[Bur] D. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, in: Conf. on Harmonic Analysis in Honor of A. Zygmund, Chicago 1981, Wadsworth, Belmont, Calif., 1983, 270–286.
[Che] S. Chevet, *Séries de variables aléatoires gaussiennes à valeurs dans $E \hat{\otimes}_\varepsilon F$. Application aux produits d'espaces de Wiener abstraits*, Sém. sur la géometrie des espaces de Banach 1977–1978, exp. 19, École Polytechnique, Paris.
[D/U] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R. I., 1977.
[E/G] R. E. Edwards and G. I. Gaudry, *Littlewood–Paley and Multiplier Theory*, Ergeb. Math. Grenzgeb. 90, Springer, New York 1977.
[Fer] D. L. Fernandez, *On Fourier multipliers of Banach-lattice-valued functions*, preprint, Campinas 1985.
[H/P] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, Providence, R. I., 1957.
[Kah] J.-P. Kahane, *Some Random Series of Functions*, Heath Math. Monographs, Lexington, Mass., 1968.
[McC] T. McConnell, *On Fourier multiplier transformations of Banach-valued functions*, Trans. Amer. Math. Soc. 285 (1984), 739–757.
[Pis] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compositio Math. 37 (1978), 3–19.
[R/T] J. L. Rubio de Francia and J. L. Torrea, *Some Banach techniques in vector-valued Fourier analysis*, Colloq. Math., to appear.

MATHEMATISCHES INSTITUT
CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL
Olshausenstr. 40, D-2300 Kiel 1, F.R.G.

Received September 14, 1987 (2362)

STUDIA MATHEMATICA, T. XCIII (1989)

Quotients and interpolation spaces of stable Banach spaces

by

J. BASTERO* (Zaragoza) and Y. RAYNAUD (Paris)

Abstract. We show the stability (in the sense of Krivine–Maurey) of the quotients of several reflexive or superreflexive stable atomic spaces. We show that the space $L(E)$ is stable provided L is a stable r.i. space and E a stable Banach space. We study the stability of interpolation spaces $[L_1, L_2]_{\theta,X}$ when L_1, L_2 are stable r.i. function spaces and X a stable atomic lattice.

Introduction. In this paper we study the stability of quotients and interpolation spaces of stable Banach spaces. The notion of stability for a Banach space was introduced by Maurey and Krivine [KM]; recall that a Banach space E is *stable* if for any bounded sequences (x_n), (y_m) and ultrafilters $\mathscr{U}$, $\mathscr{V}$ we have

$$\lim_{n,\mathscr{U}}\lim_{m,\mathscr{V}}\|x_n+y_m\| = \lim_{m,\mathscr{V}}\lim_{n,\mathscr{U}}\|x_n+y_m\|.$$

This property is clearly hereditary but does not behave well with respect to other standard operations on Banach spaces. Quotients and duals of stable Banach spaces, even reflexive, may not be stable, as was shown in [G]. Similarly, no interesting result about interpolation of stable Banach spaces was known (except the stability of Lorentz spaces, cf. [R]). In fact, if E, F form an interpolation pair of stable Banach spaces, the space $E+F$ (with norm $\|x\| = \inf\{\|e\|+\|f\| \mid e \in E,\ f \in F,\ x = e+f\}$) is a quotient of the direct sum $E \oplus_1 F$ and therefore may probably not be stable. If E is a stable Banach space, it is known that $L_p(E)$, $1 \leqslant p < \infty$, is also stable ([KM]). In fact, in all cases where a lattice L is known to be stable, the same is true for $L(E)$ if E is a stable Banach space. This is the case for L_p spaces, Orlicz spaces, Lorentz spaces. But there was no general result in this direction, except in the case of atomic lattices (i.e. spaces with 1-unconditional basis, cf. [BM] and [B]).

Here we present some positive results in these three directions. For the

* The contribution of this author was supported by the grant 0804-84 from CAICYT (Spain).

problem of stability of quotients, we consider atomic lattices with special additional properties (§I below). This allows us to show the stability of quotients of classical stable atomic lattices (l_p, l_φ, $l_{p,q}$, $d(w, p)$, $l_p(l_q)$, ...) and trace classes C_p at least when they are superreflexive. Only the case of quotients of l_p was previously known (cf. [R] or [C]). Interpolation spaces of these spaces are *ipso facto* stable.

