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W R O C Ł A W S K A D R U K A R N I A N A U K O

The maximum path theorem and extreme points of James' space

by

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Abstract. The maximum path theorem gives conditions on the costs of the edges of the graph of the relation $x > y$ on N so there is a unique "biggest" path of maximal cost. This theorem is used to characterize the extreme points of James' quasi-reflexive Banach space.

We derive a result in graph theory and apply it to the theory of Banach spaces. The maximum path Theorem 2.1 gives conditions on the costs of the edges so there is a unique "biggest" path of maximal cost on the graph of the relation $x > y$ over N. This graph is acyclic and so by negating the costs, we can find all maximal paths by well-known algorithms (see Section 1) for finding minimal paths. Algorithms for finding shortest paths have been extensively studied (see [4] and [7, pp. 41 ff.]).

This maximum path is used to characterize the extreme points of James' quasi-reflexive Banach space J (Proposition 3.1 and Corollary 3.2). Although the set of these extreme points is small (closed and nowhere dense by Corollary 3.3), the algorithmic nature of the maximum path theorem allows us to construct interesting examples of extreme points in J (Examples 3.4).

In Section 4, we briefly consider exposedness and smoothness. Each extreme point is exposed and a necessary condition for a point to be smooth are in Proposition 4.1.

§ 1. **The maximal path algorithm.** This section gives most of the basic definitions needed for the maximum path theorem in the next section. Besides including the algorithm, this section also tries to put the maximum path theorem in context by showing what is true in general and giving a couple of examples.

The setting is graph theory. The graph's vertices are the natural numbers $N = \{0, 1, 2, \ldots\}$ and there is a directed edge from j to i if $i < j$ and that edge has a cost $C_{ij} \geqslant 0$. The algorithm below will find a path of maximal

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cost between any pair of vertices. Proposition 1.2 will show how this implies a maximal cost path for the whole graph (the path could have infinite length).

This algorithm is similar to many well-known algorithms, Dijkstra's shortest path algorithm [2, p. 19] for example. The algorithm works since an optimal path must be locally optimal, i.e. optimal on each of its "subpaths" and since we are applying it to a graph with no circuits. The algorithm takes $O(n^2)$ time to compute the maximal path from n to 0.

The algorithm defines three functions for $0 \leqslant i \leqslant j < \infty$. The function $N(i, j)$ is the cost of the maximal path from j to i (N is for norm). The function $B(i, j)$ is the first vertex $\neq j$ on this path (B is for back). The function $E(i, j)$ gives the set of vertices on this path (E is for extreme).

ALGORITHM 1.1. Initialize by setting $N(i, i) = 0$ and $E(i, i) = \{i\}$ ($B(i, i)$ is undefined). For $i < j$, these functions are inductively defined by

(1) $$N(i, j) = \max\{N(i, k) + C_{kj}\colon\ i \leqslant k < j\},$$

(2) $$B(i, j) = \max\{k\colon\ i \leqslant k < j,\ N(i, j) = N(i, k) + C_{kj}\},$$

(3) $$E(i, j) = E\big(i, B(i, j)\big) \cup \{j\}.$$

Since $\big(N(i, n)\big)_n$ is increasing, define $N(i, \infty) = \lim_n N(i, n)$. Also define $C_{k\infty} = \limsup_j C_{kj}$ and note that

$$N(i, k) + C_{kj} \leqslant N(i, j) \leqslant N(i, \infty),$$

so that $N(i, k) + C_{k\infty} \leqslant N(i, \infty)$ is always true.

We are identifying subsets $A \subset \mathbf{N}$ with paths in the graph. Let $S(A)$ be the cost of the path A computed in the following strange manner. If $\big(a(k)\big)_{k=1}^{j}$ is the increasing list of the elements of a finite set A, then

$$S(A) = C_{a(1)\infty} \quad \text{if } j = 1, \text{ or}$$

$$S(A) = \sum_{k=1}^{j-1} C_{a(k)a(k+1)} + C_{a(j)\infty} \quad \text{if } j \geqslant 2.$$

If $\big(a(k)\big)_k$ is the increasing list of the elements of an infinite set A, then

$$S(A) = \sum_k C_{a(k)a(k+1)}.$$

Note that for finite sets A we are implicitly adding the vertex "∞" to the path.

The following proposition is perhaps well known. We give a proof both for comparison with Theorem 2.1 and for completeness. Readers used to "Banach space trees" are warned that our trees are usually not binary and they grow down with the root at the top, i.e. a tree is modeled on a tree of descendants (of the root).

PROPOSITION 1.2. *For each i, there is $A \subset \{i, i+1, \ldots\}$ so that $S(A) = N(i, \infty)$.*

Proof. We use the algorithm to also construct a tree. The vertices are $\{i, i+1, \ldots\}$ and i is the root. If $i, i+1, \ldots, k-1$ have been added to the tree make k a child of $B(i, k)$. There are two cases to consider.

Case I: Some vertex t in this tree has infinitely many children $(m(k))$. Note that $E(i, t)$ is the unique path in the tree between t and the root i. Since $\big(N(i, n)\big)_n$ is increasing,

$$N(i, \infty) = \lim_k N\big(i, m(k)\big) = \lim_k \big(N(i, t) + C_{tm(k)}\big) = N(i, t) + \lim_k C_{tm(k)}.$$

Let $t < n < m(k)$. Since $N(i, t) + C_{tm(k)} = N\big(i, m(k)\big) \geqslant N(i, n) \geqslant N(i, t) + C_{tn}$ we have $C_{tn} \leqslant C_{tm(k)} \leqslant C_{tm(k+1)}$. Hence $\limsup_n C_{tn} = \lim_k C_{tm(k)}$ and $S\big(E(i, t)\big) = N(i, \infty)$.

