big).$$

Now (11) implies

$$\sum_{i=1}^{s} \sum_{k=i}^{s} \Phi^*(i)/i^3 \leqslant s\sum_{i=1}^{s} \Phi^*(i)/i^3 \leqslant c\Phi^*(s)/s$$

and (9) and (12) show that

$$\sum\nolimits_2' \leqslant c\sum_{k=1}^{s} |R_k|,$$

which completes the proof.

COROLLARY 1. *For any $p \in [1, 2)$ there exists a basis $B \subset \mathbf{B}(\mathbf{R}^N)$ such that $B \in D(L^p)(\mathbf{R}^N)$.*

THEOREM 3. *There exists a basis $B \subset \mathbf{B}(\mathbf{R}^2)$ such that B differentiates $L^2(\mathbf{R}^2)$ and does not differentiate $L^p(\mathbf{R}^2)$ for $p < 2$.*

Proof. Let $E = \{x\colon |x| < 1\}$, $m_n = [n/\ln n]$, $m_n \in N$, $n \geqslant 2$. Let $\mathscr{R}_n$ consist of $R^0 = [0, n] \times [0, 1]$ together with the rectangles obtained from R^0 by rotation through the angles $k(\pi/4) m_n^{-1}$, $k = 1, \ldots, m_n$. Clearly,

$$|R| = n \quad \text{for } R \in \mathscr{R}_n, \tag{16}$$

$$\Big|\bigcup_{R \in \mathscr{R}_n} R\Big| \geqslant c \sum_{R \in \mathscr{R}_n} |R| = cnm_n \geqslant cn^2/\ln n, \tag{17}$$

$$|R \cap E|/|R| \geqslant \pi/(4n) \quad \text{for } R \in \mathscr{R}_n. \tag{18}$$

Let $\mathscr{S} = \{R_1, \ldots, R_s\} \subset \mathscr{R}_n$, and let $\theta_1 < \ldots < \theta_s$ be the respective rotation angles. Then

$$\int \Big(\sum_{k=1}^{s} \chi_{R_k}\Big)^2 = \int \sum_k \chi_{R_k} + 2 \int \sum_k \sum_{j<k} \chi_{R_k} \chi_{R_j}$$
$$= \sum_k |R_k| + 2 \sum_k \sum_{j<k} |R_k \cap R_j|.$$

Let us estimate the second term on the right-hand side. We have

$$|R_k \cap R_j| = \tfrac{1}{2} \cot(\theta_k - \theta_j) \sim m_n/(k-j) \tag{19}$$

and hence

$$\sum_k \sum_{j<k} |R_k \cap R_j| \leqslant m_n \sum_k \sum_{j<k} 1/(k-j) \leqslant cm_n s \ln s$$
$$\leqslant c(n/\ln n) s \ln s \leqslant csn = c \sum_{k=1}^{s} |R_k|.$$

Therefore, for any system $\mathscr{S} \subset \mathscr{R}_n$ we have the estimate

$$\int \Big(\sum_{R \in \mathscr{S}} \chi_R\Big)^2 \leqslant c \sum_{R \in \mathscr{S}} |R|. \tag{20}$$

Put $\sigma = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \ldots$, $E_n = E$, $\lambda_n = n$, $n \geqslant 2$. The reasoning analogous to the proof of Lemma A in [4] now yields the desired basis B.

Since by (20), B has the V_2-covering property, it follows that B differentiates $L^2(\mathbf{R}^2)$ (see [1]). On the other hand, by (16)–(18), B differentiates no class $\Phi(L)(\mathbf{R}^2)$ with $\Phi(t) = o(t^2/\log t)$ as $t \to \infty$, in particular, it does not differentiate $L^p(\mathbf{R}^2)$ for $p < 2$.

The proof of Theorem 3 is complete.