In § II, we show the stability of $L(E)$ when L is a stable r.i. space (and E a stable Banach space). Note that in this kind of problems the behaviour of r.i. spaces is quite often similar to that of atomic lattices.

In the interpolation problem we also give (in §III) a positive result for atomic or r.i. spaces, under an additional "equiintegrability" assumption.

As an application we are able to embed stable spaces with unconditional basis into stable r.i. function spaces.

I. Stability of quotients

Recall that a *type* on a (separable) Banach space E is a function $\tau \in \mathbf{R}^E$ defined using a bounded sequence $(x_n)_{n=1}^{\infty}$ in E by

$$\forall x \in E, \quad \tau(x) = \lim_{n\to\infty} \|x + x_n\|.$$

Let $\mathcal{T}(E)$ denote the space of types on E, and $\mathcal{T}_0(E)$ the space of types which are *weakly null*, i.e. can be defined by means of a weakly null sequence in E. $\mathcal{T}(E)$ is equipped with the natural topology of pointwise convergence; and also with the topology of uniform convergence on bounded sets of E (TUCB). The case where $\mathcal{T}_0(E)$ is locally TUCB-compact will be of particular interest in the quotient problem.

If $(x_n)_n$ defines τ and $(y_m)_m$ defines σ, recall that $(x_n + y_m)_{n,m}$ defines the so-called *convoluted type* $\tau * \sigma$ (cf. [KM]). Write also $\|\tau\| = \lim_{n\to\infty} \|x_n\| = \tau(0)$.

1. A continuity condition. The following lemma allows us to calculate iterated limits of the quotient norm:

LEMMA 1. *Let E be a reflexive Banach space and F a subspace. Let $(\xi_n)_{n=1}^{\infty}$ and $(\eta_m)_{m=1}^{\infty}$ be two bounded sequences in E/F and $\mathcal{U}$, $\mathcal{V}$ two (nontrivial) ultrafilters on N. Let $(x_n)_{n=1}^{\infty}$, resp. $(y_m)_{m=1}^{\infty}$, be a bounded sequence of representatives of the ξ_n, resp. η_m, in E. Then we have*

$$(1) \qquad \lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|\xi_n + \eta_m\|_{E/F} = \inf_{f\in F} \inf_{(k_n)} \lim_{n,\mathcal{U}} \inf_{(h_m)} \lim_{m,\mathcal{V}} \|x_n + y_m + f + k_n + h_m\|_E$$

where $\inf_{(k_n)}$ is the infimum over all sequences $(k_n)_{n=1}^{\infty} \subset F$ with $k_n \xrightarrow{w} 0$ as $n \to \infty$, and similarly for $\inf_{(h_m)}$.

Proof. For $n, m \in N$ let $f_{nm} \in F$ be such that

$$\|x_n + y_m + f_{nm}\|_E \leqslant \|\xi_n + \eta_m\|_{E/F} + 2^{-(n+m)}.$$

We may assume the family (f_{nm}) to be bounded. Set

$$f_n = w\text{-}\lim_{m,\mathcal{V}} f_{nm}, \qquad h_{nm} = f_{nm} - f_n$$

$$f = w\text{-}\lim_{n,\mathcal{U}} f_n, \qquad k_n = f_n - f$$

(the limits being weak limits). These are elements of F. Then

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|\xi_n + \eta_m\|_{E/F} = \lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|x_n + y_m + f + k_n + h_{nm}\|_E$$

is clearly greater than the right-hand side of (1). The converse inequality is trivial. ∎

Let us introduce some notation. Let $\mathfrak{y} = (y_m)_{m=1}^{\infty}$ be a fixed weakly null sequence in E. Let $\mathcal{F}_\mathfrak{y}$ be the set of sequences $(y_m + f_m)_{m=1}^{\infty}$ where $(f_m)_{m=1}^{\infty}$ is a weakly null sequence in F. To an element $\mathfrak{h}$ of $\mathcal{F}_\mathfrak{y}$ and an ultrafilter $\mathcal{U}$ on N we associate the type $\sigma_{\mathfrak{h},\mathcal{U}} \in \mathcal{T}_0(E)$ defined by $\mathfrak{h}$ and $\mathcal{U}$.

