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Rate of convergence of conditional entropies for some maps of an interval

by

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Abstract. We consider piecewise monotone maps f of an interval admitting an invariant probability measure μ absolutely continuous with respect to the Lebesgue measure. For f from a large class of such mappings with nonpositive Schwarzian derivative we prove the exponential convergence of conditional entropies to the measure-theoretic entropy of f. Moreover, the exponential convergence in $L^1(\mu)$ of the information functions of conditional entropies is proved.

1. Introduction. Let f be a piecewise monotone map of an interval I into itself admitting an invariant probability measure μ, absolutely continuous with respect to the Lebesgue measure λ. Let $\mathcal{A}$ be the natural partition of I into the intervals of monotonicity of f. We then have $h_\mu(f) = h_\mu(f, \mathcal{A})$. The latter quantity can be computed either from the definition: $h_\mu(f, \mathcal{A}) = \lim_n n^{-1} H_\mu(\mathcal{A}_n)$, or using conditional entropies: $h_\mu(f, \mathcal{A}) = \lim_n H_\mu(\mathcal{A} \mid f^{-1} \mathcal{A}_n)$ (where $\mathcal{A}_n = \bigvee_{i=0}^{n-1} f^{-i} \mathcal{A}$). The first method gives usually very low rate of convergence (of order $1/n$; see [1], [3]). It was conjectured in [1] that the second method gives usually exponential convergence. This was proved in [3] for unimodal smooth maps with negative Schwarzian derivative, nondegenerate critical point and "fully developed chaos" (i.e. 2-to-1 everywhere except one point).

The aim of this paper is to generalize the results of [3] to the class of maps considered in [2, 4, 5]. The methods of [3] base on the fact that the density of μ has only two singularities. This made it possible to make straightforward estimates. Here, in a more general case, the situation is more complicated. In general, the singularities of the density of μ are dense in some Cantor set. Therefore we have to use quite another technique, developed in [5].

2. Assumptions and notation. Let I be a closed interval, A its finite subset containing the endpoints of I, and $f\colon I \to I$ a continuous map which is strictly monotone on components of $I \setminus A$ and satisfies the following conditions ([2]):

(i) f is of class C^3 on $I \setminus A$.

(ii) $f' \neq 0$ on $I \setminus A$.

(iii) $Sf \leqslant 0$ on $I \setminus A$, where Sf denotes the Schwarzian derivative of f.

(iv) If $f^p(x) = x$, then $|(f^p)'(x)| > 1$.

(v) There exists a neighbourhood U of A such that for all $a \in A$ and $n \geqslant 0$, $f^n(a) \in (I \setminus U) \cup A$.

(vi) For every $a \in A$ there exists a neighbourhood U_a of a and constants $\alpha, \omega, \delta > 0$, $u \geqslant 0$ such that

$$\alpha |x-a|^u \leqslant |f'(x)| \leqslant \omega |x-a|^u, \quad |f''(x)| \leqslant \delta |x-a|^{u-1}$$

for $x \in U_a$.

The assumption of continuity of f is made only for simplicity. What follows remains true in the case when f has a finite number of discontinuities (cf. [5]).

Under the above assumptions M. Misiurewicz [2] proved the existence of an f-invariant probability measure μ absolutely continuous with respect to the Lebesgue measure λ. We will denote its density by φ.

We put $B = \overline{\bigcup_{n=1}^{\infty} f^n(A)}$. This is the set of singularities of φ, which is a Cantor set of μ-measure zero (see [2]). On each component of $I \setminus B$ the density φ is convex and of class C^{r-1} if f is of class C^r (see [2] and [4]). Hence under our assumptions φ is of class C^2 on $I \setminus B$.

Let $\mathscr{A}$ be the partition of I by the points of A. We will write $\mathscr{A}_n = \bigvee_{i=0}^{n-1} f^{-i} \mathscr{A}$ for $n \in N$, and for $x \in I$, $A_n(x)$ will be the atom of $\mathscr{A}_n$ which contains x. Notice that $\mathscr{A}_1 = \mathscr{A}$ by the definition.