Remark. The basis constructed in Theorem 3 has precisely the V_2-covering property. Indeed, by (19)

$$\int \Big(\sum_{k=1}^{m_n} \chi_{R_k}\Big)^2 = \sum_k |R_k| + 2 \sum_k \sum_{j<k} |R_k \cap R_j| \geqslant c \sum_k \sum_{j<k} 1/(k-j)$$
$$\geqslant cm_n^2 \ln m_n \geqslant cn^2/\ln n \geqslant cnm_n = c \sum_{k=1}^{m_n} |R_k|.$$

4. Differentiation of $\Phi(L)(\mathbf{R}^2)$ by subbases of $\mathbf{B}_{\Theta_0}$. The bases constructed in the previous section consist of rectangles with a dense set of directions. Thus they also have rather complicated structure.

Let us now consider the question of existence of bases of rectangles with a rare set of side directions which differentiate precisely some classes $\Phi(L)(\mathbf{R}^2)$. An example here is the basis $\mathbf{B}_{\Theta_0}$, which differentiates $L^p(\mathbf{R}^2)$, $p > 1$, and does not differentiate $o(L\log^2 L)(\mathbf{R}^2)$.

It would be interesting to see whether there exist bases $B \subset \mathbf{B}_{\Theta_0}$ which differentiate precisely a class $\Phi(L)(\mathbf{R}^2)$ intermediate between $L\log^+ L(\mathbf{R}^2)$ and $L\log^2 L(\mathbf{R}^2)$.

THEOREM 4. *Let $\Phi(t) = t\psi(t)$ be an Orlicz function satisfying*

$$\psi(t) \ln t^{-1} \uparrow \infty \quad \text{as } t \to \infty,\ t \geqslant t_0, \tag{21}$$

$$\psi(t)/\ln^2 t \downarrow a \geqslant 0 \quad \text{as } t \to \infty,\ t \geqslant t_0. \tag{22}$$

Then there exists a basis $B \subset \mathbf{B}_{\Theta_0}$ such that $B \in D(\Phi(L))(\mathbf{R}^2)$.

Proof. First we show that Ψ, the inverse function to ψ, satisfies the Δ_3-condition (for the definition see (Δ_3) below). Using (22) and $2(a^2+b^2) \geqslant (a+b)^2$ we obtain

$$\frac{\psi(t)+\psi(y)}{\psi(ty)} \geqslant \frac{\ln^2 t + \ln^2 y}{\ln^2(ty)} = \frac{1}{2} \quad \text{for } t, y \geqslant t_0,$$

hence

$$\psi(\Psi(t)\Psi(y)) \leqslant 2(t+y) = \psi(\Psi(2(y+t))).$$

Therefore $\Psi(t)\Psi(y) \leqslant \Psi(2(t+y))$ for $t, y \geqslant t_0$, and finally,

$$t\Psi(t) = \Psi(\psi(t))\Psi(t) \leqslant \Psi(2(\psi(t)+t)).$$

But, by (22), $\psi(t) = o(t)$ as $t \to \infty$, and so

$$(\Delta_3)\ t\Psi(t) \leqslant \Psi(ct) \quad \text{for } t \geqslant t_0.$$

Consequently, the conjugate function Φ^* is equivalent to Ψ (see [2]). Thus, to establish the differentiation of $\Phi(L)(\mathbf{R}^2)$ it suffices to prove (ii) of Lemma A with Φ^* replaced by Ψ—this will be used below. Further, (22) implies that Φ satisfies the Δ_2-condition.

