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A remark on functional continuity of certain Fréchet algebras

by

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Abstract. Let A be a Fréchet algebra. Denote by $S(A)$ the set of all nonzero multiplicative linear functionals and by $M(A)$ the set of all continuous members of $S(A)$. As usual we endow $M(A)$ with the weak* topology. Among other things it is proved that if there are $f_1, \ldots, f_n \in A$ such that the map $M(A) \to C^n$, $x \to (x(f_1), \ldots, x(f_n))$, is discrete, then A is functionally continuous, i.e. $M(A) = S(A)$.

1. By a *Fréchet algebra A* we mean a commutative complete metrizable locally convex topological algebra over the field of complex numbers whose topology is given by a sequence (p_n) of submultiplicative seminorms, which may be assumed to be ascending:

$$p_n(f) \leqslant p_{n+1}(f) \quad \text{for each } f \in A,\ n \in \boldsymbol{N}.$$

In [5] Michael raised the question whether every Fréchet algebra A is functionally continuous. Till now only partial answers have been obtained.

(1) Perhaps the most far reaching result, due to Arens [1], Theorem 7.1, ensures that a Fréchet algebra A is functionally continuous if there are elements $f_1, \ldots, f_n \in A$ such that for each $(z_1, \ldots, z_n) \in C^n$ the set

$$\{x \in M(A)\colon\ x(f_i) = z_i,\ i = 1, \ldots, n\}$$

is compact.

In [7] Żelazko showed that $M(A) = S(A)$ if $M(A)$ is at most countable. In his proof he first reduced the problem to the case when A has a unit, then explicitly constructed an element $h \in A$ such that the Gelfand transform $\hat{h}$ is injective on $M(A)$ and finally used the following result of Arens [1], Theorem 6.32:

(2) Let A be a Fréchet algebra with unit. Let $y \in S(A)$ and $f_1, \ldots, f_n \in A$. Then there exists $x \in M(A)$ such that

$$x(f_i) = y(f_i) \quad \text{for } i = 1, \ldots, n.$$

Our first lemma is contained in the proof of Żelazko's result. For the reader's convenience we give a (different) proof.

LEMMA 1. *Let A be a Fréchet algebra. Let (x_i) be a sequence of distinct points in $M(A)$. Then there exists $h \in A$ so that $(x_i(h))$ is a sequence of distinct points in the plane.*

Proof. For $i \neq j$ set

$$A_{ij} = \{f \in A: \; x_i(f) = x_j(f)\}.$$

Obviously A_{ij} is a closed subset of A and it is easy to see that A_{ij} is nowhere dense in A. Hence there is

$$h \in A \setminus \bigcup_{i=1}^{\infty} \bigcup_{j>i} A_{ij}$$

by Baire's Theorem. ■

We need not assume for the lemma that A has a unit. In the proof of our theorem we have to be a little bit more careful.

If A has no unit we adjoin a unit 1 in the usual way. Consider the direct sum

$$A_1 = A \oplus \mathbf{C} = \{(f, \lambda): f \in A, \; \lambda \in \mathbf{C}\}$$

with the topology given by the seminorms $p_n'(f, \lambda) = p_n(f) + |\lambda|$. All elements of $S(A)$ extend to elements $\tilde{x}$ of $S(A_1)$ by setting

$$\tilde{x}(f, \lambda) = x(f) + \lambda.$$

Note that $\tilde{x} \in M(A_1)$ if $x \in M(A)$. Set

$$y_0: \; A_1 \to \mathbf{C}, \quad (f, \lambda) \to \lambda.$$

Then $y_0 \in M(A_1)$. It is not hard to prove that

(3) $$i: \; M(A) \to M(A_1) \setminus \{y_0\}, \quad x \to \tilde{x},$$

is a weak* homeomorphism. Note that $i^{-1}(y)(f) = y(f, 0)$ for all $y \in M(A_1) \setminus \{y_0\}$ and all $f \in A$.