PROPOSITION 2. *If the map $\mathcal{T}(E) \to \mathbf{R}$, $\tau \mapsto \inf_{\mathfrak{h}\in\mathcal{F}_\mathfrak{y}} \|\tau * \sigma_{\mathfrak{h},\mathcal{U}}\|$, is continuous for the simple topology of $\mathcal{T}(E)$ (for any $\mathfrak{y}$ and $\mathcal{U}$), then the stability of E implies the stability of E/F.*

Proof. Use Lemma 1 and set $y = w\text{-}\lim_{m,\mathcal{V}} y_m$ and $y'_m = y_m - y$, $\mathfrak{y}' = (y'_m)_m$ and similarly $x = w\text{-}\lim_{n,\mathcal{U}} x_n$ and $x'_n = x_n - x$, $\mathfrak{x}' = (x'_n)_n$. Then

$$\begin{aligned}\lim_{n,\mathcal{U}} \inf_{(h_m)} \lim_{m,\mathcal{V}} \|x_n + y_m + f + k_n + h_m\| &= \lim_{n,\mathcal{U}} \inf_{\mathfrak{h}\in\mathcal{F}_{\mathfrak{y}'}} \sigma_{\mathfrak{h},\mathcal{V}}(x_n + y + f + k_n)\\ &= \inf_{\mathfrak{h}\in\mathcal{F}_{\mathfrak{y}'}} \sigma_{\mathfrak{k},\mathcal{U}} * \sigma_{\mathfrak{h},\mathcal{V}}(x + y + f)\end{aligned}$$

by the continuity condition, where $\mathfrak{k} = (x'_n + k_n)_n \in \mathcal{F}_{\mathfrak{x}'}$ and $\inf_{(h_m)}$ is taken over all $(h_m) \subset F$ with $\lim_{m,\mathcal{V}} h_m = 0$. Thus

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|\xi_n + \eta_m\| = \inf_{f\in F} \inf_{\mathfrak{k}\in\mathcal{F}_{\mathfrak{x}'}} \inf_{\mathfrak{h}\in\mathcal{F}_{\mathfrak{y}'}} \sigma_{\mathfrak{k},\mathcal{U}} * \sigma_{\mathfrak{h},\mathcal{V}}(x + y + f).$$

As convolution of types is a symmetric operation, the left-hand side of this equality remains unchanged after exchanging the limits. Thus E/F is stable. ∎

2. Strong compactness condition for $\mathcal{T}_0(E)$

PROPOSITION 3. *Let E be a stable Banach space. If $\mathcal{T}_0(E)$ is locally compact for TUCB, then the continuity condition of Proposition 2 is satisfied (and consequently quotients of E are stable if E is supposed to be reflexive).*

Proof. The map $\mathcal{T}(E) \times \mathcal{T}_0(E) \to \mathbf{R}_+$, $(\tau, \sigma) \mapsto \|\tau * \sigma\|$, is jointly continuous when $\mathcal{T}(E)$ is equipped with the simple topology and $\mathcal{T}_0(E)$ with

TUCB. Therefore the map $\mathcal{T}(E) \to \mathbf{R}_+$, $\tau \mapsto \|\tau * \sigma\|$, is continuous w.r.t. τ, uniformly in σ varying in a bounded subset of $\mathcal{T}_0(E)$, under the strong compactness condition made here; hence the continuity condition of Prop. 2 follows. ∎

A trivial example of a space satisfying the strong compactness condition of Prop. 3 is the space l_p $(1 < p < \infty)$, since $\mathcal{T}_0(l_p)$ contains only the types τ_a: $x \mapsto (\|x\|^p + a^p)^{1/p}$, $a \in \mathbf{R}_+$.

Another example is given by the space spanned by a 1-symmetric (not necessarily 1-unconditional) weakly null sequence in L_p, $p \geqslant 2$ (cf. [G2]).

A more interesting example is given by Orlicz sequence spaces. We refer to [LT I] for basic facts on Orlicz sequence spaces and recall that Orlicz (sequence or function) spaces are stable (cf. [Ga]).

Proposition 4. *If l_φ is a reflexive Orlicz sequence space, then $\mathcal{T}_0(l_\varphi)$ is locally TUCB-compact.*

Proof. A type $\tau \in \mathcal{T}_0(l_\varphi)$ is given by $\tau(x) = \lim_{n\to\infty} \|x + x_n\|$ where $(x_n)_n$ is a block basic sequence in l_φ. We can suppose $\|x_n\| = \tau(0) = \|\tau\| \neq 0$.