Write $\omega(t) := \psi(2^{t/4})$. From (21) and (22) we obtain

$$\omega(t)/t \uparrow \infty \quad \text{as } t \to \infty,\ t \geqslant t_0, \tag{23}$$

$$\omega(t)/t^2 \downarrow b \geqslant 0 \quad \text{as } t \to \infty,\ t \geqslant t_0. \tag{24}$$

Without loss of generality we may assume that $b \leqslant 2^{-6}$. Then there is $j_0 \geqslant t_0$ such that

$$8 \leqslant \omega(j-1)/j \leqslant (j-1)/16 \quad \text{for } j \geqslant j_0. \tag{25}$$

Let $m_j = [\omega(j-1)/j]-1$, $m_j \in N$, $j \geqslant j_0$. By (23), $\omega(j) \geqslant (j/(j-1))\,\omega(j-1)$, and hence for $j \geqslant j_0$

$$m_j+1 \leqslant (1/(j-1))\,\omega(j-1) = (j/(j-1)-1)\,\omega(j-1) \leqslant \omega(j)-\omega(j-1).$$

Consequently,

$$\sum_{j=j_0}^{s} (m_j+1) \leqslant \omega(s) \quad \text{for } s \geqslant j_0. \tag{26}$$

On the other hand, (24) yields $\omega(j) \leqslant (j/(j-1))^2\,\omega(j-1)$, and since $1/j \geqslant (1/4)(2j-1)/(j-1)^2$, $j \geqslant 3$, we have

$$\begin{aligned} m_j+1 &\geqslant (\omega(j-1)/j)-1 \geqslant (\tfrac{1}{4}(2j-1)/(j-1)^2)\,\omega(j-1)-1 \\ &= \tfrac{1}{4}(j^2/(j-1)^2-1)\,\omega(j-1)-1 \geqslant \tfrac{1}{4}(\omega(j)-\omega(j-1))-1. \end{aligned}$$

Finally, (25) implies

$$\begin{aligned} \sum_{j=j_0}^{s} (m_j+1) &\geqslant \tfrac{1}{4} \sum_{j=j_0}^{s} (\omega(j)-\omega(j-1))-(s+1) \\ &\geqslant \tfrac{1}{4}\,\omega(s)-\omega(j_0-1)-(s+1) \\ &\geqslant \tfrac{1}{8}\,\omega(s)-\omega(j_0-1), \end{aligned}$$

and hence for some $n_0 \geqslant j_0$

$$\sum_{j=j_0}^{n} (m_j+1) \geqslant \omega(n)/16 \quad \text{for } n \geqslant n_0. \tag{27}$$

As before, we now construct the desired basis with the use of Lemma A.

First we introduce some auxiliary sets and obtain certain preliminary estimates.

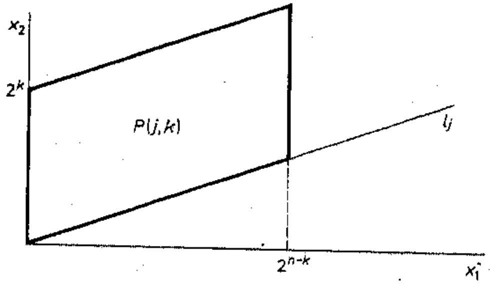

Fig. 4

Fix $n \geqslant n_0$, $n \in N$. Let l_j be the line $x_2 = 2^{j-n} x_1$, $j = j_0, \ldots, n$, in the (x_1, x_2)-plane. Figures 4 and 5 define the families of parallelograms $\{P(j, k)\colon j = j_0, \ldots, n;\ k = 0, \ldots, m_j\}$ and $\{D_j\colon j = j_0, \ldots, n\}$. Let

$$\exp(t) := 2^t.$$

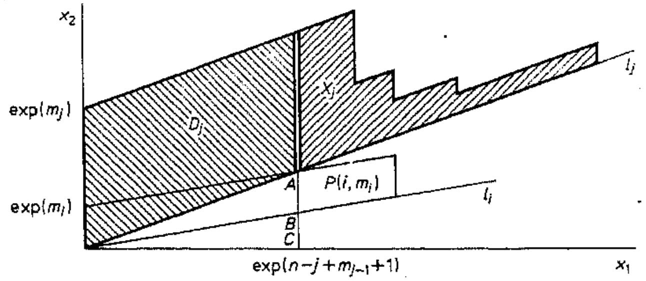

Fig. 5

Clearly, $|D_j| = \exp(n-j+m_{j-1}+1+m_j)$.