THEOREM 1. *Let A be a Fréchet algebra. Suppose there are $f_1, \ldots, f_n \in A$ such that for each $x \in M(A)$ the set*

$$\{y \in M(A): \; y(f_i) = x(f_i), \; i = 1, \ldots, n\}$$

is at most countable. Then $S(A) = M(A)$.

Proof. Let $x_0 \in S(A)$ be an arbitrary element.

(i) Assume first that A has a unit. Choose x by (2) such that $x(f_i) = x_0(f_i)$, $i = 1, \ldots, n$. The set

$$T = \{x \in M(A): \; y(f_i) = x(f_i), \, i = 1, \ldots, n\}$$

is at most countable, hence – by Lemma 1 – we can find $f_{n+1} \in A$ so that the Gelfand transform $\hat{f}_{n+1}$ is injective on T. So – using again (2) – there is exactly one element $x_1 \in M(A)$ such that

(4) $$x_1(f_i) = x_0(f_i), \quad i = 1, \ldots, n+1.$$

Hence $x_1 = x_0$ by (2) and (4).

(ii) If A has no unit we first choose $f_{n+1} \in A$ with

(5) $$x_0(f_{n+1}) \neq 0.$$

By (3), (2) and our assumption the set

$$\{y \in M(A_1): \; y(f_i, 0) = \tilde{x}_0(f_i, 0), \; i = 1, \ldots, n+1\}$$

is at most countable. As in (i) we can show that $\tilde{x}_0 \in M(A_1)$. Since $\tilde{x}_0 \neq y_0$ by (5), $x_0 \in M(A)$ by (3). ■

(6) For each $n \in N$ denote by K_n the set of all elements of $M(A)$ which are continuous with respect to the seminorm p_n. It can be shown that (K_n) is an admissible exhaustion of $M(A)$, i.e. each K_n is compact and if $K \subset M(A)$ is a compact set then there is $l \in N$ with $K \subset K_l$ (cf. [5] and [8]). A Hausdorff space with an admissible exhaustion is called *hemicompact*.

COROLLARY 1. *Let A be a Fréchet algebra. Suppose there are $f_1, \ldots, f_n \in A$ such that the map*

$$M(A) \to \mathbf{C}^n, \quad x \to (x(f_1), \ldots, x(f_n)),$$

is discrete. Then $M(A) = S(A)$.

Proof. (i) Assume first that A has a unit. Let $x \in M(A)$ be an arbitrary element. Since $M(A)$ is hemicompact the set

$$\{y \in M(A): \; y(f_i) = x(f_i), \; i = 1, \ldots, n\}$$

is at most countable and our assertion follows from Theorem 1.

(ii) Now assume that A has no unit. Let $x_0 \in S(A)$ be an arbitrary element. Choose $f_{n+1} \in A$ with

(7) $$x_0(f_{n+1}) \neq 0.$$

Then trivially

(8) $$M(A) \to \mathbf{C}^{n+1}, \quad x \to (x(f_1), \ldots, x(f_{n+1})),$$

is again a discrete map. Set

$$T = \{y \in M(A_1): \; y(f_i, 0) = \tilde{x}_0(f_i, 0), \; i = 1, \ldots, n+1\},$$

$$U = \{y \in M(A_1): \; |y(f_{n+1}, 0)| < \tfrac{1}{2}|x_0(f_{n+1})|\}.$$

U is an open neighbourhood of y_0 in $M(A_1)$ and

(9) $$T \subset M(A_1) \setminus U \quad \text{by (7)}.$$

$i^{-1}(T)$ is a discrete subset of $M(A)$ by (8), hence T is a discrete subset of $M(A_1)$ by (3) and (9). It follows that T is at most countable. Now show as above that $\tilde{x}_0 \in M(A_1) \setminus \{y_0\}$, i.e. $x_0 \in M(A)$. ∎