3. Statement of the results. The partition $\mathscr{A}$ is the natural partition of I into the pieces of monotonicity of f. Hence the entropy of f can be computed as the limit of the conditional entropies $H_\mu(\mathscr{A} \mid f^{-1} \mathscr{A}_n)$. We will prove the following

THEOREM 1. *$H_\mu(\mathscr{A} \mid f^{-1} \mathscr{A}_n)$ tends exponentially to $h_\mu(f)$ as $n \to \infty$.*

By the Rokhlin formula we have

$$h_\mu(f) = \int_I \log \frac{\varphi(f(x))|f'(x)|}{\varphi(x)} d\mu.$$

We also have the following integral fomulas for the conditional entropies:

$$H_\mu(\mathscr{A} \mid f^{-1} \mathscr{A}_n) = -\int_I \log \frac{\mu\big(A_1(x) \cap f^{-1}(A_n(f(x)))\big)}{\mu\big(f^{-1}(A_n(f(x)))\big)} d\mu(x)$$

$$= \int_I \log \frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))} d\mu(x).$$

Hence to prove Theorem 1 it is enough to prove

THEOREM 2. *The information functions $\log \frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}$ tend to $\log \frac{\varphi(f(x))|f'(x)|}{\varphi(x)}$ exponentially in $L^1(\mu)$.*

Theorem 2 and the Borel–Cantelli Lemma imply that the functions $\log \frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}$ converge exponentially to $\log \frac{\varphi(f(x))|f'(x)|}{\varphi(x)}$ μ-almost everywhere.

The next sections contain a full proof of Theorem 2.

4. Construction of good atoms. Fix some neighbourhood U of A satisfying assumptions (v) and (vi). The backward images of A being dense in I, we can assume U is a union of intervals with endpoints in $\bigcup_{i=1}^{N} f^{-i}(A)$ for some sufficiently large $N \in N$ and each of these intervals contains exactly one point of A.

For any $n \in N$ we define

$$E_n = \{x \in I \colon f^i(x) \notin U \text{ for } i = 0, 1, \ldots, n-1\},$$

which is a neighbourhood of the set $B \setminus A$.

LEMMA 1. *For all $n \in N$ and $m \geqslant n+N$, the set E_n is the union of some atoms of the partition $\mathscr{A}_m$.*

Proof. By the choice of U and the definition of E_n, each E_n is a finite union of intervals with endpoints in $\bigcup_{i=0}^{n+N-1} f^{-i}(A)$. ■

Let $F = \sup_{x \in I} |f'(x)|$ (obviously $F > 1$). Let $d_1 > 0$, $\gamma_1 \in (0, 1)$ be the constants from [5, Th. 2]. Fix $p \in N$ such that $\gamma_1 F^{1/p} < 1$. If $n \geqslant (2+N)p$, then $[n/p] - N \geqslant 1$ and we define $\mathscr{E}_n = E_{[n/p]-N}$.

LEMMA 2. *For every $n \geqslant (2+N)p$ the set $\mathscr{E}_n$ is the union of some atoms of the partition $\mathscr{A}_n$.*

Proof. By Lemma 1, $\mathscr{E}_n$ is the union of some atoms of the partition $\mathscr{A}_{[n/p]}$. But it is also the union of some atoms of the partition $\mathscr{A}_n$ because $\mathscr{A}_n$ is finer than $\mathscr{A}_{[n/p]}$. ■

Now we fix some $k \in N$. In view of [5, Th. 2] there exist constants $c_0, c_1 > 0$ such that for every $n \in N$ we can choose a neighbourhood $\mathscr{K}_n$ of A which is the union of some atoms of $\mathscr{A}_n$ and

$$\lambda(\mathscr{K}_n) \leqslant c_1 \gamma_1^{n/k}, \quad \operatorname{dist}(I \setminus \mathscr{K}_n, A) \geqslant c_0 \gamma_1^{n/k}. \tag{1}$$

Let $\mathscr{B}_n = \mathscr{E}_n \cup f^{-1}\mathscr{E}_n \cup \mathscr{K}_n \cup (I \setminus \operatorname{supp}\mu)$. Atoms of $\mathscr{A}_n$ contained in $\mathscr{B}_n$ will be called *bad*. Atoms of $\mathscr{A}_n$ which are not bad will be called *good*.