To each $x \in l_\varphi$ we associate the Orlicz function φ_x: $\varphi_x(t) = \sum_{i=1}^{\infty} \varphi(t|x_i|)$. Then

$$\tau(x) = \inf\{\varrho > 0:\ \varphi_x(1/\varrho) + \lim_n \varphi_{x_n}(1/\varrho) \leqslant 1\}.$$

The functions ψ_n: $\psi_n(t) = \varphi_{x_n}(t/\|x_n\|)$ belong to the class $C_{\varphi,1}$ ([LT I], p. 140) which is norm compact in $C([0, 1])$. Let ψ be in the closure of $(\psi_n)_n$. We have

$$\tau(x) = \inf\{\varrho > 0:\ \varphi_x(1/\varrho) + \psi(\|\tau\|/\varrho) \leqslant 1\};$$

in fact, $\tau(x)$ is the unique solution of the implicit equation

$$\varphi_x(1/\tau(x)) + \psi(a/\tau(x)) = 1 \quad \text{where } a = \|\tau\|. \tag{2}$$

To each pair $(\psi, a) \in C_{\varphi,1} \times \mathbf{R}_+$ we can associate the function $\tau = \tau_{\psi,a}$: $l_\varphi \to \mathbf{R}_+$ such that $\tau(x)$ is the unique solution of (2) (note that, as is easily seen, $a \vee \|x\| \leqslant \tau_{\psi,a}(x) \leqslant a + \|x\|$). This map $C_{\varphi,1} \times \mathbf{R}_+ \to \mathbf{R}_+^{l_\varphi}$ is continuous for the norm topology (of $C([0, 1])$) on $C_{\varphi,1}$ and TUCB on $\mathbf{R}_+^{l_\varphi}$. Indeed, the function $F(\lambda; \psi, a; x) = \varphi_x(\lambda) + \psi(\lambda a)$ is continuous w.r.t. (ψ, a) uniformly in $\lambda \leqslant 1/a$ and in x varying in a bounded subset B of l_φ; and moreover $\partial F/\partial\lambda$ is bounded from below (at $\lambda = \tau_{\psi,a}(x)$) independently of $x \in B$:

$$\frac{\partial F}{\partial \lambda} = \varphi_x'(\lambda) + a\psi'(\lambda a) \geqslant \frac{1}{\lambda}\big(\varphi_x(\lambda) + \psi(\lambda a)\big) = \frac{1}{\lambda} \geqslant \frac{1}{a + \|x\|}.$$

Thus the set $\{\tau_{\psi,a}\}_{\psi \in C_{\varphi,1}, |a| \leqslant A}$ is TUCB-compact for every $A < \infty$. ∎

It is not hard to see that if $\mathcal{T}_0(E)$ is locally TUCB-compact, the same is true for $\mathcal{T}_0(E/F)$ (for all quotients of E).

A simple example where $\mathcal{T}_0(E)$ is not locally TUCB-compact is given by the space $l_p(l_q)$, $p \neq q$ (cf. [G3] or [R2], Prop. A2).

Other simple counterexamples are the Lorentz sequence spaces $l_{p,q}$, $p \neq q$, and the Schatten trace classes C_p, $p \neq 2$ (containing $l_p(l_2)$). These spaces are known to be stable ([R], [R3]).

In the case of $l_{p,q}$ consider the types τ_n defined by the sequence $x_{n,k} = n^{-1/p} \sum_{l=1}^{n} e_{k+l}$ (where $(e_n)_n$ is the natural basis of $l_{p,q}$). The sequence $(\tau_n)_n$ converges simply in $\mathcal{T}_0(l_{p,q})$ to the type τ such that $\tau(x) = (\|x\|^q + 1)^{1/q}$; but this convergence does not hold for TUCB because $\tau_n * \tau_n$ does not converge to $\tau * \tau$ (since $\|\tau_n * \tau_n\| = 2^{1/p}$ and $\|\tau * \tau\| = 2^{1/q}$).

These counterexamples lead to give another (less restrictive) condition.

3. Case of superreflexive atomic spaces. The "continuity condition" of no. 1 may be written as

$$\lim_{n,\mathcal{U}} \inf_{\mathfrak{h}=(h_m)\in\mathcal{F}_0} \lim_{m,\mathcal{V}} \|z + y_m + x_n + h_m\| = \inf_{\mathfrak{h}=(h_m)\in\mathcal{F}_0} \lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} \|z + y_m + x_n + h_m\|$$

for every $z \in E$ and any weakly null sequences $(x_n)_{n=1}^{\infty}$, $(y_m)_{m=1}^{\infty}$ in E. Here $\mathcal{F}_0$ is the space of weakly null sequences $\mathfrak{h} = (h_m)_{m=1}^{\infty}$ in F.