Case II: All vertices in the tree have finitely many children. Now the infinity lemma [9, p. 381] applies and yields the existence of an infinite branch $L \subset \{i, i+1, \ldots\}$. Note $L = \bigcup_{j \in L} E(i, j)$ and hence if $L = \big(m(k)\big)_k$, then $S(L) = \lim_k N\big(i, m(k)\big) = N(i, \infty)$. Readers not familiar with the infinity lemma may pick $m(k+1)$ as any child of $m(k)$ with infinitely many descendants. ■

EXAMPLES 1.3. (A) For $i \geqslant 1$ define $C_{2i-1,2i+1} = C_{2i,2i+2} = 2^{-i}$ and otherwise $C_{ij} = 0$. If $A = \{2i-1\colon\ i \geqslant 1\} \cup \{0\}$ and $B = \{2i\colon\ i \geqslant 0\}$, then both $S(A) = S(B) = N(0, \infty) = 1$. That is, different maximal paths need not have any vertices in common other than the destination vertex. Also, there are two almost distinct maximal paths from $2n$ to 1.

(B) Define $C_{12} = 1$; $C_{1n} = 2 - 1/n$, for $n \geqslant 3$; $C_{2n} = 1 - 2/n$ for $n \geqslant 3$; and otherwise $C_{ij} = 0$. In this example $A = \{1\}$ and $B = \{1, 2\}$ are both maximal paths for $i = 1$. Yet for $k \geqslant 3$, $N(1, k) = C_{1k} > C_{12} + C_{2k} = N(1, 2) + N(2, k)$. That is, the path B only catches up to this cost of the path A in the limit.

(C) Let $C_{ij} \equiv 0$; then any $A \subset \mathbf{N}$ is a path of maximal cost.

(D) Let $C_{ij} = i$; then any infinite $A \subset \mathbf{N}$ is a path of maximal cost. If A is finite and $n = \max A$, then $S(A) \leqslant n + n(n-1)/2 < \infty = N(0, \infty)$.

(E) Let $C_{01} = C_{12} = 0$, $C_{02} = 1$, $C_{n,n+1} = 1/n$ for $n \geqslant 2$ and $C_{ij} = 0$ otherwise. Let $A = \{0, 2, 3, 4, \ldots\}$, $B = \mathbf{N}$ and $A_n = \{i \in A\colon\ i \leqslant n\}$, $B_n = \{i \in B\colon\ i \leqslant n\}$. Observe $S(B_n) < S(A_n)$ but $S(A) = S(B)$.

§ 2. The maximum path theorem

THEOREM 2.1. *If the costs C_{ij} satisfy the glueing condition, are of bounded variation and are nice at infinity, then for each i, there is $A \subset \{i, i+1, \ldots\}$ so that $S(A) = N(i, \infty)$, and for each t with $i < t < \infty$ and $t \notin A$ there is then a $\delta = \delta_t > 0$ so that if $B \subset \{i, i+1, \ldots\}$ and $t \in B$ then $S(B) \leqslant N(i, \infty) - \delta$.*

Before defining the terms in the hypothesis, let us observe that A is the maximum path. It follows from the theorem that if $B \subset \{i, i+1, \ldots\}$ and $S(B) = N(i, \infty)$, then $B \subset A$. That is, A is the "longest" or "biggest" path of maximal cost, and is obviously unique. Example 1.3C shows that the inclusion $B \subset A$ can be proper. Indeed, there $A = \{i, i+1, \ldots\}$ and any nonempty $B \subset A$ has $S(B) = N(i, \infty) = 0$. We collect these observations in the following statement.

Corollary 2.2. *If the hypothesis of the theorem holds, then for each i, there is a unique $A \subset \{i, i+1, \ldots\}$ so that $S(A) = N(i, \infty)$, and for any $B \subset \{i, i+1, \ldots\}$ with $S(B) = N(i, \infty)$ we then have $B \subset A$.*

Definition 2.3. (A) The costs C_{ij} are of *bounded variation* if

$$\sup\Big\{\sum_{k=1}^{n} C_{i(k)i(k+1)}\colon\ n \geqslant 1,\ 0 \leqslant i(1) < i(2) < \ldots < i(n+1)\Big\} < \infty.$$

(B) The costs C_{ij} satisfy the *glueing condition* if given $0 \leqslant r < s < t < u$ so that both $r < k < t$ implies $C_{rk} + C_{kt} < C_{rt}$ and $s < k < u$ implies $C_{sk} + C_{ku} < C_{su}$, we have $C_{rt} + C_{su} - C_{st} \leqslant C_{ru}$.

(C) The costs C_{ij} are *nice at infinity* if given $r < s$ and an increasing sequence $(m(k))$ of integers $> s$ so that for all k, $B(r, m(k)) = r$ and $B(s, m(k)) = s$, the sequence $(C_{rm(k)} - C_{sm(k)} - N(r, s))_k$ is bounded away from zero.

Remarks. 1. Observe that the C_{ij} are of bounded variation if and only if $N(i, \infty) < \infty$ for each i. This hypothesis is almost not needed in the theorem. It is used to prevent things like in Example 1.3E where the inferior path B_n catches A_n in the limit. However, the theorem is still true for Example 1.3E. We will delay the use of bounded variation in the proof as long as possible.

2. The glueing condition is necessary as Example 1.3A shows. As we will see in the course of the proof, this property is well named. That is, it requires different maximal paths between different pairs of vertices to have a "common part" (see I, J below). The glueing condition says under certain conditions, even allowing a negative cost for backing up an edge $(-C_{st})$ will not increase the cost.

3. Frankly, the nice condition at infinity was added on just to make the proof work. It is designed to prevent what happens in Example 1.3B. We note that the conclusion of the theorem is still true for Example 1.3B. However, this condition is satisfied by the application we have in mind. And this application requires a slightly stronger statement which does require this condition. (See the proposition below.) Also it yields a construction of the set A in the conclusion of Theorem 2.1.

In addition to the $N(i, j)$, $B(i, j)$, $E(i, j)$, $N(i, \infty)$ and $S(A)$ notation defined in Section 1, we need a bit more. Define

$$E(i, \infty) = \bigcup_{j \geqslant i} \bigcap_{n \geqslant j} E(i, n) = \liminf_{j} E(i, j).$$

It is possible that $E(i, \infty)$ is a finite set, even a singleton. When $E(i, \infty)$ is finite define

$$F(i, \infty) = \{m > r = \max E(i, \infty)\colon\ B(i, m) = r\}.$$

The following proposition formally implies Theorem 2.1.

