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Analytic Toeplitz algebras and intertwining operators *

by

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Abstract. For φ in H^∞, let T_φ be the analytic Toeplitz operator with symbol φ and let $G = \varphi(D)$. In this article we will characterize the weak-star closed algebra A generated by T_φ^*. It turns out that A equals the space of all bounded analytic functions on an appropriate domain G_0 containing G. We also show that if nonzero operators X and Y intertwine two analytic Toeplitz operators with symbols φ and ψ then $G_0 = E_0$, where $G = \varphi(D)$ and $E = \psi(D)$. Finally, it is shown that if X and Y are operators with dense range intertwining two analytic Toeplitz operators T_φ and T_ψ with φ univalent, then the two analytic Toeplitz operators have the same essential spectrum.

1. Introduction. For φ in H^∞, let T_φ be the analytic Toeplitz operator with symbol φ and let $G = \varphi(D)$. In Section 3, we use the results in [8] to characterize the weak-star closed algebra A generated by T_φ^*. In fact, A equals the space of all bounded analytic functions on an appropriate domain G_0 containing G. The main appearance of such domains in the literature is in connection with the Sarason hull of the scalar-valued spectral measure for the minimal normal extension of subnormal operators. For the exact meaning of these terms see [2].

Even though there is some overlap between Section 3 of the present paper and our paper [8] we mention that this characterization of the algebra does not appear in the literature. Moreover, the operators T considered in [8] have the property that $\dim\ker(T-\lambda)$ is a constant for all λ in an appropriate domain and in the proof of the characterization we use the fact that the commutant $\{T\}'$ of such operators is completely known. However, these properties are not in general true for analytic Toeplitz operators.

In Section 3 we show that if nonzero operators X and Y intertwine two analytic Toeplitz operators with symbols φ and ψ then $G_0 = E_0$, where $G = \varphi(D)$ and $E = \psi(D)$.

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In conclusion we show that if X and Y are operators with dense range intertwining two analytic Toeplitz operators T_φ and T_ψ such that φ is univalent, then the two analytic Toeplitz operators have the same essential spectrum.

2. Preliminaries. This section includes the necessary definitions and notation. For K a compact subset of the plane, let $R(K)$ denote the algebra of all continuous complex-valued functions on K which can be approximated uniformly on K by rational functions whose poles all lie outside K. We say that $R(K)$ is a *Dirichlet algebra* on ∂K if the real parts of the functions in $R(K)$ when restricted to ∂K are dense in the space of continuous real-valued functions on ∂K. A compact subset K of the plane is a *spectral set* for $T \in L(H)$ if it contains the spectrum of T, $\sigma(T)$, and $\|f(T)\| \leqslant \max\{|f(z)|: z \in K\}$ for all rational functions f with poles off K. If E is an open subset of the plane, then $H^\infty(E)$ denotes the space of bounded analytic functions in E.

For G a domain (an open connected subset of the plane) and α an ordinal number, the set G_α can be defined as in Sarason [6, p. 525]. However, the set G considered in [6] is simply connected and we will not assume any restriction of this sort. The definition of G_α in the general case appears in [8, p. 234]. For the benefit of the reader we will reiterate the necessary definitions.

If B is a bounded domain in the plane, then the *Carathéodory hull* (or *C*-hull) of B is the complement of the closure of the unbounded component of the complement of the closure of B. We denote the *C*-hull of B by B^*. Loosely speaking, B^* can be described as the interior of the outer boundary of B, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leqslant \sup\{|p(z)|: z \in B\}$ for all polynomials p. The components of B^* are simply connected; in fact, it is a simple matter to show that each of these components has a connected complement. We denote by B_1 the component of B^* that contains B.

Again let B be a bounded domain in the plane. For any simply connected domain E containing B we can define the *relative hull of B in E*, or the *E-hull of B*, to be the interior of the set of all points z_0 in E such that $|f(z_0)| \leqslant \sup\{|f(z)|: z \in B\}$ for every function f bounded and analytic in E.