Let us rewrite this condition in the ultrapower $\tilde{E} = E^{\mathbf{N}}/\mathcal{U}$. We obtain

$$\lim_{n,\mathcal{U}} \inf_{\tilde{h}\in\tilde{F}_0} \|z + x_n + \tilde{y} + \tilde{h}\| = \inf_{\tilde{h}\in\tilde{F}_0} \lim_{n,\mathcal{U}} \|z + x_n + \tilde{y} + \tilde{h}\| \tag{3}$$

for every $\tilde{y} \in \tilde{E}_0$ (the space of those $\tilde{y} \in \tilde{E}$ which have weakly null representatives in $E^{\mathbf{N}}$); here $\tilde{F}_0$ is the subspace of elements of $\tilde{E}$ defined by sequences belonging to $\mathcal{F}_0$.

The left-hand side of (3) is clearly smaller than the right-hand one. So it suffices to prove that for each sequence $(\tilde{h}^n)_n$ in $\tilde{F}_0$ we have

$$\lim_{n,\mathcal{U}} \|z + x_n + \tilde{y} + \tilde{h}^n\| \geqslant \inf_{\tilde{h}\in\tilde{F}_0} \lim_{n,\mathcal{U}} \|z + x_n + \tilde{y} + \tilde{h}\|.$$

We can restrict ourselves to bounded sequences $(\tilde{h}^n)_n$. As $\tilde{F}_0$ is a closed subspace of the reflexive space $\tilde{E}$, such a sequence has a weak limit $\tilde{h}^\infty$ (along $\mathcal{U}$). So it suffices that

$$\lim_{n,\mathcal{U}} \|z + x_n + \tilde{y} + \tilde{h}^n\| \geqslant \lim_{n,\mathcal{U}} \|z + x_n + \tilde{y} + \tilde{h}^\infty\|.$$

So we are led to the following criterion:

Proposition 5. *Let E be a superreflexive stable Banach space. Suppose E satisfies the following condition:*

(c) *For every ultrapower $\tilde{E}$ of E, every bounded sequence (x_n) in E and* *every bounded sequence* $(\tilde{u}_n)$ *in* $\tilde{E}_0$

$$\lim_{n,\mathscr{U}} \|x_n+\tilde{u}_n\| \geqslant \lim_{n,\mathscr{U}} \|x_n+\tilde{u}_\infty\|$$

where $\tilde{u}_\infty$ *is the weak limit* $w\text{-}\lim_{n,\mathscr{U}}\tilde{u}_n$.

Then E *satisfies the continuity condition of Proposition* 2 (*and hence its quotients are stable*).

Note that in the case where E is a Banach lattice, $\tilde{E}_0$ and E are disjoint sublattices of $\tilde{E}$.

Let us see how the criterion of Prop. 5 applies to the spaces $d(w, p)$ and c_p.

COROLLARY 6. *The superreflexive spaces* $d(w, p)$, $1 < p < \infty$, *have stable quotients.*

We refer to [LT I] for the definition and basic results on the spaces $d(w, p)$.

LEMMA 7. *An ultrapower* $\tilde{E}$ *of a space* $E = d(w, p)$ $(p < \infty)$ *can be split as* $\tilde{E} = E_1 \oplus_p E_2$, *where:*

(i) E_1 *and* E_2 *are disjoint bands which are members of a p-direct sum* (*i.e.* $x_1 \in E_1$, $x_2 \in E_2 \Rightarrow \|x_1+x_2\|^p = \|x_1\|^p+\|x_2\|^p$).

(ii) E_1 *is an atomic lattice* (*in fact, a space* $d(I; w, p)$ *on an uncountable set of indices* I) *containing* E *as a band.*

This decomposition is similar to that of ultrapowers of Orlicz spaces given in [DCK]. Elements of E_2 are those elements $\tilde{x} \in \tilde{E} = E^I/\mathscr{U}$ which can be represented by a family $(x_i)_{i\in I}$ with $\lim_{i,\mathscr{U}}\|x_i\|_{c_0} = 0$. On the other hand, $\tilde{x} \in E_1$ when $(x_i)_{i\in I}$ may be chosen "equisupported" (i.e. $\forall \varepsilon > 0$, $\exists \tilde{x}' \in \tilde{E}$ with $\|\tilde{x}-\tilde{x}'\| < \varepsilon$ and $\exists N \in \boldsymbol{N}$ s.t. $\forall i \in I$, x_i' has a support of cardinal $\leqslant N$). We refer to [R3, Lemma 5] where a similar result is stated for sequences in a symmetric sequence space (not containing c_0). ∎