Proposition 2.3. *Under the hypothesis of Theorem 2.1, the set $A = E(i, \infty)$ satisfies the conclusion.*

Proof. The proof is long-winded and broken up into a sequence of lemma-like "claims". We group the claims by increasing strength of their hypotheses. The first collection are true for any costs C_{ij}.

Claims. *For any $C_{ij} \geqslant 0$:*

(A) *$E(i, j) \subset \{k\colon i \leqslant k \leqslant j\}$ and if $s \neq i$ and $s \in E(i, j)$ then $B(i, s) = \max\{k < s\colon k \in E(i, j)\}$.*

(B) *If $i \leqslant j \leqslant k$, then $N(i, j) + N(j, k) \leqslant N(i, k)$.*

(C) *If $i \leqslant j < k$ and $m = B(j, k)$, then $N(i, j) + N(j, k) = N(i, k)$ if and only if both $N(i, j) + N(j, m) = N(i, m)$ and $N(i, m) + C_{mk} = N(i, k)$.*

(D) *If D is a finite set with $i = \min D < j = \max D$ and $(d(k))_{k=1}^{n}$ is the list of the elements of D in increasing order, then $\sum_{k=1}^{n-1} C_{d(k)d(k+1)} \leqslant N(i, j)$. Hence if $D \subset \{i, i+1, \ldots\}$, then $S(D) \leqslant N(i, \infty)$.*

(E) *Let $0 \leqslant i \leqslant j$ and suppose $N(i, j) + N(j, m) = N(i, m)$ for $j \leqslant m < k$; then $B(i, k) \leqslant B(j, k)$ and either $N(i, j) + N(j, k) = N(i, k)$ or $B(i, k) < j$.*

Proofs. (A, B, C, D) are all done by induction. The proofs are straightforward and are omitted.

(E) Let $m = B(j, k)$ and let $m < n < k$ so that $N(j, n) + C_{nk} < N(j, k)$. Now

$$N(i, n) + C_{nk} = N(i, j) + N(j, n) + C_{nk} < N(i, j) + N(j, k) \leqslant N(i, k),$$

so that $B(i, k) \leqslant m = B(j, k)$.

Now let $n = B(i, k)$ and suppose $j \leqslant n \leqslant m = B(j, k)$. We have

$$\begin{aligned} N(i, k) &= N(i, n) + C_{nk} = N(i, j) + N(j, n) + C_{nk} \\ &\leqslant N(i, j) + N(j, k), \end{aligned}$$

and the reverse inequality is in (B). This completes (E).

MORE CLAIMS. *Suppose the costs C_{ij} satisfy the glueing condition. Then:*

(F) *Let $0 \leqslant i \leqslant j < n$ and suppose $N(i,j)+N(j,k) = N(i,k)$ for $j \leqslant k < n$, but $N(i,j)+N(j,n) < N(i,n)$. Then for all $m \geqslant n$, $N(i,j)+N(j,m) < N(i,m)$ and either $B(i,m) \geqslant n$ or $B(i,m) < j$.*

(G) *Let $0 \leqslant i < j < k$. Then $j \in E(i,k)$ if and only if $N(i,j)+N(j,k) = N(i,k)$.*

(H) *Let $0 \leqslant m \leqslant i < j < k$. If $N(i,j)+N(j,k) < N(i,k)$, then $N(m,j)+N(j,k) < N(m,k)$.*

(I) *If $0 \leqslant m \leqslant i \leqslant j \leqslant n$, then $E(m,n) \cap \{k: i \leqslant k \leqslant j\} \subset E(i,j)$; furthermore, if $i, j \in E(m,n)$, then $E(m,n) \cap \{k: i \leqslant k \leqslant j\} = E(i,j)$.*

(J) *$E(i,\infty) = \{k \geqslant i$: for $j \geqslant k$, $k \in E(i,j)\}$. If $i < j$, then $E(i,\infty) \cap \{k \geqslant j\} \subset E(j,\infty)$, and if $j \in E(i,\infty)$, then $E(i,\infty) \cap \{k \geqslant j\} = E(j,\infty)$.*

(K) *If $t \notin E(i,\infty)$ and $i < t < k \in E(i,\infty)$, then for some s and $\delta > 0$, we have for $n \geqslant s$*

$$N(i,n)-\delta \geqslant N(i,t)+N(t,n).$$

(L) *The proposition is true for infinite $E(i,\infty)$ or for finite $E(i,\infty)$ with $i < t < \max E(i,\infty)$.*

Proofs. (F) Suppose the first conclusion is false and let $m \geqslant n$ be the smallest integer with $N(i,j)+N(j,m) = N(i,m)$. Let $p = B(j,m)$, so that $p \geqslant j$. By the equality above and part (C) we have $N(i,j)+N(j,p) = N(i,p)$, and hence $j \leqslant p < n$. Let $q = B(i,n)$, so by (E) we have $i \leqslant q < j$. Thus $q < p < n < m$.

We are set up to apply the glueing condition. Note $p = B(j,m)$ (resp. $q = B(i,n)$) implies $C_{pk}+C_{km} < C_{pm}$ for $p < k < m$ (resp. $C_{qk}+C_{kn} < C_{qn}$ for $q < k < n$). Thus $C_{qn}+C_{pm}-C_{pn} \leqslant C_{qm}$.

However,

$$N(i,n) = N(i,q)+C_{qn} > N(i,p)+C_{pn},$$
$$N(i,q)+C_{qm} \leqslant N(i,p)+C_{pm} = N(i,m).$$

Subtracting we get $C_{qn}-C_{qm} > C_{pn}-C_{pm}$, a contradiction.

A similar contradiction is obtained by assuming $B(i,n) = q < j \leqslant p = B(i,m) < n < m$. This completes (F).