By the definition of ω and m_j and (25), (26)

$$\begin{aligned} \int_{D_j} \Psi\Big(\sum_{i=j_0}^{j} \sum_{k=0}^{m_i} \chi_{P(i,k)}\Big) &\leqslant \Psi\Big(\sum_{i=j_0}^{j} (m_i+1)\Big)|D_j| \\ &\leqslant \Psi(\omega(j))\exp(n-j+m_{j-1}+1+m_j) \leqslant \exp(j/4+n-j+2m_j+1) \\ &\leqslant \exp(j/4+n-j+2(j-1)/16+1) \leqslant \exp(n+1-j/2). \end{aligned} \tag{28}$$

Write $X_j := \bigcup_{k=0}^{m_j} P(j, k) \setminus D_j$ (see Fig. 5 where the case $i = j-1$ is represented). Clearly,

$$|AB| \geqslant \tfrac{1}{2}|AC| = \tfrac{1}{2}\exp(j-n)\exp(n-j+m_{j-1}+1) = \exp(m_{j-1}),$$

and since $m_i \leqslant m_{j-1}$ for $j_0 \leqslant i \leqslant j-1$, we have $X_j \cap X_i = \emptyset$ for $i \neq j$, $i, j = j_0, \ldots, n$.

We now define $\mathcal{R}_n$ to be the family of rectangles

$$\{R(j, k)\colon j = j_0, \ldots, n;\ k = 0, \ldots, m_j\}$$

where $R(j, k) = ABCD$ and $P(j, k) = AEFD$ in Fig. 6. Obviously,

$$|R(j, k)| = |P(j, k)| = 2^n \quad \text{for } j = j_0, \ldots, n,\ k = 0, \ldots, m_j. \tag{29}$$

Set $Y := \bigcup_{R \in \mathcal{R}_n} R \setminus \{(x_1, x_2)\colon x_1 \geqslant 0\}$. It can easily be seen that Y is an isosceles right-angled triangle with hypotenuse of length $\exp(m_n)$ lying on the x_2-axis. From (25) we obtain $|Y| \leqslant \exp(2m_n) \leqslant \exp(n/8)$, and hence

$$\int_{Y} \Psi\Big(\sum_{R \in \mathcal{R}_n} \chi_R\Big) \leqslant \Psi\Big(\sum_{j=j_0}^{n} (m_j+1)\Big)|Y| \leqslant \exp(n/4)\exp(n/8) \leqslant 2^n. \tag{30}$$

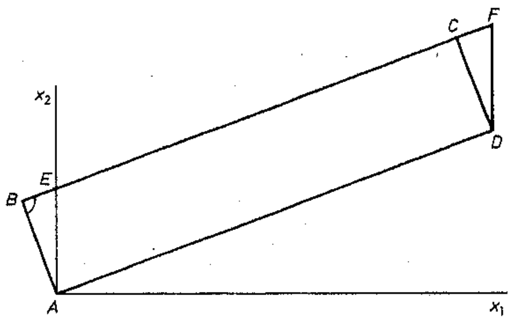

Fig. 6

Let $E := \{(x_2, x_1)\colon 0 \leqslant x_1 \leqslant 1,\ \ x_1 \leqslant x_2 \leqslant 1\}$. Then $E \subset R(j, k)$ and therefore

(31) $$|R \cap E|/|R| = |E|/|R| = 2^{-n-1} \quad \text{for } R \in \mathscr{R}_n.$$

Moreover, from the construction of $\mathscr{R}_n$ it follows that for some $c \in (0, 1)$

$$\Big|R \setminus \bigcup_{R' \in \mathscr{R}_n, R' \neq R} R'\Big| \geqslant c|R| \quad \text{for } R \in \mathscr{R}_n.$$