LEMMA 8. *If* X *is an atomic lattice not containing* c_0, *then*

$$\lim_{n,\mathscr{U}} \|x_n+u_n\| \geqslant \lim_{n,\mathscr{U}} \|x_n+u_\infty\|$$

for every pair of bounded sequences $(x_n)_{n=1}^\infty$, $(u_n)_{n=1}^\infty$ *with* x_n, u_n *disjoint* $(\forall n)$ *and* $u_\infty = w\text{-}\lim_{n,\mathscr{U}} u_n$.

This follows easily from the fact that there is a decomposition $u_n = v_n+w_n$ with v_n, w_n disjoint and $v_n \xrightarrow[n,\mathscr{U}]{} u_\infty$ in the norm topology. ∎

We now verify easily that the spaces $d(w, p)$, $1 < p < \infty$, satisfy condition (c) of Prop. 5 when they are superreflexive.

Indeed, we decompose $\tilde{u}_n = \tilde{u}_n'+\tilde{u}_n''$ $(\tilde{u}_n' \in E_1,\ \tilde{u}_n'' \in E_2)$ and set $\tilde{u}_\infty' = \lim_{n,\mathscr{U}}\tilde{u}_n'$, $\tilde{u}_\infty'' = \lim_{n,\mathscr{U}}\tilde{u}_n''$. We have

$$\begin{aligned}\lim_{n,\mathscr{U}} \|x_n+\tilde{u}_n\|^p &= \lim_{n,\mathscr{U}} \|x_n+\tilde{u}_n'\|^p + \lim_{n,\mathscr{U}} \|\tilde{u}_n''\|^p \\ &\geqslant \lim_{n,\mathscr{U}} \|x_n+\tilde{u}_\infty'\|^p + \|\tilde{u}_\infty''\|^p \quad \text{(by Lemma 8)} \\ &= \lim_{n,\mathscr{U}} \|x_n+\tilde{u}_\infty\|^p. \ ∎\end{aligned}$$

Remark 9. 1. It could be shown similarly that the dual spaces $d(w, p)^*$ also satisfy condition (c) of Prop. 5.

2. Cor. 6 has a vectorial version: if E satisfies (c) then so does $d_{(w,p)}(E)$.

3. Lemma 8 is easily extended to Banach spaces having a boundedly complete monotone Schauder FDD.

COROLLARY 10. *The spaces* c_p, $1 < p < \infty$, *have stable quotients.*

We refer to [GK], [MC] for the definition and properties of $c_p = c_p(H)$.

We proceed analogously to the preceding proof.

Ultrapowers $\tilde{E}$ of $E = c_p$ can be decomposed as $\tilde{E} = E_1 \oplus_p E_2$ where $E_1 = c_p(\mathscr{H})$, $\mathscr{H}$ being a nonseparable Hilbert space. Let $(e_\alpha)_{\alpha\in A}$ be a hilbertian basis of $\mathscr{H}$; we may consider that $N \subseteq A$ and that $(e_\alpha)_{\alpha\in N}$ spans H; the embedding $H \subset \mathscr{H}$ induces the natural embedding of $E = c_p$ into E_1. We decompose: $\tilde{u}_n = \tilde{u}_n'+\tilde{u}_n''$, $\tilde{u}_n' \in E_1$, $\tilde{u}_n'' \in E_2$.

The condition $\tilde{u}_n' \in \tilde{E}_0$ means that $\tilde{u}_n' \in \operatorname{span}\{e_\alpha \otimes e_\beta \mid \alpha, \beta \notin N\}$. The elements x_n, $\tilde{u}_n'$ $(n \in N)$ belong to a subspace of $c_p(\mathscr{H})$ isomorphic to c_p. Using the so-called *shell decomposition* $(S_n)_{n=1}^\infty$ of c_p $(S_n = \operatorname{span}\{e_i \otimes e_n, e_n \otimes e_i \mid i \leqslant n\})$ which is known to be a monotone boundedly complete Schauder FDD, [A], and Remark 9.3 we conclude as in the case of $d(w, p)$. ∎

PROPOSITION 11. *Let* $E = l(X)$ *be a superreflexive space where* X *is a Banach space and* l *an atomic lattice both satisfying condition* (c) *of Prop.* 5. *Then* $l(X)$ *also satisfies condition* (c).