(G) Suppose $N(i,j)+N(j,k) = N(i,k)$. Then by (F), $N(i,j)+N(j,n) = N(i,n)$ for $j \leqslant n \leqslant k$. We show $j \in E(i,k)$ by induction on k. Certainly $j \in E(i,j)$. Let $n = B(j,k) \geqslant j$. We have

$$N(i,n)+C_{nk} = N(i,j)+N(j,n)+C_{nk} = N(i,j)+N(j,k) = N(i,k).$$

Thus $m = B(i,k) \geqslant n \geqslant j$. Now $j \leqslant m < k$ implies by (F) and the induction hypothesis that $j \in E(i,m) \subset E(i,k)$.

The converse is also proved by induction on k. Note that both $N(i,j)+N(j,j) = N(i,j)$ and $j \in E(i,j)$. So suppose $j \in E(i,k)$ and the result is true for smaller k. Let $m = B(i,k)$. Since $E(i,k) = E(i,m) \cup \{k\}$, we have $j \in E(i,m)$. Thus $N(i,j)+N(j,m) = N(i,m)$. Hence

$$N(i,k) = N(i,m)+C_{mk} = N(i,j)+N(j,m)+C_{mk} \leqslant N(i,j)+N(j,k).$$

The reverse inequality is in (B) so $N(i,j)+N(j,k) = N(i,k)$ and (G) is done.

(H) Suppose i is the largest integer $< j$ with $N(i,j)+N(j,k) < N(i,k)$. Suppose the result is false and let n be the largest integer $< i$ with $N(n,j)+N(j,k) = N(n,k)$. The idea is to contradict the glueing condition.

Suppose for some m, $n < m \leqslant i$ and $m \in E(n,j)$. Then $N(n,m)+N(m,j) = N(n,j)$ by (G). And hence

$$N(n,k) = N(n,j)+N(j,k) = N(n,m)+N(m,j)+N(j,k)$$
$$< N(n,m)+N(m,k) \leqslant N(n,k),$$

a contradiction. Thus if p is the second smallest element of $E(n,j)$ we have $i < p \leqslant j$.

The maximal condition on i implies that $N(p,j)+N(j,k) = N(p,k)$. Now $N(n,j) = N(n,p)+N(p,j)$ by (G) and $N(n,p) = C_{np}$. If $p \in E(i,k)$, then

$$N(i,k) = N(i,p)+N(p,k) = N(i,p)+N(p,j)+N(j,k)$$
$$\leqslant N(i,j)+N(j,k),$$

which is a contradiction to our initial assumption. Thus $p \notin E(i,k)$.

Let $r < p < s$ so that $\{t: r \leqslant t \leqslant s\} \cap E(i,k) = \{r,s\}$. If $i < r$, then by assumption $N(r,j)+N(j,k) = N(r,k)$. However,

$$N(i,k) = N(i,r)+N(r,k) = N(i,r)+N(r,j)+N(j,k)$$
$$\leqslant N(i,j)+N(j,k),$$

which is a contradiction by part (B). Thus $r = i$.

Now $N(i,k) = N(i,s)+N(s,k) = C_{is}+N(s,k)$ and $N(i,k) > N(i,p)+N(p,k) \geqslant C_{ip}+N(p,k)$. Also $N(n,k) = N(n,p)+N(p,k) = C_{np}+N(p,k)$ and $N(n,k) \geqslant N(n,s)+N(s,k) \geqslant C_{ns}+N(s,k)$. Thus

$$C_{is}+N(s,k) > C_{ip}+N(p,k),$$
$$C_{ns}+N(s,k) \leqslant C_{np}+N(p,k).$$

Subtracting yields $C_{is}-C_{ns} > C_{ip}-C_{np}$ or $C_{is}+C_{np}-C_{ip} > C_{ns}$, which contradicts the glueing hypotheses. Thus (H) is done.

(I) If k satisfies $i \leqslant k \leqslant j$ and $k \notin E(i,j)$, then $N(i,k)+N(k,j) < N(i,j)$ by (G). Hence by (H) and (F), $N(m,k)+N(k,n) < N(m,n)$ and by (G) we have $k \notin E(m,n)$.

Suppose $i = m$ or $j = n$, $i, j \in E(m,n)$ and $t \in E(i,j)$. Say if $j = n$, then

$$N(m,n) = N(m,i)+N(i,n) = N(m,i)+N(i,t)+N(t,n)$$
$$= N(m,t)+N(t,n),$$

hence $t \in E(m, n)$. Now in the general case $i, j \in E(m, n)$ implies $j \in E(i, n)$. Thus

$$N(m, n) = N(m, i)+N(i, n) = N(m, i)+N(i, j)+N(j, n)$$
$$= N(m, i)+N(i, t)+N(t, j)+N(j, n) \leqslant N(m, t)+N(t, n).$$

Appeals to (B) and (G) complete (I).

(J) If $k \in E(i, \infty)$, then for some $j \geqslant k$, $k \in E(i, n)$ for each $n \geqslant j$. By (I), $k \in E(i, n)$ for each $n \geqslant k$. If $n \geqslant j$, $m \geqslant n$ and $n \in E(i, \infty)$, then $n \in E(i, m)$, hence by (I), $n \in E(j, m)$ so that $n \in E(j, \infty)$. If in addition $j \in E(i, \infty)$, then $j \in E(i, m)$ for $m \geqslant j$. So if $n \in E(j, \infty)$ and $m \geqslant n$, then $n \in E(j, m)$ and $n \in E(i, m)$ by (I), thus $n \in E(i, \infty)$. Now (J) is complete.

(K) Let $r, s \in E(i, \infty)$ so that $r < t < s$ and $\{n\colon r \leqslant n \leqslant s\} \cap E(i, \infty) = \{r, s\}$. Observe that if $j \geqslant s$, then $s \in E(i, j)$ and $B(i, s) = \max\{k < s\colon k \in E(i, j)\}$ by (A). Hence $B(i, s) = r$, $N(i, s) = N(i, r)+C_{rs} = N(i, r)+N(r, s)$. And since $t \notin E(i, s)$, $N(i, t)+N(t, s) < N(i, s)$, by (G).