LEMMA 3. *For every $n \geqslant (2+N)p$ the set $\mathscr{B}_n$ is the union of some atoms of the partition $\mathscr{A}_{n+1}$.*

Proof. In view of Lemma 2 and the definition of $\mathscr{K}_n$, the set $\mathscr{E}_n \cup f^{-1}\mathscr{E}_n \cup \mathscr{K}_n$ is the union of some atoms of the partition $\mathscr{A}_{n+1}$. This set is a neighbourhood of B. The set $I \setminus \operatorname{supp}\mu$ is a finite union of some open intervals with endpoints in B (see [2]). These intervals being contained in the interior of $\mathscr{B}_n$, the endpoints of components of $\mathscr{B}_n$ are the endpoints of some components of $\mathscr{E}_n \cup f^{-1}\mathscr{E}_n \cup \mathscr{K}_n$, which completes the proof. ∎

LEMMA 4. *There exist $\gamma_2 \in (0, 1)$ and $c_2 > 0$ such that $\mu(\mathscr{B}_n) \leqslant c_2 \gamma_2^n$.*

Proof. This follows from (1), [2, Prop. 2.1] and [5, Ths. 2, 3]. ∎

LEMMA 5. *There exists $c > 0$ such that*

$$\lambda(\mathscr{B}_n \cap \operatorname{supp}\mu) \leqslant \frac{c_2}{c}\gamma_2^n.$$

Proof. Let $c = \inf_{\operatorname{supp}\mu}\varphi$. By [2, Th. (6.3)], $c > 0$, and using Lemma 3 we get

$$c\lambda(\mathscr{B}_n \cap \operatorname{supp}\mu) \leqslant \int_{\mathscr{B}_n \cap \operatorname{supp}\mu} \varphi\, d\lambda = \mu(\mathscr{B}_n) \leqslant c_2\gamma_2^n. \ ∎$$

5. Estimations on good atoms. Now we will investigate the behaviour of f', φ and $\varphi \circ f$ on good atoms.

Set $G_n = I \setminus \mathscr{B}_n$. We will assume $n \geqslant (2+N)p$ in the whole section.

LEMMA 6. *There exist $\gamma_3 \in (0, 1)$ and $c_3 > 0$ such that for any x, y belonging to a good atom of $\mathscr{A}_{n+1}$*

$$\left|\frac{f'(x)}{f'(y)} - 1\right| \leqslant c_3\gamma_3^n.$$

Proof. Let x, y belong to a good atom of $\mathscr{A}_{n+1}$. By assumption (iii), $|f'|$ has no positive strict local minima, so the infimum of $|f'|$ on (x, y) is $|f'(x)|$ or $|f'(y)|$, say $|f'(x)|$. Hence $|f(x) - f(y)| \geqslant |x-y|\,|f'(x)|$ and

(2)
$$\left|\frac{f'(y)}{f'(x)} - 1\right| \leqslant \frac{\sup_I |f''|}{(f'(x))^2}|f(x) - f(y)|.$$

Now we estimate $|f'(x)|$ from below. If $x \notin U$, then $|f'(x)|$ is bounded from below by a positive constant. If $x \in U$, then in view of (vi)

$$|f'(x)| \geqslant \alpha\big(\operatorname{dist}(G_n, A)\big)^u,$$

where $\alpha > 0$, $u \geqslant 0$ are constants independent of x. It follows from (1) that $\operatorname{dist}(G_n, A) \geqslant c_0\gamma_1^{n/k}$, so

$$|f'(x)| \geqslant \alpha c_0^u (\gamma_1^{u/k})^n.$$

But k was fixed arbitrarily and we can assume that k is so large that $u/k < \frac{1}{4}$. We thus obtain

$$|f'(x)| \geqslant \alpha c_0^u \gamma_1^{n/4},$$

which together with (2) and [5, Th. 2] gives the desired estimate. ∎

LEMMA 7. *There exist $\gamma_4 \in (0, 1)$ and $c_4 > 0$ such that for any x, y belonging to a good atom of $\mathscr{A}_{n+1}$*

$$\left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \leqslant c_4\gamma_4^n, \qquad \left|\frac{\varphi\circ f(x)}{\varphi\circ f(y)} - 1\right| \leqslant c_4\gamma_4^n.$$

Proof. Let x, y belong to a good atom of $\mathscr{A}_{n+1}$. By the definition of good atoms, $x, y \notin E_{[n/p]-N}$. An argument similar to that we used in the proof of [5, Lemma 12] proves the estimate

(3)
$$|\varphi(x) - \varphi(y)| \leqslant \varphi'(v)\,|x-y| \leqslant dF^{n/p}\gamma_1^n,$$

where $d > 0$ is some constant independent of n, x, y and of the atom including x, y. By the definition of good atoms, also $f(x), f(y) \notin E_{[n/p]-N}$ and using the same argument gives the estimate