Now let G be a bounded domain in the plane. We have already defined G_1 to be the component of the *C*-hull of G that contains G. We now define inductively for every countable ordinal number α a simply connected domain G_α containing G as follows. If α has an immediate predecessor we let G_α be the component of the $G_{\alpha-1}$-hull of G that contains G. If α has no immediate predecessor we define G_α to be the component of the interior of $\bigcap_{\beta<\alpha} G_\beta$ that contains G. (It is easily verified that G_α then has a connected complement, and so is simply connected.) It is shown in [6, p. 525] that there is a least

countable ordinal γ such that $G_\gamma = G_{\gamma+1}$. We call γ the *order* of G. Obviously, $G_\alpha = G_\gamma$ for all $\alpha \geqslant \gamma$. For convenience we set $G_0 = G_\gamma$.

It is evident that for every α, $R(\bar{G}_\alpha)$ is a Dirichlet algebra. Therefore it makes sense to talk about $H^\infty(\partial\bar{G}_\alpha)$, the weak-star closure of $R(\bar{G}_\alpha)$ in $L^\infty(m)$, where m is the harmonic measure on $\partial\bar{G}_\alpha$ (for the definition of harmonic measure see [2, p. 332]). We will also identify the two spaces $H^\infty(\partial G_\alpha)$ and $H^\infty(G_\alpha)$ bearing in mind that $\partial G_\alpha = \partial\bar{G}_\alpha$ (see [7, Lemma 4.2, p. 5]).

Let D denote the open unit disk, let H^2 denote the Hardy space of functions f analytic on D with $\int |f(re^{it})|^2\,dt$ bounded independently of r, let H^∞ be the space of bounded analytic functions on D, and for φ in H^∞ let T_φ denote the operator on H^2 defined by $T_\varphi f = \varphi f$. The operator T_φ is said to be an *analytic Toeplitz operator*.

If $\psi \in H^2$, then the function $\tilde{\psi}$ defined by $\tilde{\psi}(z) = \overline{\psi(\bar{z})}$ is also in H^2. For $|\lambda| < 1$, define $k_\lambda \in H^2$ by the relation $k_\lambda(z) = (1-\lambda z)^{-1}$. Now, let $\varphi \in H^\infty$ and set $T = T_{\tilde{\varphi}}^*$ and $G = \varphi(D)$, the range of φ. Then $Tk_\lambda = \varphi(\lambda) k_\lambda$. This notation will be retained throughout the rest of the paper. We also observe that the linear subspace consisting of all functions of the form $f = \sum_{i=1}^k c_i k_{\lambda_i}$, for $c_i \in C$, $\lambda_i \in D$ $(i = 1, \ldots, k)$ is dense in H^2.

3. The algebra generated by a Toeplitz operator. The next lemma is a preliminary effort in characterizing the weak-star closed algebra generated by T, where T is as before. The proof is a slight modification of [8, Lemma 3.1] and hence will not be included.

(3.1) LEMMA. *Let T be as before and let α be an ordinal number. For f in $H^\infty(\partial G_\alpha)$ define $f(T)$ by*

$$f(T)\Big[\sum_{i=1}^{k} c_i k_{\lambda_i}\Big] = \sum_{i=1}^{k} c_i f(\varphi(\lambda_i)) k_{\lambda_i}.$$

Then $f(T)$ extends to a bounded operator on H^2. Furthermore, if we set $\Phi_\alpha(f) = f(T)$ then Φ_α is an isometry from $H^\infty(\partial G_\alpha)$ into $L(H^2)$.

For any operator $X \in L(H)$ let $A_1(X)$ denote the WOT (weak operator topology) sequential closure of $A_0(X) = \{p(X): p \text{ is a polynomial}\}$. We now define inductively for every ordinal number α a set $A_\alpha(X)$ as follows. If $A_\alpha(X)$ is defined for some ordinal number α, let $A_{\alpha+1}(X)$ denote the WOT sequential closure of $A_\alpha(X)$. If α is a limit ordinal and $A_\beta(X)$ is defined for all $\beta < \alpha$, let $A_\alpha(X)$ be the WOT sequential closure of $\bigcup_{\beta<\alpha} A_\beta(X)$. It is a well-known property of weak-star topologies [1] that the spaces $A_\alpha(X)$ eventually become constant; that is, there is a least countable ordinal α_0 such that $A_\alpha(X) = A_{\alpha_0}(X) = A(X)$, the weak-star closed algebra generated by X, for $\alpha \geqslant \alpha_0$.