But $r \in E(i, \infty)$ so $r \in E(i, k)$ for $k \geqslant r$, in particular when $k = t$. Thus by (G), $N(i, r)+N(r, t) = N(i, t)$. We have $N(i, r)+N(r, t)+N(t, s) < N(i, s) = N(i, r)+C_{rs}$ or

$$\delta = C_{rs}-\big(N(r, t)+N(t, s)\big) > 0.$$

Now suppose $n \geqslant s$. We have $s \in E(i, n) \cap \{k\colon t \leqslant k \leqslant n\} \subset E(t, n)$ so that $s \in E(t, n)$. By (G),

$$N(i, t)+N(t, n) = N(i, r)+N(r, t)+N(t, s)+N(s, n),$$
$$N(i, n) = N(i, r)+N(r, s)+N(s, n), \qquad N(r, s) = C_{rs}.$$

Thus $N(i, n)-\big(N(i, t)+N(t, n)\big) = C_{rs}-\big(N(r, t)+N(t, s)\big) = \delta$. Thus (K) is complete.

(L) Use the fact that the C_{ij} are of bounded variation, so that the inequality in (K) passes to the limit. We have $N(i, \infty)-\delta \geqslant N(i, t)+N(t, \infty)$. If $B \subset \{i, i+1, \ldots\}$ so that $t \in B$, it follows that $S(B \cap \{k\colon i \leqslant k \leqslant t\}) \leqslant N(i, t)$ and $S(B \cap \{k \geqslant t\}) \leqslant N(t, \infty)$ by (D). Therefore

$$S(B) \leqslant N(i, \infty)-\delta = S\big(E(i, \infty)\big)-\delta.$$

And hence (L) and this set of claims is done.

Still more claims. *Let C_{ij} satisfy the glueing condition and suppose $E(i, \infty)$ is a finite set with $r = \max E(i, \infty)$. Then:*

(M) *$E(r, \infty) = \{r\}$ and $F(i, \infty) = F(r, \infty)$.*

(N) *$F(i, \infty)$ is an infinite set, $\{C_{rs}\colon s \in F(i, \infty)\}$ is strictly increasing and $N(i, \infty) = N(i, r)+\lim\{C_{rs}\colon s \in F(i, \infty)\}$.*

(O) *Suppose $r < t < m < s$, with $m \in F(i, \infty)$ and $s \in E(t, \infty)$. Let $n = \min\{k > m\colon k \in E(t, \infty)\}$. Then either $n \in F(i, \infty)$ or $m \in E(t, \infty)$. Thus there is a $\delta > 0$ so that $k \geqslant n$ implies*

$$N(i, k)-\delta \geqslant N(i, t)+N(t, k).$$

Note such m and s exist if $E(t, \infty)$ is an infinite set.

(P) *Suppose $r < t$ and $s = \max E(t, \infty) < \infty$. Then $F(i, \infty) \cap F(t, \infty)$ has infinitely many elements.*

(Q) *The proposition is true.*

Proofs. (M) Since $r \in E(i, \infty)$, $E(i, m) \cap \{k\colon r \leqslant k \leqslant m\} = E(r, m)$. Thus $B(i, m) = r$ if and only if $B(r, m) = r$. Also $E(r, \infty) = E(i, \infty) \cap \{k \geqslant r\} = \{r\}$ by (J). So much for (M).

(N) If $n > r$, then since $r < t \leqslant n$ implies $t \notin E(i, \infty)$ there is some k so that $j \geqslant k$ and $r < t \leqslant n$ implies $t \notin E(i, j)$ by (I). Let $s = \min\{m > r\colon m \in E(i, k)\}$. By (A), it follows that $B(i, s) = r$. Since $s > n$, we see that $F(i, \infty)$ is infinite.

If $s, t \in F(i, \infty)$ with $s < t$, then

$$N(i, t) = N(i, r)+C_{rt} > N(i, s)+C_{st} = N(i, r)+C_{rs}+C_{st}.$$

Hence $C_{rt} > C_{rs}+C_{st} \geqslant C_{rs}$. Thus $\{C_{rs}\colon s \in F(i, \infty)\}$ is strictly increasing and the last result follows since $\big(N(i, n)\big)_n$ is nondecreasing.

(O) By (I), $E(i, n) \cap \{k\colon i \leqslant k \leqslant m\} \subset E(i, m)$ so that $B(i, n) \geqslant m$ or $B(i, n) = r$. The last equality is because $r \in E(i, \infty)$ and it implies $n \in F(i, \infty)$. On the other hand, the definition of n implies $B(i, n) \leqslant m$ so the first inequality implies $m = B(i, n) \in E(t, \infty)$.

In either case we have an $n \in E(t, \infty) \cap F(i, \infty)$, with $n > t$. Now

$$N(i, n) = C_{rn}+N(i, r) > N(i, t)+N(t, n)$$
$$\geqslant N(i, r)+N(r, t)+N(t, n),$$

by (H). Thus $\delta = C_{rn}-\big(N(r, t)+N(t, n)\big) > 0$. Now if $k \geqslant n$, then $n \in E(t, k)$ so that $N(t, k) = N(t, n)+N(n, k)$. So

$$N(i, k) \geqslant N(i, n)+N(n, k) = N(i, r)+C_{rn}+N(n, k)$$
$$= \delta+N(i, r)+N(r, t)+N(t, n)+N(n, k).$$

But $r \in E(i, \infty)$, $n \in E(t, \infty)$ yield $N(i, k) \geqslant \delta+N(i, t)+N(t, k)$.

(P) Suppose $m > s$ and $m \in F(i, \infty)$. Suppose $m \notin F(t, \infty)$ and let $n \in F(t, \infty)$ be as small as possible with $n > m$. Since $B(t, n) = s$, we have $B(i, n) < m$ by (I). Hence $B(i, n) = r$, since $B(i, m) = r$ and $r \in E(i, \infty)$. Thus $n \in F(i, \infty)$ and (P) is complete.