(4)
$$|\varphi\circ f(x) - \varphi\circ f(y)| \leqslant dF^{n/p}\gamma_1^n.$$

We put $\gamma_4 = F^{1/p}\gamma_1$ ($\gamma_4 \in (0, 1)$). Good atoms are contained in $\operatorname{supp}\mu$, a set invariant under f, so $\varphi(x), \varphi(y), \varphi\circ f(x), \varphi\circ f(y) \geqslant c$ ($c = \inf_{\operatorname{supp}\mu}\varphi > 0$). Applying this to (3) and (4) we obtain the desired estimates. ∎

6. Proof of Theorem 2. Recall that we have to estimate

$$\int_I \left|\log\frac{\mu\big(A_n(f(x))\big)}{\mu(A_{n+1}(x))} - \log\frac{\varphi\circ f\cdot|f'|}{\varphi}(x)\right| d\mu(x).$$

We assume $n \geqslant (2+N)p$. First we will prove

LEMMA 8. *There exist constants $\gamma_5 \in (0, 1)$, $c_5 > 0$ such that*

$$\int_{G_n} \left|\log\frac{\mu\big(A_n(f(x))\big)}{\mu(A_{n+1}(x))} - \log\frac{\varphi\circ f\cdot|f'|}{\varphi}(x)\right| d\mu(x) \leqslant c_5\gamma_5^n.$$

Proof. If $x \in G_n$, then by the definition of G_n, $A_{n+1}(x)$ is a good atom, so in particular its endpoints do not belong to A and $f\big(A_{n+1}(x)\big) = A_n\big(f(x)\big)$.

Hence

$$\frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}=\frac{\mu(f(A_{n+1}(x)))}{\mu(A_{n+1}(x))}=\frac{\varphi\circ f(x)\cdot|f'(x)|}{\varphi(x)}\times\frac{\int_{A_{n+1}(x)}\frac{\varphi\circ f(y)}{\varphi\circ f(x)}\cdot\frac{|f'(y)|}{|f'(x)|}d\lambda(y)}{\int_{A_{n+1}(x)}\frac{\varphi(y)}{\varphi(x)}d\lambda(y)}.$$

If we set

$$J_n(x)=\frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}\cdot\frac{\varphi(x)}{\varphi\circ f(x)\cdot|f'(x)|},$$

we obtain

$$J_n(x)=\frac{\int_{A_{n+1}(x)}\frac{\varphi\circ f(y)}{\varphi\circ f(x)}\cdot\frac{|f'(y)|}{|f'(x)|}d\lambda(y)}{\int_{A_{n+1}(x)}\frac{\varphi(y)}{\varphi(x)}d\lambda(y)}.$$

In view of Lemmas 6 and 7 there exist constants $\gamma_5\in(0,1)$, $c_5>0$ such that

$$|\log J_n(x)|\leqslant c_5\gamma_5^n,$$

which completes the proof. ■

Now we will estimate integrals over bad atoms.

LEMMA 9. *There exist constants* $\gamma_6\in(0,1)$, $c_6>0$ *such that*

$$\int_{\mathscr{B}_n}\log\frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}d\mu(x)\leqslant c_6\gamma_6^n.$$

Proof. Let $\mathscr{D}_n$ be the union of all atoms of $\mathscr{A}_n$ which contain images of atoms belonging to $\mathscr{B}^n\cap\operatorname{supp}\mu$. In this proof Δ will always denote an atom of $\mathscr{A}_n$ contained in $\mathscr{D}_n$. We have

$$\int_{\mathscr{B}_n}\log\frac{\mu(A_n(f_n(x)))}{\mu(A_{n+1}(x))}d\mu=\int_{\mathscr{B}_n\cap\operatorname{supp}\mu}\log\frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}d\mu$$
$$=\sum_{\Delta}\int_{f^{-1}\Delta}-\log\frac{(A(x)\cap f^{-1}(A_n(f(x))))}{\mu(f^{-1}(A_n(f(x))))}d\mu$$
$$=\sum_{\Delta}\mu(f^{-1}\Delta)\left[\frac{1}{\mu(f^{-1}\Delta)}\int_{f^{-1}\Delta}-\log\frac{\mu(A(x)\cap f^{-1}\Delta)}{\mu(f^{-1}\Delta)}d\mu\right].$$

Now we remark that the expression in brackets is equal to the entropy of the partition $\mathscr{A}$ restricted to the set $f^{-1}\Delta$ with respect to the probability measure on $f^{-1}\Delta$ induced by μ. Hence this expression is bounded by $\log(\operatorname{card}\mathscr{A})$ and we obtain

$$\int_{\mathscr{B}_n}\log\frac{\mu(A_n(f(x)))}{\mu(A_{n+1}(x))}d\mu\leqslant\operatorname{card}\mathscr{A}\cdot\mu(\mathscr{D}_n).$$

To complete the proof it suffices to estimate $\mu(\mathscr{D}_n)$.