Then (27) gives

$$\Big|\bigcup_{R \in \mathscr{R}_n} R\Big| \geqslant c \sum_{R \in \mathscr{R}_n} |R| = c2^n \sum_{j=j_0}^{n} (m_j+1) \geqslant c2^n \psi(2^{n/4}).$$

On the other hand, (22) implies that $\psi(u^4)/\ln^2(u^4) \leqslant \psi(u)/\ln^2 u$ for $u \geqslant j_0$, therefore $\psi(u) \geqslant \psi(u^4)/16$, and hence $\psi(2^{n/4}) \geqslant \frac{1}{16}\psi(2^n)$. Finally,

(32) $$\Big|\bigcup_{R \in \mathscr{R}_n} R\Big| \geqslant c2^n \psi(2^n) = c\Phi(2^n)|E| \quad \text{for } n \geqslant n_0.$$

Further, we aim at obtaining the inequality

(33) $$\int_{\mathbf{R}^2 \setminus Y} \Psi\big(\tfrac{1}{2} \sum_{R \in \mathscr{S}} \chi_R\big) \leqslant c \sum_{R \in \mathscr{S}} |R|$$

for an arbitrary subcollection $\mathscr{S} \subset \mathscr{R}_n$. To avoid technicalities, we consider the most important special case: $\mathscr{S} = \mathscr{R}_n$. Then

$$\int_{\mathbf{R}^2 \setminus Y} \Psi\big(\sum_{R \in \mathscr{R}_n} \chi_R\big) = \int_D \Psi\big(\sum_{R \in \mathscr{R}_n} \chi_R\big) + \int_X \Psi\big(\sum_{R \in \mathscr{R}_n} \chi_R\big) =: I + J$$

where $D := \bigcup_{j=j_0}^n D_j$, $X := \bigcup_{j=j_0}^n X_j$. Let us estimate I. From (28) and (29) we obtain

$$I \leqslant \sum_{j=j_0}^{n} \int_{D_j} \Psi\big(\tfrac{1}{2} \sum_{i=j_0}^{j} \sum_{k=1}^{m_i} \chi_{R(i,k)}\big) \leqslant \sum_{j=j_0}^{n} \exp(n+1-j/2)$$
$$\leqslant c2^n \leqslant c \sum_{j=j_0}^{n} \sum_{k=1}^{m_j} |R(j, k)|$$

and therefore

(34) $$I \leqslant c \sum_{R \in \mathscr{R}_n} |R|.$$

It remains to estimate J. We have

$$J \leqslant \sum_{j=j_0}^{n} \int_{\mathbf{R}^2} \Psi\big(\tfrac{1}{2} \sum_{k=1}^{m_j} \chi_{P(j,k)}\big) =: \sum_{j=j_0}^{n} J_j.$$

Let

$$Q(j, k) = \begin{cases} P(j, k) \setminus \bigcup_{i=k+1}^{m_j} P(j, i) & \text{for } k = 1, \ldots, m_j - 1, \\ P(j, m_j) & \text{for } k = m_j. \end{cases}$$

It is easy to see that

$$J_j = \sum_{k=1}^{m_j} \int_{Q(j,k)} \Psi\big(\tfrac{1}{2} \sum_{i=1}^{k} \chi_{P(j,i)}\big) =: \sum_{k=1}^{m_j} J_j^k.$$

Define now

$$T(j, k, h) = \begin{cases} \big(P(j, k-h+1) \setminus P(j, k-h)\big) \cap Q(j, k) & \text{for } h = 1, \ldots, k-1, \\ P(j, 1) \cap Q(j, k) & \text{for } h = k. \end{cases}$$

Obviously,

$$Q(j, k) = \bigcup_{h=1}^{k} T(j, k, h),$$

$$|T(j, k, h)| \leqslant \exp(k-h+1)\exp(n-k) = \exp(n-h+1),$$

$$\sum_{i=1}^{k} \chi_{P(j,i)}(x) = h \quad \text{for } x \in T(j, k, h).$$

By (21),

$$J_j^k \leqslant \sum_{h=1}^{k} \int_{T(j,k,h)} \Psi(h/2) \leqslant c \sum_{h=1}^{k} \exp(h/2)\exp(n-h+1) \leqslant c2^n.$$