(Q) From (L), we may assume $\max E(i, \infty) = r < \infty$ and $t > r$. As in the proof of (L), if the hypothesis in (O) is true, then the proposition is true (again using the bounded variation property of C_{ij}). Thus we may assume $s = \max E(t, \infty) < \infty$. By (P), there is a strictly increasing sequence $(m(k)) \subset F(i, \infty) \cap F(t, \infty)$. Now by (M), $E(r, \infty) = \{r\}$, $E(s, \infty) = \{s\}$, $F(i, \infty) = F(r, \infty)$ and $F(t, \infty) = F(s, \infty)$.

Thus by the nice-at-infinity condition,

$$\delta = \inf\{C_{rm(k)} - C_{sm(k)} - N(r, s)\} > 0.$$

Hence

$$\begin{aligned} N(i, m(k)) &= N(i, r) + C_{rm(k)} \geqslant N(i, r) + N(r, s) + C_{sm(k)} + \delta \\ &\geqslant N(i, r) + N(r, t) + N(t, s) + C_{sm(k)} + \delta \\ &= N(i, t) + N(t, m(k)) + \delta. \end{aligned}$$

By (N) and bounded variation, $N(i, \infty) \geqslant N(i, t) + N(t, \infty) + \delta$. And once again as in the proof of (L) this implies the proposition. Hence (Q), this sequence of claims, the proposition and the theorem are all true. ∎

§ 3. Extreme points of J^{**}. James' quasi-reflexive Banach space J ([8] or [10, p. 25]) is a famous counterexample. In this section we characterize the extreme points of J^{**}, its bidual.

The space J is the collection of all real null sequences with bounded square variation. We will often identify the sequence (a_i) with the (sometimes formal) sum $\sum a_i e_i$ where the e_i are the usual unit vector basis. The norms on J which we are interested in are defined via

$$2\left\|\sum a_i e_i\right\|^2 = \sup\Big\{\sum_{i=1}^{k} (a_{p(i)} - a_{p(i+1)})^2\Big\},$$

$$2\left|\left|\left|\sum a_i e_i\right|\right|\right|^2 = \sup\Big\{\sum_{i=1}^{k} (a_{p(i)} - a_{p(i+1)})^2 + (a_{p(k+1)} - a_{p(1)})^2\Big\}$$

where the sup is over all sequences $(p(i))_1^{k+1}$ with $0 \leqslant p(1) < p(2) < \ldots < p(k+1)$. By convention $a_0 = 0$ always.

The space J^{**} is the collection of all real sequences with bounded square variation. It is one dimension larger than J in that it also includes the constant sequences like $\sum e_n$. If $p = \|\cdot\|$ or $|||\cdot|||$ and if $\sum a_n e_n \in J^{**}$, then the corresponding norm in J^{**} is given by $p(\sum a_n e_n) = \lim_N p(\sum_1^N a_n e_n)$.

In the norm $|||\cdot|||$, J is isometric to J^{**} [10, p. 25] while in $\|\cdot\|$, the only onto isometries on J are plus or minus the identity [1].

An element x in a normed space X is said to be *extreme* if whenever $x = (y+z)/2$ and $y \neq z$, then $\max(\|y\|, \|z\|) > \|x\|$.

PROPOSITION 3.1. *Let $x = \sum a_n e_n \in J^{**}$ and let $C_{ij} = (a_i - a_j)^2$ for $0 \leqslant i < j$. Then the C_{ij} satisfy the hypotheses of Theorem 2.1 and x is an extreme point of the unit ball of J^{**} with $\|\cdot\|$ if and only if $E(0, \infty) = \mathbf{N}$.*

Proof. The bounded variation of C_{ij} follows from the fact that the norm is $< \infty$. If $r < s < m$ and $B(r, m) = r$, then either $a_r < a_s < a_m$ or $a_r > a_s > a_m$. Take the first say. Then the hypothesis of the nice-at-infinity condition implies $a_r < a_s < a_{m(1)} < a_{m(2)} < \ldots$ and

$$N(r, s) + (a_{m(1)} - a_s)^2 < (a_{m(1)} - a_r)^2.$$

Since $(a_{m(k)} - a_r)^2 - (a_{m(k)} - a_s)^2$ is increasing as $k \to \infty$, C_{ij} is nice at infinity.

To see that the glueing condition is true, let $0 \leqslant r < s < t < u$ and $m = a_r$, $n = a_s$, $p = a_t$ and $q = a_u$. The hypothesis on the glueing condition implies $m > n > p > q$ or $m < n < p < q$ and the conclusion is that

$$(m-p)^2 + (n-q)^2 \leqslant (m-q)^2 + (n-p)^2.$$

We may assume $q = 0$ by translating by $-q$. So this last inequality is equivalent to $-2mp \leqslant -2np$, which is true under either hypothesis.

Now suppose there is a $t \notin E(0, \infty)$. We claim there is a $d > 0$ s.t. both $\|x \pm de_t\| \leqslant 1$. If $(p(i))_1^{k+1}$ is a sequence which misses t, then the approximation to the norm is the same for $x \pm de_t$ as it is for x. On the other hand, if $(p(i))_1^{k+1}$ includes t, then by Proposition 2.3 it falls short of norming x by some fixed $\delta_t > 0$. Using this sequence to norm $x \pm de_t$ can increase the norm squared by at most

$$2\left[(2\sup|a_n| + d)^2 - (2\sup|a_n|)^2\right],$$

which is $< \delta$ for small enough d. Hence x is not an extreme point.

Conversely; if $E(0, \infty) = \mathbf{N}$ but x is not extreme, then there are $y = \sum y_n e_n$ and $z = \sum z_n e_n$ so that $a_n = (y_n + z_n)/2$ and $\|y\|, \|z\| \leqslant \|x\|$. The triangle inequality implies $\|y\| = \|z\| = \|x\|$.