There are two types of atoms in $\mathscr{D}_n$: 1) atoms which are exactly images of some atoms of $\mathscr{B}_n\cap\operatorname{supp}\mu$, and 2) atoms which only contain such images. The atoms of the second type can appear as images of some atoms with one endpoint in A. Hence, there are only a finite number of atoms of the second type, no more than $\operatorname{card}\mathscr{A}$, and in view of Lemma 5, [5, Th. 2] and assumption (i), we have

$$\lambda(\mathscr{D}_n)\leqslant\lambda(f(\mathscr{B}_n\cap\operatorname{supp}\mu))+\operatorname{card}\mathscr{A}\cdot d_1\gamma_1^n\tag{5}$$
$$\leqslant F\lambda(\mathscr{B}_n\cap\operatorname{supp}\mu)+\operatorname{card}\mathscr{A}\cdot d_1\gamma_1^n\leqslant F\frac{c_2}{c}\gamma_2^n+\operatorname{card}\mathscr{A}\cdot d_1\gamma_1^n.$$

(Recall that $d_1>0$, $\gamma_1\in(0,1)$ are constants from [5, Th. 2].)

Now, from (5) and [5, Th. 3] we obtain the desired estimate of $\mu(\mathscr{D}_n)$, which completes the proof. ■

LEMMA 10. *There exist constants* $c_6>0$, $\gamma_6\in(0,1)$ *such that*

$$\left|\int_{\mathscr{B}_n}\log|f'|\,d\mu\right|\leqslant c_6\gamma_6^n.$$

Proof. We have

$$\int_{\mathscr{B}_n}\log|f'|\,d\mu=\int_{\mathscr{B}_n\cap U}\log|f'|\,d\mu+\int_{\mathscr{B}_n\setminus U}\log|f'|\,d\mu$$

(recall that U is the neighbourhood of A fixed in Section 4). The function $|f'|$ being bounded from below and above by positive constants on $\mathscr{B}_n\setminus U$, the integral over $\mathscr{B}_n\setminus U$ is bounded by $\mu(\mathscr{B}_n)$ multiplied by a positive constant. In view of Lemma 4, it remains to estimate the integral over $\mathscr{B}_n\cap U$.

Recall that U is a union of components, each of which is an interval containing exactly one point of A. So the number of these intervals is finite, not greater than $\operatorname{card}A$, and to estimate the integral over $\mathscr{B}_n\cap U$ it suffices to estimate the integral over $\mathscr{B}_n\cap V$, where V is one of the components of U.

Let $a\in A\cap V$. By the definition of U, V may contain exactly one point from B and it can only be a.

Assume $a\in B$. If N was fixed sufficiently large, then for every $x\in V$, a is the point of B nearest to x and in view of [5, Lemma 11] there exist constants $L_0>0$, $\xi_0\in(0,1)$ such that for every $x\in V$

$$\varphi(x)\leqslant L_0/|x-a|^{\xi_0}.\tag{6}$$

We can also assume that N is so large that $V \subset U_a$, where U_a is the neighbourhood of a from assumption (vi). Therefore using (vi) and (6) we obtain

$$\Big|\int_{\mathscr{B}_n \cap V} \log|f'|\,d\mu\Big| = \Big|\int_{\mathscr{B}_n \cap V \cap \operatorname{supp}\mu} \log|f'|\,d\mu\Big|$$
$$\leqslant L_0 \int_{\mathscr{B}_n \cap V \cap \operatorname{supp}\mu} \frac{|\log(\alpha|x-a|^u)|}{|x-a|^{\xi_0}}\,d\lambda(x)$$
$$\leqslant L_0 \int_{\mathscr{B}_n \cap V \cap \operatorname{supp}\mu \cap \{x\in I:|x-a|>\gamma_2^n\}} \frac{|\log(\alpha|x-a|^u)|}{|x-a|^{\xi_0}}\,d\lambda(x)$$
$$+2L_0 \int_a^{a+\gamma_2^n} \frac{|\log(\alpha|x-a|^u)|}{|x-a|^{\xi_0}}\,d\lambda(x).$$