EXAMPLE. Let (X, p) be a hemicompact manifold spread over C^n, i.e. X is a hemicompact space and $p\colon X \to C^n$ is a local homeomorphism. Then $\mathcal{O}_X$—the algebra of all holomorphic functions on X—becomes a Fréchet algebra when endowed with the compact-open topology ([3], Theorem 5, p. 158). Note that $p_i \in \mathcal{O}_X$ for $i = 1, \ldots, n$ if $p = (p_1, \ldots, p_n)$. It is known (cf. for instance [2], Theorem 2.5, p. 176) that $(M(\mathcal{O}_X), \hat{p})$ is again a manifold spread over C^n, with

$$\hat{p}\colon M(\mathcal{O}_X) \to C^n, \quad x \to (x(p_1), \ldots, x(p_n)).$$

Clearly $\hat{p}$ is discrete since it is a local homeomorphism. Hence Corollary 1 yields a proof for $M(\mathcal{O}_X) = S(\mathcal{O}_X)$ not depending on the deep imbedding theorem for n-dimensional Stein manifolds.

2. In this section we want to describe a sort of reduction principle for the functional continuity problem. Throughout the rest of the paper we only consider Fréchet algebras A with unit.

Let I be a proper closed ideal in A. Then A/I endowed with the quotient topology is again a Fréchet algebra with unit. (The quotient topology is determined by the system of seminorms $q_n(f+I) = \inf\{p_n(f+g)\colon g \in I\}$.)

Denote by $V(I)$ the set of all elements $x \in M(A)$ with $x(f) = 0$ for all $f \in I$. Since the natural map $j\colon A \to A/I$ is a surjective and continuous homomorphism, the adjoint spectral map

$$j^*\colon M(A/I) \to M(A), \quad y \to y \circ j,$$

is continuous and injective. We have $j^*(y)(f) = 0$ for each $f \in I$ and each $y \in M(A/I)$, hence $j^*(M(A/I)) \subset V(I)$. The inverse inclusion also follows, since for $x \in V(I)$

$$\tilde{x}\colon A/I \to C, \quad f + I \to x(f),$$

defines an element of $M(A/I)$ and $j^*(\tilde{x}) = x$. In fact, it is not hard to prove that $j^*\colon M(A/I) \to V(I)$ is a homeomorphism for $V(I)$ endowed with the relative weak* topology.

Let $F = (f_1, \ldots, f_n) \in A^n$. Set

$$\hat{F}\colon M(A) \to C^n, \quad x \to (x(f_1), \ldots, x(f_n)).$$

For $z = (z_1, \ldots, z_n) \in \hat{F}(M(A))$ define

$$\hat{F}^{-1}(z) = \{x \in M(A)\colon \hat{F}(x) = z\},$$

$$I(F, z) = \{f \in A\colon x(f) = 0 \text{ for all } x \in \hat{F}^{-1}(z)\}.$$

$I(F, z)$ is obviously a closed ideal in A and $V(I(F, z)) = \hat{F}^{-1}(z)$.

THEOREM 2. *Let A be a Fréchet algebra with unit. Suppose there is $F = (f_1, \ldots, f_n) \in A^n$ such that $A/I(F, z)$ is functionally continuous for each $z \in \hat{F}(M(A))$. Then $S(A) = M(A)$.*

Proof. Let $x_0 \in S(A)$ be an arbitrary element. Set

$$z_i = x_0(f_i), \quad i = 1, \ldots, n.$$

Then $z = (z_1, \ldots, z_n) \in \hat{F}(M(A))$ by (2). Let $f \in I(F, z)$ be an arbitrary element. Choose $y \in M(A)$ so that

$$y(f) = x_0(f), \quad y(f_i) = x_0(f_i), \quad i = 1, \ldots, n.$$

Then $y \in \hat{F}^{-1}(z)$, so $0 = y(f) = x_0(f)$. Hence

$$\tilde{x}_0\colon A/I(F, z) \to C, \quad f + I(F, z) \to x_0(f),$$

defines an element of $S(A/I(F, z))$, which is continuous by our hypothesis. This implies

$$x_0(f) = \tilde{x}_0(f + I(F, z)) = j^*(\tilde{x}_0)(f)$$

for all $f \in A$, hence $x_0 \in M(A)$. ∎

Remarks. (i) It follows from [5], Lemma 12.3 b), that the converse of Theorem 2 is also valid, more precisely, if A is functionally continuous then $A/I(F, z)$ is functionally continuous for all $z \in \hat{F}(M(A))$.