Consider the sequences (ξ_n), (η_n) and (ζ_n) in l_2, the space of squared summable sequences with $\left\|\sum \xi_n e_n\right\|_2^2 = \sum \xi_n^2$. Define

$$\xi_1 = \lim_n a_n, \qquad \eta_1 = \lim_n y_n, \qquad \zeta_1 = \lim_n z_n,$$

$$\xi_{n+1} = a_n - a_{n+1}, \qquad \eta_{n+1} = y_n - y_{n+1}, \qquad \zeta_{n+1} = z_n - z_{n+1}.$$

It follows that $2\|x\|_J^2 = \left\|\sum \xi_n e_n\right\|_2^2$ and that $2\|y\|_J^2 \geqslant \left\|\sum \xi_n e_n\right\|_2^2$ and $2\|z\|_J^2 \geqslant \left\|\sum \xi_n e_n\right\|_2^2$. But since $\xi_n = (\eta_n + \zeta_n)/2$ and every point in l_2 is extreme, at least one of $\left\|\sum \eta_n e_n\right\|_2$ or $\left\|\sum \zeta_n e_n\right\|_2$ is strictly larger than $\left\|\sum \xi_n e_n\right\|_2$. This contradiction completes the proof. ∎

COROLLARY 3.2. *If $\sum a_n e_n \in J^{**}$ is not extreme in the norm $\|\cdot\|$, then for some m and $\delta > 0$*

$$\left\|\sum a_n e_n\right\| = \left\|\sum a_n e_n \pm \delta e_m\right\|.$$

Remarks. 1. Proposition 3.1 does not hold for $|||\cdot|||$. Indeed, it is easy to check that $e_2 - e_3$ is extreme in $\|\cdot\|$ but not in $|||\cdot|||$.

2. If $|||\sum a_n e_n||| = 1$ and $\lim_n a_n = 1$, then Proposition 3.1 does hold for $|||\cdot|||$, i.e. $\sum a_n e_n$ is extreme if and only if $E(0, \infty) = N$.

Corollary 3.3. *The set of extreme points in J or J^{**} with the norm $\|\cdot\|$ is a closed nowhere dense set.*

Proof. If $x = \sum a_i e_i$ is not extreme and $t \notin E(0, \infty)$ for x, then there is $\varepsilon > 0$ so that if $\|x-y\| < \varepsilon$, then $t \notin E(0, \infty)$ for y. It follows that y is not extreme.

If $x = \sum a_i e_i$ is extreme, then

$$x_1^n = \sum_{i=1}^{n} a_i e_i + (2e_{n+1} + e_{n+2})/n \to x$$

and the x_n are not extreme. ∎

Example 3.4. Define $a_{2n} = 1-(2/3)^n$ and $a_{2n-1} = 1-2^{n-1}/3^n$ $(a_0 = 0)$ and let $x_n = \sum_{i=1}^{2n} a_i e_i + \sum_{i=2n+1}^{\infty} e_i$ and $x_\infty = \sum a_i e_i$. Then x_n is extreme in $\|\cdot\|$ and $|||\cdot|||$ for $1 \leqslant n \leqslant \infty$.

Proof. Note that if $0 < a > b < c$, then $ae_1 + be_2 + c\sum_{i=3}^{\infty} e_i$ is extreme if and only if $c^2 \leqslant a^2 + (a-b)^2 + (b-c)^2$, or $c \leqslant (a^2+b^2-ab)/b = a + (a-b)^2/b$. In particular, the choice $a = 2/3$, $b = 1/3$ and $c = 1$ works and so x_1 is extreme.

Suppose x_m is extreme; then we will show that x_{m+1} is extreme. Apply the algorithm with costs $C_{ij} = (b_i - b_j)^2$ where $x_{m+1} = \sum b_i e_i$. Since x_m is extreme, if x_{m+1} is not extreme, then it must be the case that $B(0, 2m+1) = j \neq 2m$. Let $K = N(j, 2m)$ and hence

$$(a_{2m+1} - a_j)^2 - (a_{2m+1} - a_{2m})^2 - K > 0.$$

But $f(y) = (y-a_j)^2 - (y-a_{2m})^2 - K$ is increasing for $y > a_{2m} > a_j$. But using $y = 1$ this implies that x_m is not extreme, i.e. the algorithm applied to costs derived from x_m also has $B(0, 2m+1) \neq 2m$.

Finally, if x_∞ is not extreme, then again $B(0, j+1) \neq j$ for some j. But this would also be the case for x_j and hence x_j would not be extreme. ∎

Remark. In some sense these results say (in a vague way) that a vector is extreme if and only if it is locally extreme. However, it seems "locally" cannot be replaced with some bounded number.

For example, if $\sum b_n e_n$ is extreme, then neither $b_n < b_{n+1} < b_{n+2}$ nor $b_n > b_{n+1} > b_{n+2}$ is possible, but the converse is false since $(1+\varepsilon)e_1 + (1-\varepsilon)e_2 + 2e_3$ is not extreme. Another example is when $d = \sqrt{6}/3$; then $(1+d)e_1 + de_2 + (1+2d)e_3 + 2de_4 + (1+3d)e_5$ is extreme but $(1+d)e_1 + de_2 + (1+3d)(e_3+e_4+e_5)$ is not extreme. Thus the process in Example 3.4 does not create all extreme points even of the form $\sum a_n e_n$ with $\|\sum a_n e_n\| = 1$ and $\lim a_n = 1$.

§ 4. **Support functionals.** If $P = (p(i))_{i=0}^{\infty}$ is an increasing integer sequence with $p(0) = 0$, then define $T_P\colon J \to l_2$ by

$$T_P\left(\sum a_n e_n\right) = \sum_{i=1}^{\infty} (a_{p(i-1)} - a_{p(i)})\, u_{p(i)}/\sqrt{2},$$

where (u_i) is the usual basis for l_2. Besides requiring $a_0 = 0$, let us add the point ∞ to the integers and require $a_\infty = 0$. If $P = (p(i))_{i=0}^{k}$ is a finite increasing integer sequence with $k \geqslant 2$ and $0 = p(0)$, $p(k) = \infty$, we also define $T_P\colon J \to l_2$ by

$$T_P\left(\sum a_n e_n\right) = \sum_{i=1}^{k} (a_{p(i-1)} - a_{p(i)})\, u_{p(i)}/\sqrt{2}.$$

Clearly $\|T_P x\| \leqslant \|x\|$ and for each $x \in J$, there is a P with $\|T_P x\| = \|x\|$.