By Lemma 5 the first integral does not exceed

$$\left(\frac{c_2}{c}\gamma_2^n\right)\frac{|\log(\alpha\gamma_2^{nu})|}{\gamma_2^{n\xi_0}},$$

which is equal to $n(\gamma_2^{1-\xi_0})^n$ multiplied by a positive constant.

It remains to estimate the second integral. By a suitable change of variable we can write it as

$$\int_0^{\gamma_2^n} \frac{|\log y|}{y^{\xi_0}}\,d\lambda(y)$$

multiplied by a positive constant and we have

$$\int_0^{\gamma_2^n} \frac{|\log y|}{y^{\xi_0}}\,d\lambda(y) = \sum_{k=n}^{\infty} \int_{\gamma_2^{k+1}}^{\gamma_2^k} \frac{|\log y|}{y^{\xi_0}}\,d\lambda(y) \tag{7}$$
$$\leqslant \sum_{k=n}^{\infty} (\gamma_2^k - \gamma_2^{k+1})\frac{(k+1)\log\gamma_2}{(\gamma_2^{k+1})^{\xi_0}},$$

which completes the proof in the case $a \in B$. If $a \notin B$, then φ is bounded on V by a positive constant and the estimations become easier. ∎

LEMMA 11. $\log\varphi \in L^1(\mu)$.

Proof. Let, for every $n \in N$, E_n and $\mathscr{K}_n$ be the sets defined in Section 4. The set $E_n \cup \mathscr{K}_n$ is a neighbourhood of the set of singularities of φ and $I\setminus(E_n \cup \mathscr{K}_n)$ has a finite number of connected components, so φ is bounded on $I\setminus(E_n \cup \mathscr{K}_n)$. But φ is also bounded from below by a positive constant on $\operatorname{supp}\mu$, so the integral $\int_{I\setminus(E_n\cup\mathscr{K}_n)}|\log\varphi|\,d\mu$ is finite for every $n \in N$.

Now we will estimate the integral over $\mathscr{K}_n$. Fix n so large that $\mathscr{K}_n \subset U$. We can assume $\mathscr{K}_n$ is a union of intervals, each of which contains exactly one point from A. Let V be one of these intervals, $a \in V$.

If $a \notin B$, then the whole interval V is far from B and φ is bounded on V. Hence the integral $\int_V |\log\varphi|\,d\mu$ is finite.

If $a \in B$, then we can assume that n was fixed so large that for every $x \in V$, a is the point of B nearest to x. Now, using (6) and (7) we prove easily that the integral $\int_V |\log\varphi|\,d\mu$ is finite.

The number of intervals V does not exceed $\operatorname{card} A$, so we have proved the integrability of $\log\varphi$ on $\mathscr{K}_n$.

It remains to prove the integrability of $\log\varphi$ on E_n. We have $E_{k+1} \subset E_k$ for every $k \in N$, and

$$\int_{E_n} |\log\varphi|\,d\mu = \sum_{k=n}^{\infty} \int_{E_k\setminus E_{k+1}} |\log\varphi|\,d\mu. \tag{8}$$

We have to show that the above series is convergent.

Let $x \in E_k \setminus E_{k+1}$ and let $b \in B$ be nearest to x. We assume k is so large that b cannot belong to A. Hence $|x-b| \geqslant \operatorname{dist}(B\setminus A,\ I\setminus E_{k+1})$ and it is proved in [5, proof of Th. 3] that there exists a constant $h > 0$ such that

$$\operatorname{dist}(B\setminus A,\ I\setminus E_{k+1}) \geqslant h/F^{k+1}. \tag{9}$$

Therefore $\varphi(x) \leqslant (L_0/h)F^{\xi_0(k+1)}$ for $x \in E_k \setminus E_{k+1}$, and

$$\int_{E_k\setminus E_{k+1}} |\log\varphi|\,d\mu = \int_{(E_k\setminus E_{k+1})\cap\operatorname{supp}\mu} |\log\varphi|\,d\mu$$
$$\leqslant \max\left\{|\log c|,\ \left|\log\left(\frac{L_0}{h}(F^{\xi_0})^{k+1}\right)\right|\right\}\mu(E_k).$$