Proof. An ultrapower $\tilde{l}$ of l can be split as

$$\tilde{l} = l \oplus \tilde{l}_0$$

(where l is the canonical image of l in $\tilde{l}$, and $\tilde{l}_0$ is the space of elements having a weakly null representative). Indeed, we have an evident projection π: $\tilde{l} \to l$: $\pi(\tilde{x}) = w\text{-}\lim_{i,\mathscr{U}} x_i$ (when $(x_i)_i$ represents $\tilde{x}$) which is a band projection.

To this decomposition there corresponds a decomposition of the ultrapower $\tilde{E}$ of $E = l(X)$:

$$\tilde{E} = E_1 \oplus E_2 \tag{4}$$

where $E_1 = l(\tilde{X})$, $\tilde{X}$ being the corresponding ultrapower of X. Let N be the random norm (in the sense of [LR]) on $\tilde{E}$ with values in $\tilde{l}$; E_2 is the space of $\tilde{y} \in \tilde{E}$ with $N(y)$ belonging to $\tilde{l}_0$.

Let (x_n), $(\tilde{u}_n)$ be as in Prop. 5, with $\tilde{u}_n = \tilde{u}'_n + \tilde{u}''_n$ $(\tilde{u}'_n \in E_1,\ \tilde{u}''_n \in E_2)$. We have

$$\lim_{n,\mathscr{U}} \|x_n + \tilde{u}_n\| = \lim_{n,\mathscr{U}} \|N(x_n + \tilde{u}'_n) + N(\tilde{u}''_n)\|_{\tilde{l}} \geqslant \lim_{n,\mathscr{U}} \|N(x_n + \tilde{u}'_n) + w\text{-}\lim_{n,\mathscr{U}} N(\tilde{u}''_n)\|_{\tilde{l}}$$

(by condition (c) for l). But $w\text{-}\lim_{n,\mathscr{U}} N(\tilde{u}''_n) \geqslant N(\tilde{u}''_\infty)$ where $\tilde{u}''_\infty = w\text{-}\lim_{n,\mathscr{U}} \tilde{u}''_n$. Thus

$$\lim_{n,\mathscr{U}} \|x_n + \tilde{u}_n\| \geqslant \lim_{n,\mathscr{U}} \|N(x_n + \tilde{u}'_n) + N(\tilde{u}''_\infty)\| = \lim_{n,\mathscr{U}} \|x_n + \tilde{u}'_n + \tilde{u}''_\infty\|.$$

Applying the decomposition (4) to $E_1^N/\mathscr{U}$ we decompose $x_n = x_n^{(1)} + x_n^{(2)}$ and $\tilde{u}'_n = \tilde{u}'^{(1)}_n + \tilde{u}'^{(2)}_n$, with

$$N(x_n^{(1)}) \xrightarrow[n,\mathscr{U}]{\|\ \|} w\text{-}\lim_{m,\mathscr{U}} N(x_m), \qquad N(x_n^{(1)} + \tilde{u}'^{(1)}_n) \xrightarrow[n,\mathscr{U}]{\|\ \|} w\text{-}\lim_{m,\mathscr{U}} N(x_m + u_m)$$

(norm convergence) and

$$N(x_n^{(1)}),\ N(\tilde{u}'_n) \quad \text{disjoint from} \quad N(x_n^{(2)}),\ N(\tilde{u}'^{(2)}_n).$$

We have $N(x_n^{(2)} + \tilde{u}'^{(2)}_n) \geqslant N(x_n^{(2)})$ and thus

$$\lim_{n,\mathscr{U}} \|x_n + \tilde{u}'_n + \tilde{u}''_\infty\| = \lim_{n,\mathscr{U}} \|N(x_n^{(1)} + \tilde{u}'^{(1)}_n) + N(x_n^{(2)} + \tilde{u}'^{(2)}_n) + N(\tilde{u}''_\infty)\|_{\tilde{l}}$$

$$\geqslant \lim_{n,\mathscr{U}} \|w\text{-}\lim_{m,\mathscr{U}} N(x_m + \tilde{u}'_m) + N(x_n^{(2)}) + N(\tilde{u}''_\infty)\|.$$