Using transfinite induction we will show that for each ordinal number α there exists an isometric isomorphism Φ_α: $H^\infty(\partial G_\alpha) \to A_\alpha(T)$. To see this let $f \in H^\infty(\partial G_\alpha)$. Then invoking Lemma (3.1), there is an operator $f(T)$ in $L(H^2)$ such that $\|f(T)\| = \|f\|_\infty$. The proof that $f(T)$ is actually in $A_\alpha(T)$ is along the same lines as [8, p. 237] and hence will be omitted. It remains to show that Φ_α is actually onto which is the content of the next lemma. In the proof of [8, Lemma 3.2] we use the fact that the commutant $\{A\}'$ with A having a generalized Bergman kernel is completely characterized. However, this is not the case for the commutant of an analytic Toeplitz operator and we therefore give a proof of the fact that Φ_α is onto in the next lemma.

(3.2) LEMMA. *For every ordinal number α, Φ_α is an isometric isomorphism from $H^\infty(\partial G_\alpha)$ onto $A_\alpha(T)$.*

Proof. We apply transfinite induction to show that for every ordinal number α, Φ_α is onto.

To show that Φ_1 is onto, let $R \in A_1(T)$. Then by definition of $A_1(T)$ there exists a sequence $\{p_n\}$ of polynomials such that $p_n(T) \to R$ (WOT). Now $\|p_n(T)\| \leq M$ for some $M > 0$ and $\|p_n\|_{G_1} = \|p_n\|_G = \|p_n(T)\| \leq M$. Since $\{p_n\}$ forms a normal family in $H^\infty(G_1)$, by dropping to a subsequence if need be, we may assume that $\{p_n\}$ converges uniformly on compact subsets of G_1 to a function ψ in $H^\infty(G_1)$. But

$$p_n(T)\Big[\sum_{i=1}^{k} c_i k_{\lambda_i}\Big] = \sum_{i=1}^{k} c_i p_n(\varphi(\lambda_i)) k_{\lambda_i}$$

converges weakly to $R\sum_{i=1}^k c_i k_{\lambda_i}$ and in norm to $\sum_{i=1}^k c_i \psi(\varphi(\lambda_i)) k_{\lambda_i}$. Therefore $R = \psi(T)$, $\psi \in H^\infty(G_1)$.

For a nonlimit ordinal α assume $\Phi_{\alpha-1}$ is onto, let $S \in A_\alpha(T)$ and choose a sequence $\{S_n\}$ in $A_{\alpha-1}(T)$ such that $S_n \to S$ (WOT). By the induction hypothesis $S_n k_\lambda = \psi_n(\varphi(\lambda)) k_\lambda$, $\lambda \in D$, where $\psi_n \in H^\infty(G_{\alpha-1})$. We have $\|\psi_n\|_{G_{\alpha-1}} = \|\psi_n\|_G = \|S_n\| \leq M$, for some $M > 0$. By using a normal family argument, we may assume that $\{\psi_n\}$ converges uniformly on compact subsets of $G_{\alpha-1}$ to a function ψ in $H^\infty(G_{\alpha-1})$. It is easy to see that $S = \psi(T)$, $\psi \in H^\infty(G_\alpha)$.

Suppose α is a limit ordinal and let $X \in \bigcup_{\beta<\alpha} A_\beta(T)$. Then $X \in A_\beta(T)$ for some $\beta < \alpha$. Also $Xk_\lambda = \psi(\varphi(\lambda)) k_\lambda$, $\lambda \in D$, where $\psi \in H^\infty(G_\beta)$ by the induction hypothesis. Since $G_\alpha \subset G_\beta$ we have $\psi \in H^\infty(G_\alpha)$ and $X = \psi(T)$.

If there is a sequence $\{A_n\}$ in $\bigcup_{\beta<\alpha} A_\beta(T)$ such that $A_n \to A$ (WOT), then $A_n k_\lambda = \psi_n(\varphi(\lambda)) k_\lambda$, $\lambda \in D$, where $\psi_n \in H^\infty(G_\alpha)$ by the previous argument. Now $\|A_n\| \leq M$ for some $M > 0$, hence $\|\psi_n\|_{G_\alpha} \leq M$. By a normal family argument we may assume that ψ_n converges uniformly on compact subsets of G_α to a function ψ in $H^\infty(G_\alpha)$. It is easy to see that $A = \psi(T)$. Hence Φ_α is onto. ∎

For the proof of the next theorem see [8, Theorem 3.3].

