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An example of a nonseparable Banach algebra without nonseparable commutative subalgebras

by

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Abstract. The aim of the paper is to construct a Banach algebra as in the title. It provides the answer to a question raised by W. Żelazko.

Let A be a set which we will call the *alphabet*. With the letters of this alphabet we form *words*, finite sequences with values in A. Let S denote the set of all words, i.e. $S = \bigcup_{n\in N} A^n$ where by definition $A^0 = \{\emptyset\}$, the empty word. If $s \in A^n$, we say that the *length* of s, $l(s)$, is equal to n. The product $s_1 s_2$ of two words is obtained by first writing the letters of s_1 and then those of s_2. We also put $\emptyset s = s\emptyset = s$ for all $s \in S$. The set S equipped with that product is called the *free semigroup* (with unit) *over the alphabet* A. It is noncommutative if $\operatorname{card} A \geqslant 2$.

Consider $\boldsymbol{R}^S$, the vector space of all real functions defined on S. In $\boldsymbol{R}^S$ we may also define multiplication. For $s \in S$, let δ_s denote the function equal to 1 at s and zero elsewhere. Thus any $x \in \boldsymbol{R}^S$ is represented as $\sum_{s\in S} x(s)\,\delta_s$. For $s_1, s_2 \in S$ define $\delta_{s_1} * \delta_{s_2} = \delta_{s_1 s_2}$ and for $x, y \in \boldsymbol{R}^S$ let

$$(1)\qquad x * y = \sum_{s_1, s_2 \in S} x(s_1)\, y(s_2)\, \delta_{s_1} * \delta_{s_2} = \sum_{s\in S} \Big(\sum_{0 \leqslant k \leqslant l(s)} x(s|k)\, y(s-k) \Big) \delta_s$$

($s|k$ is the word obtained by omitting the letters after the kth and $s-k$ is the sequence obtained by omitting the first k letters). The inner sum is finite and $x * y \in \boldsymbol{R}^S$. It is a matter of simple checking that the multiplication $*$ is associative and distributive with respect to addition, with $\delta_\emptyset$ as the unit element.

Thus $(\boldsymbol{R}^S, +, *)$ is an algebra with unit, and if $\operatorname{card} A \geqslant 2$ it is not commutative.

This algebra does not have divisors of zero. Indeed, let $x, y \in \boldsymbol{R}^S \setminus \{0\}$ and take an $s_1 \in S$ of minimal length such that $x(s_1) \neq 0$; s_2 is defined similarly for y. We have

$$x*y(s_1 s_2) = \sum_{k \leqslant l(s_1)+l(s_2)} x(s_1 s_2 | k)\, y(s_1 s_2 - k) = x(s_1)\, y(s_2) \neq 0,$$

thus $x*y \neq 0$.

Let $l^1(S)$ be as usual the Banach space of those $x \in \boldsymbol{R}^S$ for which $\|x\| = \sum_{s \in S} |x(s)| < \infty$; $\| \ \|$ is the norm in $l^1(S)$, and by (1), $\|x*y\| \leqslant \|x\|\,\|y\|$. Hence $(l^1(S), +, *)$ is a Banach algebra. We will now show that if S is uncountable (equivalently A is uncountable), then $l^1(S)$ is the algebra we are looking for.

For a set $X \subseteq S$, we write $\mathrm{Lett}(X)$ for the set of all letters appearing in words from X.

LEMMA. *If* $x, y \in l^1(S) \setminus \{0\}$, $x(\emptyset) = y(\emptyset) = 0$ *and* $x*y = y*x$, *then* $\mathrm{Lett}(\mathrm{supp}\, x) = \mathrm{Lett}(\mathrm{supp}\, y)$.

Proof. Assume, by contradiction, that e.g. $\mathrm{Lett}(\mathrm{supp}\, y) \nsubseteq \mathrm{Lett}(\mathrm{supp}\, x)$, i.e. there is a letter a which appears in some word s with $y(s) \neq 0$, but $x(t) = 0$ for all words t containing a.

Let s_1 be a word of minimal length with $x(s_1) \neq 0$.

If a letter a appears in a word s, set

$$r_a(s) = \min\{k \in N:\ s(k) = a\},$$

and define

$$r_a = \min\{r_a(s):\ a \text{ appears in } s \text{ and } y(s) \neq 0\}.$$

Let s_2 be a word of minimal length such that a appears in s_2, $y(s_2) \neq 0$ and $r_a(s_2) = r_a$. We thus obtain

$$\begin{aligned} y*x(s_2 s_1) &= \sum_{0<k<l(s_2)+l(s_1)} y(s_2 s_1 | k)\, x(s_2 s_1 - k) \\ &= \sum_{0<k<r_a} \ldots + \sum_{r_a \leqslant k < l(s_2)} \ldots + y(s_2)\, x(s_1) + \sum_{l(s_2)<k<l(s_2)+l(s_1)} \ldots \\ &= y(s_2)\, x(s_1), \end{aligned}$$

while

$$\begin{aligned} x*y(s_2 s_1) &= \sum_{0<k<l(s_2)+l(s_1)} x(s_2 s_1 | k)\, y(s_2 s_1 - k) \\ &= \sum_{0<k<r_a} \ldots + \sum_{r_a \leqslant k < l(s_2)+l(s_1)} \ldots = 0. \end{aligned}$$

Hence $x*y \neq y*x$ and this contradiction proves the lemma.

Assume now that X is a commutative subalgebra of $l^1(S)$. If $x_0 \in X$ is not a multiple of the unit, then $I_0 = \mathrm{Lett}(\mathrm{supp}\, x_0)$ is countable. By the lemma for any $y \in X$, $\mathrm{Lett}(\mathrm{supp}\, y) \subseteq I_0$ (note that x and y commute iff $\delta_\emptyset + x$ and y commute). Hence $X \subseteq l^1(S_0)$ where S_0 is the subsemigroup generated by I_0. S_0 is countable, thus $l^1(S_0)$ is separable, and so is X.

Remark 1. The lemma says more, namely the commutant of any element $x \in l^1(S)$, not a multiple of the unit, is separable.

Remark 2. It is easy to see that for any semigroup $\mathscr{S}$ we may define on the Banach space $l^1(\mathscr{S})$ the "$*$" multiplication and $l^1(\mathscr{S})$ becomes a Banach algebra. If we take for $\mathscr{S}$ the set of countable ordinal numbers with addition defined as addition of ordinal types (see [KM] for precise definition), then we obtain a Banach algebra with the property as in the title of this note. It was in fact the first example solving the problem. It seems possible that any uncountable semigroup with only countable commutative subsemigroups works too.

Reference

[KM] K. Kuratowski and A. Mostowski, *Set Theory*, PWN, Warszawa 1976.

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