This completes the proof because $\mu(E_k)$ decreases exponentially. ∎

Remark. Notice that in the proof of Lemma 11 we have obtained in fact the exponential convergence to zero of $\int_{E_n}|\log\varphi|\,d\mu$ and $\int_{\mathscr{K}_n}|\log\varphi|\,d\mu$. ∎

LEMMA 12. *There exist constants $c_7 > 0$, $\gamma_7 \in (0, 1)$ such that*

$$\left|\int_{\mathscr{B}_n} \log\frac{\varphi\circ f}{\varphi}\,d\mu\right| \leqslant c_7\gamma_7^n.$$

Proof (due to M. Misiurewicz). Let $\div$ denote the symmetric difference of sets. Observe that

$$\left|\int_{\mathscr{B}_n} \log\frac{\varphi\circ f}{\varphi}\,d\mu\right| = \left|\int_{\mathscr{B}_n}\log(\varphi\circ f)\,d\mu - \int_{\mathscr{B}_n}\log\varphi\,d\mu\right|$$
$$= \left|\int_{\mathscr{B}_n}\log(\varphi\circ f)\,d\mu - \int_{f^{-1}(\mathscr{B}_n)}\log(\varphi\circ f)\,d\mu\right|$$
$$\leqslant \int_{\mathscr{B}_n \div f^{-1}(\mathscr{B}_n)} |\log(\varphi\circ f)|\,d\mu$$

$$\leqslant \int\limits_{f^{-1}(\mathscr{K}_n)} |\log(\varphi\circ f)|\,d\mu + \int\limits_{(\mathscr{B}_n \div f^{-1}(\mathscr{B}_n))\setminus f^{-1}(\mathscr{K}_n)} |\log(\varphi\circ f)|\,d\mu$$
$$= \int\limits_{\mathscr{K}_n} |\log\varphi|\,d\mu + \int\limits_{(\mathscr{B}_n \div f^{-1}(\mathscr{B}_n))\setminus f^{-1}(\mathscr{K}_n)} |\log(\varphi\circ f)|\,d\mu,$$

and the first integral in the last sum converges exponentially to zero (see Remark). It remains to estimate the second integral.

Write

$$\mathscr{C}_n = (\mathscr{B}_n \div f^{-1}(\mathscr{B}_n))\setminus f^{-1}(\mathscr{K}_n),$$
$$\mathscr{C}_n^A = \{x\in\mathscr{C}_n\colon \text{the point of } B \text{ nearest to } f(x) \text{ is in } A\},$$
$$\mathscr{C}_n^B = \mathscr{C}_n\setminus\mathscr{C}_n^A.$$

Notice that $f^{-1}(\mathscr{E}_n)\cap(\mathscr{B}_n \div f^{-1}(\mathscr{B}_n)) = \emptyset$.

If $x\in\mathscr{C}_n^A\cap\operatorname{supp}\mu$ and $a\in A\cap B$ is the point of B nearest to $f(x)$, then in view of [5, Lemma 11] and (1), $|\log\varphi(f(x))|$ does not exceed n multiplied by a positive constant. Hence, using Lemma 4 we obtain the estimate of $\int_{\mathscr{C}_n^A}|\log\varphi\circ f(x)|\,d\mu$ by $n\gamma_2^n$ multiplied by a positive constant.

It remains to estimate the integral over $\mathscr{C}_n^B$. If $x\in\mathscr{C}_n^B\cap\operatorname{supp}\mu$, then $f(x)\notin\mathscr{E}_n$, the point b of B nearest to $f(x)$ belongs to $B\setminus A$. Hence

$$|f(x)-b| \geqslant \operatorname{dist}(I\setminus\mathscr{E}_n, B\setminus A) = \operatorname{dist}(I\setminus E_{[n/p]-N}, B\setminus A)$$
$$\geqslant \frac{h}{F^{[n/p]-N}} \quad \text{(in view of (9))},$$

which together with [5, Lemma 11] gives the estimate of $|\log\varphi\circ f(x)|$ by n multiplied by a positive constant. Therefore using Lemma 4 we obtain the estimate of $\int_{\mathscr{C}_n^B}|\log\varphi\circ f(x)|\,d\mu$ by $n\gamma_2^n$ multiplied by a positive constant. ■

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