(ii) The following result (cf. [5], Lemma 12.3 a)) is a special case of Theorem 2: A is functionally continuous if $A/R(A)$ is functionally continuous, where $R(A)$ denotes the radical of A.

The next lemma will enable us to give a slight generalization of Arens' result (1) (see Corollary 2 below).

LEMMA 2. *Let A be a Fréchet algebra with unit. Suppose*

$$M(A) = \bigcup_{i \in I} L_i$$

with each L_i compact and open. Then A is functionally continuous.

Proof. Since $M(A)$ is hemicompact we can assume that I is at most countable.

If $M(A)$ is compact our assertion follows from (1), if not we can w.l.o.g. assume that for each $n \in N$:

(i) L_n is a proper subset of L_{n+1}.

(ii) All elements of L_n are continuous with respect to the seminorm p_n (cf. (6)).

Since all elements of L_n are homomorphisms, (ii) means that

(10) $$|x(f)| \leqslant p_n(f) \quad \text{for all } f \in A \text{ and all } x \in L_n.$$

By Shilov's idempotent theorem, which is also valid for Fréchet algebras (cf. [6]), there are $f_n \in A$ so that

$$x(f_n) = \begin{cases} 0 & \text{if } x \notin L_n, \\ 1 & \text{if } x \in L_n. \end{cases}$$

By (10), $p_n(f_n) \geqslant 1$. Set

$$g_n = f_n/(2^n p_n(f_n)).$$

Then $g = \sum_{n=1}^{\infty} g_n$ defines an element of A and

$$x(g) = \sum_{k=n}^{\infty} 1/(2^k p_k(f_k)) \quad \text{if } x \in L_n \setminus L_{n-1} \text{ (with } L_0 = \emptyset).$$

By hypothesis the sets $\{x \in M(A)\colon x(g) = c\}$ are compact for each $c \in C$ and our assertion follows from (1). ∎

The hypothesis of Lemma 2 implies that $M(A)$ is locally compact. Since it is easy to construct Fréchet algebras A with countable but not locally compact spectrum $M(A)$, Żelazko's result cannot be deduced from Lemma 2. Hence—with regard to the next result—we note that hypothesis (i) of Corollary 2 is not a consequence of (ii) and *vice versa*.

COROLLARY 2. *Let A be a Fréchet algebra with unit. Suppose there is $F \in A^n$ such that for each $z \in C^n$ at least one of the following conditions is fulfilled*:

(i) *$\hat{F}^{-1}(z)$ is at most countable.*

(ii) *$\hat{F}^{-1}(z)$ is the union of compact sets which are open in the relative weak* topology on $\hat{F}^{-1}(z)$.*

Then A is functionally continuous.

3. In the last section we endow $S(A)$ with the Gelfand topology, i.e. the coarsest topology such that all Gelfand transforms are continuous functions on $S(A)$. An open neighbourhood basis of an element $x \in S(A)$ is given by

sets of the form

$$\{y \in S(A)\colon |y(f_i) - x(f_i)| < \varepsilon,\ i = 1, \ldots, r\},$$

$\varepsilon > 0$ and $f_1, \ldots, f_r \in A$.

As a consequence of (2), $M(A)$ is a dense subset of $S(A)$ (cf. [8], Corollary 12.28).