(3.3) THEOREM. *Let $\varphi \in H^\infty$, $T = T_\varphi^*$ and $G = \varphi(D)$. Then there is a norm isometric, weak-star homeomorphic algebra isomorphism Φ from $H^\infty(\partial G_0)$ $(= H^\infty(G_0))$ onto $A(T)$ that takes a polynomial p to $p(T)$. In fact, Φ is a functional calculus.*

Note. We would like to point out that if $f \in H^\infty(G_0)$ then $f \circ \varphi \in H^\infty$ and $f(T) = T^*_{\tilde{f} \circ \varphi}$. Therefore $A(T)$ consists of coanalytic Toeplitz operators.

4. Intertwining operators. Let $\varphi, \psi \in H^\infty$. Deddens [3, 4] has shown that if $Y \neq 0$ is a bounded operator satisfying the condition $YT_\varphi = T_\psi Y$ then $\sigma(T_\psi) \subset \sigma(T_\varphi)$. Therefore if $X \neq 0$ is a bounded operator satisfying the condition $XT_\psi^* = T_\varphi^* X$ then $X^* T_{\bar\varphi} = T_{\bar\psi} X^*$, so $\sigma(T_{\bar\psi}) \subset \sigma(T_{\bar\varphi})$, from which it follows that $\sigma(T_\psi^*) \subset \sigma(T_\varphi^*)$.

The idea of the next two results is taken from [2, pp. 219–220].

(4.1) LEMMA. *Let A_1, A_2 be algebras of coanalytic Toeplitz operators and let C_1, C_2 be their WOT sequential closures consisting of coanalytic Toeplitz operators. If F: $A_1 \to A_2$ is a contractive monomorphism and $X \neq 0$ is an operator such that $XF(A) = AX$ for every A in A_1, then F extends to a contractive monomorphism $\tilde{F}$: $C_1 \to C_2$ such that $X\tilde{F}(C) = CX$ for every C in C_1.*

Proof. Let $C \in C_1$ and choose a sequence $\{A_n\}$ in A_1 such that $A_n \to C$ (WOT). Now $\|A_n\| \leq M$ for some $M > 0$, so $\|F(A_n)\| \leq M$. Thus, there are a D in C_2 and a subsequence $\{A_{n_k}\}$ such that $F(A_{n_k}) \to D$ (WOT). Since $XF(A_{n_k}) = A_{n_k} X$, $XD = CX$. By the above result of Deddens [3, 4] we conclude that $\sigma(D) \subset \sigma(C)$. Hence $\|D\| \leq \|C\|$. If we set $\tilde{F}(C) = D$, then $\tilde{F}$ is the desired extension. ∎

(4.2) THEOREM. *Let $\varphi, \psi \in H^\infty$ and set $T = T_\varphi^*$, $S = T_\psi^*$. Suppose there exist $X \neq 0$, $Y \neq 0$ satisfying $XS = TX$ and $YT = SY$. There is an isometric isomorphism F: $A(T) \to A(S)$ such that:*

(a) *$F(T) = S$.*
(b) *$XF(A) = AX$ and $YA = F(A)Y$ for all A in $A(T)$.*
(c) *F is a weak-star homeomorphism.*

Proof. Let p be a polynomial. Then $Xp(S) = p(T)X$ and $Yp(T) = p(S)Y$. Define F_0: $A_0(T) \to A_0(S)$ and G_0: $A_0(S) \to A_0(T)$ by $F_0(p(T)) = p(S)$ and $G_0(p(S)) = p(T)$. So $F_0 = G_0^{-1}$. Applying transfinite induction, Lemma 4.1 and the Krein–Shmul'yan Theorem we obtain the result. ∎

(4.3) Corollary. *For* $\varphi, \psi \in H^\infty$ *let* $T = T_\varphi^*$, $S = T_\psi^*$, $G = \varphi(D)$ *and* $E = \psi(D)$. *Suppose there exist* $X \neq 0$, $Y \neq 0$ *satisfying* $XS = TX$ *and* $YT = SY$. *Then* $G_0 = E_0$.