Then

(35) $$\sum_{j=j_0}^{n} \sum_{k=1}^{m_j} J_j^k \leqslant c \sum_{j=j_0}^{n} \sum_{k=j}^{m_j} 2^n = c \sum_{j=j_0}^{n} \sum_{k=1}^{m_j} |R(j, k)|$$

and the required estimate follows from (30) and (33)–(35).

Thus the collection $\sigma = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \ldots$ is defined. Let $E_n = E$, $\lambda_n = 2^n$, $n \geqslant n_0$. Then (29), (33), (31) and (32) imply (i)–(iv) of Lemma A, which completes the proof of Theorem 4.

Corollary 2. *For any* $p \in [1, 2]$ *there exists a basis* $B \subset \boldsymbol{B}_{\Theta_0}$ *such that* $B \in D(L(\log^+ L)^p)(\boldsymbol{R}^2)$.

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DEPARTMENT OF MATHEMATICS
ODESSA STATE UNIVERSITY
Petra Velikogo 2, 270000 Odessa, U.S.S.R.

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L^p-Multiplier transference induced by representations in Hilbert space

by

EARL BERKSON (Urbana, Ill.), T. A. GILLESPIE (Edinburgh), and PAUL S. MUHLY (Iowa City, Ia.)

Abstract. Let $(\mathcal{M}, \mu)$ be a measure space, and S a representation of a locally compact abelian group G by measure-preserving transformations of the points of $\mathcal{M}$. Under suitable further hypotheses on μ, G, and S, the Coifman–Weiss Transference Theorem for Multipliers provides a machinery whereby S can be made to transfer "normalized" $L^p(G)$-multiplier transforms, along with their bounds, to $L^p(\mu)$, for all finite p. We show below that multiplier transference can be freed of technical restrictions so that its broader, structurally simpler nature emerges in the following form: whenever an arbitrary locally compact abelian group G has a uniformly bounded strongly continuous representation R in $L^2(\mu)$ (μ an arbitrary measure) such that R has a uniformly bounded $L^p(\mu)$-version for some $p \in (1, \infty)$, then R will transfer continuous $L^p(G)$-multipliers to $L^p(\mu)$. The added generality is illustrated in some elementary examples and in a short proof of the Homomorphism Theorem for Multipliers. An application to generalized analyticity is presented in the last section.

1. Introduction. The Coifman–Weiss theory of transference methods unifies and expands diverse streams of thought in general analysis by transferring operators affiliated with groups, along with their bounds, to spaces in which the groups act (see [6] for an expository account of the theory's nature and lineage). In this article we shall be concerned with a generalization, to a wider context, of the Coifman–Weiss Transference Theorem for Multipliers (see Theorem (2.1) below for this generalization).

Suppose that $(\mathcal{M}, \mu)$ is a measure space, and S is a representation of a locally compact abelian group G by measure-preserving transformations of the points of $\mathcal{M}$. Suppose also that the unitary representation R of G in $L^2(\mathcal{M}, \mu)$ implemented by S is strongly continuous. The Coifman–Weiss Transference Theorem for Multipliers ([5, Theorem 3.7]) provides a method for using R to transfer "normalized" multiplier transforms from $L^p(G)$ to $L^p(\mathcal{M}, \mu)$ without increasing their operator norms. In order to support this multiplier transference method, various technical hypotheses are imposed in [5]: the group G is assumed to be σ-compact, $(\mathcal{M}, \mu)$ is taken to be σ-finite, and joint measurability in $(u, \omega) \in G \times \mathcal{M}$ is implicitly assumed for functions of the form $f(S_u \omega)$ $(f \in L^p(\mathcal{M}, \mu))$. Recently, it was shown in [4, Proposition