Let (v_i), (f_i) be the coefficient functionals to (u_i), (e_i) respectively. Simple algebra shows

$$T_P^*\left(\sum b_n v_n\right) = \begin{cases} \sum_{i=1}^{\infty} (b_{p(i+1)} - b_{p(i)})\, f_{p(i)}/\sqrt{2}, & \text{or} \\ \sum_{i=1}^{k-1} (b_{p(i+1)} - b_{p(i)})\, f_{p(i)}/\sqrt{2} \end{cases}$$

depending on the finite or infinite size of P. Let $I\colon l_2 \to l_2$ be the map for which $Iu_n = v_n$ and consider $S_P = T_P^* I T_P\colon J \to J^*$. We have $\|S_P\| = 1$, $(S_P x)(x) = \|T_P x\|^2$, and

$$S_P\left(\sum a_n e_n\right) = \begin{cases} \sum_{i=1}^{\infty} (2a_{p(i)} - a_{p(i-1)} - a_{p(i+1)})\, f_{p(i)}/2, & \text{or} \\ \sum_{i=1}^{k-1} (2a_{p(i)} - a_{p(i-1)} - a_{p(i+1)})\, f_{p(i)}/2 \end{cases}$$

again depending on if P is finite or infinite.

Suppose $x \in J$ with $\|x\| = 1$ and P is so that $\|T_P x\| = 1$. Then $x^* = S_P x$ is a support functional at x. A *support functional* at x is an x^* in the dual with norm one so that $x^*(x) = 1$. Indeed, $x^*(x) = 1$, so that $\|x^*\| \geqslant 1$ and $\|x^*\| \leqslant \|S_P\|\,\|x\| = 1$. The unit vector x is said to be a *smooth point* if there is a unique support functional at x. The unit vector x is said to be an *exposed point* if there is a support functional x^* with the property that $x^*(y) < 1$ for $\|y\| \leqslant 1$ and $y \neq x$. We collect a couple of easy results in the next proposition.

Proposition 4.1. *Let $x \in J$ with $\|x\| = 1$.*

(A) *x is extreme if and only if x is exposed.*

(B) *If there are $P \neq Q$ so that $\|T_P x\| = \|T_Q x\| = 1$, then x is not a smooth point.*

Proof. Exposed points are extreme so suppose x is an extreme point of J. Let $P = N$, then $\|T_P x\| = 1$, and consider $x^* = S_P x$. If $y \in J$ with $\|y\| \leqslant 1$, then $\|T_P y\| \leqslant 1$, $x^*(y) = \langle T_P x, T_P y\rangle < 1$ if $T_P y \neq T_P x$, where $\langle\cdot,\cdot\rangle$ is the inner product on l_2. But since $P = N$, $x \neq y$ if and only if $T_P x \neq T_P y$. So (A) is complete.

Now (B) would be true if $S_P x \neq S_Q x$. Suppose $j \in P \setminus Q$, $0 < j < \infty$. Then the f_j coefficient of $S_Q x$ is zero and the f_j coefficient of $2S_P x$ is $2a_{p(i)} - a_{p(i-1)} - a_{p(i+1)}$, for some i. If $2a_{p(i)} - a_{p(i-1)} - a_{p(i+1)} \neq 0$, then we are done. Otherwise, $a_{p(i)} = (a_{p(i-1)} + a_{p(i+1)})/2$, so if $a_{p(i)} \neq a_{p(i-1)}$, then the norm of x would be increased by deleting the singleton $\{p(i)\}$ from P. This would contradict the assumption that $\|T_P x\| = 1$. Thus $a_{p(i)} = a_{p(i-1)} = a_{p(i+1)}$. We will complete the proof by showing there is another support functional different from $S_P x$.

Either there is $n < i-1$ or $m > i+1$ so that $a_{p(n)} \neq a_{p(n+1)} = a_{p(n+2)} = \ldots = a_{p(i)}$ or $a_{p(i)} = \ldots = a_{p(m-1)} \neq a_{p(m)}$ since $x \neq 0$. It follows that the coefficient of either $f_{p(n+1)}$ or $f_{p(m-1)}$ is nonzero in $S_P x$. Let y^* be $S_P x$ with the nonzero coefficient of either $f_{p(n+1)}$ or $f_{p(m-1)}$ moved to $f_{p(i)}$. Clearly, $y^* \neq S_P x$, $y^*(x) = 1$ and since (f_n) is invariant under spreading [3] we have $\|y^*\| = \|S_P x\| = 1$. ■

Remarks. 1. Note that f_2 does not expose the extreme point e_2. Here $S_N e_2 = (-f_1 + 2f_2 - f_3)/2$.

2. If x is in the linear span of $\{e_i\}$, then x is not smooth. Indeed, if $x = \sum a_i e_i$, then eventually $a_i = 0$ and so there are uncountably many P with $\|T_P x\| = \|x\|$. Thus the set of smooth points is nowhere dense. (The set of smooth points is always dense in the unit sphere of a separable space; see [5, p. 42].)

Open questions. 1. We do not know if the converse of Proposition 4.1 is true. That is, if $\|x\| = 1 = \|T_P x\| = \|T_Q x\|$ implies $P = Q$, then is x smooth? The difficulty seems to be knowing how vectors in J^* are normed.

2. We also do not know if each extreme point of J is strongly exposed. (Here a norm one extreme point x is strongly exposed if for each $\varepsilon > 0$ there is a $\delta > 0$ so that $\|y\| \leqslant 1$ and $S_N x(y) > 1 - \delta$ imply $\|x - y\| < \varepsilon$.) The difficulty here is if $z = x - y = \sum a_n e_n$ is so that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_m \geqslant a_{m+1} \geqslant \ldots$; then in certain cases $\|x \pm z\| \sim \|x\|$. (This is much like the construction of l_∞^n's in J; see [6].)

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