Proof. Let $\Phi_T\colon H^\infty(G_0) \to A(T)$ and $\Phi_S\colon H^\infty(E_0) \to A(S)$ be the functional calculi defined in Theorem (3.3) and let $F\colon A(T) \to A(S)$ be the map defined in Theorem (4.2). Then $\Phi_S^{-1} \circ F \circ \Phi_T\colon H^\infty(G_0) \to H^\infty(E_0)$ is an algebra isomorphism which is the identity on the polynomials. Therefore the position function $z \to z$ has the same spectrum in the two spaces $H^\infty(G_0)$ and $H^\infty(E_0)$. So $\bar{G}_0 = \bar{E}_0$. It follows that $\operatorname{int}(\bar{G}_0) = \operatorname{int}(\bar{E}_0)$. But $\bar{G}_0 = \operatorname{int}(\bar{G}_0)$ and $\bar{E}_0 = \operatorname{int}(\bar{E}_0)$, from which the conclusion is immediate. ■

Recall that for $f \in H^\infty$ and $z_0 \in \partial D$ we define the *cluster set* $C(f, z_0)$ of f at z_0 in either of the following two equivalent ways:

(i) $C(f, z_0)$ is the set of points α in $\boldsymbol{C}$ such that there exists a sequence $\{z_n\} \subset D$ such that $\lim_{n\to\infty} z_n = z_0$ and $\lim_{n\to\infty} f(z_n) = \alpha$.

(ii) $C(f, z_0) = \bigcap_{r>0} \overline{f(D \cap B(z_0, r))}$.

In the next result we use the fact that if $f \in H^\infty$ then the essential spectrum $\sigma_e(T_f)$ of T_f is given by $\bigcup_\theta C(f, e^{i\theta})$ ([5]).

(4.4) Proposition. *Let* $\varphi, \psi \in H^\infty$, *and let* φ *be univalent. Let* $X, Y \in L(H^2)$ *be operators with dense range such that* $XT_\varphi = T_\psi X$ *and* $YT_\psi = T_\varphi Y$. *Then* $\sigma_e(T_\varphi) = \sigma_e(T_\psi)$.

Proof. Since X^* and Y^* are one-to-one and φ is univalent we conclude that $\dim\ker(T_\varphi^* - \bar{\lambda}) = \dim\ker(T_\psi^* - \bar{\lambda}) = 1$ or 0 for every $\lambda \in \boldsymbol{C}$. It follows that the number of zeros of $\psi - \lambda$ in D is at most 1. Hence ψ is univalent.

Now let $G = \varphi(D)$ and $\Omega = \psi(D)$. By a result of Deddens [4, Theorem 2] $G = \Omega$. We now show that φ and ψ have the same set of cluster values. That is, $\bigcup_\theta C(\varphi, e^{i\theta}) = \bigcup_\theta C(\psi, e^{i\theta})$ or equivalently $\sigma_e(T_\varphi) = \sigma_e(T_\psi)$.

Note that if $\alpha = \lim \varphi(z_n)$ and $|z_n| \to 1$, then, by univalence, α is not an interior point of $\varphi(D)$. Let $\varphi(z_n) = \psi(\omega_n)$. If ω_n has a cluster point on the circle we are done. But if $|\omega_n| \leqslant r < 1$ for all n then $\omega_{n_k} \to \omega$ for some subsequence and $\alpha = \varphi(\omega)$ is interior to $\psi(D)$, a contradiction. Hence we conclude that $\sigma_e(T_\varphi) \subset \sigma_e(T_\psi)$. Since this argument is reversible we obtain $\sigma_e(T_\psi) \subset \sigma_e(T_\varphi)$. Combining the two inclusions we have $\sigma_e(T_\varphi) = \sigma_e(T_\psi)$. ■

Note. The referee has pointed out that H. Wang [9] proves Proposition (4.4) under the stronger hypothesis that one of the symbols is a weak-star generator of H^∞ and therefore obtaining the stronger result of unitary equivalence. The author would like to thank the referee for his helpful comments.

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