LEMMA 3. *Let A be a Fréchet algebra with unit and let (K_n) be an admissible compact exhaustion of $M(A)$. Suppose there is an at most countable subset $T \subset A$ with the following property: For each sequence (x_i) in $M(A)$ with $x_i \notin K_i$ for all $i \in N$ there is $f \in T$ (depending on (x_i)) such that*

$$\sup\{|x_i(f)|\colon i \in N\} = \infty.$$

Then $M(A) = S(A)$.

Proof. Let $T = \{f_i\colon i \in N\}$. Endow $S(A)$ with the coarsest topology such that the Gelfand transforms of all elements of T are continuous. If $x \in S(A)$ then sets of the form

$$U_{n,x} = \{y \in S(A)\colon |y(f_i) - x(f_i)| < 1/n,\ i = 1, \ldots, n\}$$

are an open neighbourhood basis of x in this topology.

Fix an arbitrary $x \in S(A)$. By (2), $U_{n,x} \cap M(A) \neq \emptyset$. Assume first that we can find $x_n \in (M(A) \cap U_{n,x}) \setminus K_n$ for each $n \in N$. Clearly (x_n) converges to x in the new topology. Choose $f \in T$ such that $\sup\{|x_i(f)|\colon i \in N\} = \infty$. Then $(x_n(f))$ does not converge to $x(f)$, a contradiction.

Hence there exists $n \in N$ so that

(11) $$M(A) \cap U_{n,x} \subset K_n.$$

Let (y_α) be a net in $M(A)$ which converges to x in the Gelfand topology. It follows from (11) that a subnet converges to a point $y \in K_n$. This implies $y = x$, since $S(A)$ is a Hausdorff space. ∎

The next proposition shows that the hypothesis of Lemma 3 is perhaps less technical than it may look at first sight.

PROPOSITION 1. *Let A be a Fréchet algebra with unit and let (K_n) be an admissible compact exhaustion of $M(A)$. Let (x_n) be a sequence in $M(A)$ with $x_n \notin K_n$ for each $n \in N$. Then there exists $f \in A$ so that*

$$\sup\{|x_n(f)|\colon n \in N\} = \infty.$$

Proof. Considering subsequences of (x_n) and of the system of seminorms (p_n) if necessary, we can w.l.o.g. assume by (6) that for each $n \in N$:

(i) $x_n \in K_{n+1} \setminus K_n$.

(ii) K_n is the set of all elements of $M(A)$ which are continuous with respect to the seminorm p_n.

Choose inductively elements $f_n \in A$ so that:

(a) $p_n(f_n) \leqslant 1/2^n$.
(b) $|x_n(f_n)| \geqslant n + \left|\sum_{k=1}^{n-1} x_n(f_k)\right|$.

Then $f = \sum_{n=1}^{\infty} f_n$ defines an element of A by (a) and

$$|x_n(f)| \geqslant |x_n(f_n)| - \left|\sum_{k=1}^{n-1} x_n(f_k)\right| - \left|\sum_{k=n+1}^{\infty} x_n(f_k)\right| \geqslant n-1. \quad \blacksquare$$

Remark. Proposition 1, known from the theory of several comple: variables, has the following consequence: If $S(A)$ is a k-space, then $M(A$ $= S(A)$ (cf. also [4], Proposition 3).

Recall that a Hausdorff space is called a *k-space* if every subse intersecting each compact set in a closed set is itself closed. Locally compac spaces or first countable spaces are examples of k-spaces. If X is a topologi cal space and Y is a k-space, then every continuous map $f\colon X \to Y$ which i proper (meaning that the inverse images of compact sets are compact) i closed. Now consider the map

$$M(A) \to \dot{S}(A), \quad x \to x.$$

It is clearly continuous and it is also proper. (Otherwise there would b a compact set $K \subset S(A)$ such that $K \cap M(A)$ is not compact. Using Proposi tion 1 we then find $f \in A$ such that the Gelfand transform $\hat{f}$ is unbounded o $K \cap M(A)$, a contradiction with the compactness of K.) It follows that th map is closed. But this implies $M(A) = S(A)$, since $M(A)$ is a dense subset c $S(A)$.

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