The weak convergence appearing here is coordinatewise convergence. Therefore, as X satisfies (c), it is easy to see that

$$w\text{-}\lim_{n,\mathscr{U}} N(x_n + \tilde{u}'_n) \geqslant w\text{-}\lim_{n,\mathscr{U}} N(x_n + \tilde{u}'_\infty) \quad \text{where} \quad \tilde{u}'_\infty = w\text{-}\lim_{n,\mathscr{U}} \tilde{u}'_n.$$

Thus

$$\lim_{n,\mathscr{U}} \|x_n + \tilde{u}'_n + \tilde{u}''_\infty\| \geqslant \lim_{n,\mathscr{U}} \|w\text{-}\lim_{m,\mathscr{U}} N(x_m + \tilde{u}'_\infty) + N(x_n^{(2)}) + N(\tilde{u}''_\infty)\|$$

$$= \lim_{n,\mathscr{U}} \|N(x_n^{(1)} + \tilde{u}'_\infty) + N(x_n^{(2)}) + N(\tilde{u}''_\infty)\|$$

$$= \lim_{n,\mathscr{U}} \|x_n + \tilde{u}'_\infty + \tilde{u}''_\infty\| = \lim_{n,\mathscr{U}} \|x_n + \tilde{u}_\infty\|. \ \blacksquare$$

Remark 12. 1. It is not hard to see that a reflexive Banach space E with $\mathscr{T}_0(E)$ locally TUCB-compact satisfies condition (c) of Prop. 5.

2. An easy generalization of Cor. 6 shows that a symmetric sequence space L with $\mathscr{T}_{00}(L)$ locally TUCB-compact also satisfies condition (c). Here $\mathscr{T}_{00}(L)$ is the space of types on L which can be defined by a c_0-null sequence.

II. Stability of rearrangement invariant function spaces

In this part we restrict our attention to rearrangement invariant function spaces (r.i. spaces). We use standard terminology related with this concept as may be found in [LT II].

L will denote a r.i. space on a measure space (Ω, Σ, m) which is one of the spaces N, $[0, 1]$, or $[0, \infty)$ with the natural measure. Let us give the following

DEFINITION 13. A subset $M \subseteq L$ is said to be *equinormable* in L if for each $\varepsilon > 0$ the following conditions hold:

(i) There is $R > 0$ such that $\|f\chi_{\{|f| > R\}}\| < \varepsilon$ for all $f \in M$.

(ii) There is $a > 0$ such that for all $f \in M$ we can choose a subset $A \subseteq \Omega$ with $m(A) \leqslant a$ such that $\|f\chi_{A^c}\| < \varepsilon$.

An equinormable set is obviously bounded. A convergent sequence is equinormable. It is worthwhile to note that for a bounded subset M condition (i) is equivalent to

(i') $\lim_{m(A) \to 0} \sup_{f \in M} \|f\chi_A\| = 0.$

If L does not contain c_0, condition (ii) is equivalent to

(ii') $\lim_{\delta \to 0} \sup_{f \in M} \|f\chi_{\{|f| < \delta\}}\| = 0.$

For bounded subsets, condition (i) (resp. (ii)) is trivially satisfied when $\Omega = N$ (resp. $\Omega = [0, 1]$; in this case, equinormability is the same as equiintegrability in the sense of [LT II], 2.g.9, or [JMST], déf. 6.5).

The following lemma enlightens the usefulness of the preceding notion in studying the stability of these spaces.

If $f \in L$, we denote by f^* the nonincreasing rearrangement of $|f|$ (which is in L).

LEMMA 14. *Let L be an order continuous r.i. space. A subset M of L is equinormable iff the set $M^* = \{f^* \mid f \in M\}$ is relatively compact for the norm topology of L.*

Proof. The equinormability of M is clearly equivalent to that of M^*. In order continuous r.i. spaces, finite sets are equinormable, and hence relatively norm compact sets also.

Conversely, let M be an equinormable set of decreasing functions on $[0, \infty)$. We can suppose the elements of M to be uniformly essentially bounded by $A < \infty$ and to have support in $[0, B]$.

Let $\varepsilon > 0$; we can find $N \in N$ such that $2A\|\chi_{[0,1/N]}\| < \varepsilon$. Let $\mathscr{B}$ be the σ-algebra generated by the intervals $[k/N, (k+1)/N[$, $0 \leqslant k < BN$. If $f \in M$

then

$$|f - E^{\mathscr{B}} f| \leqslant \sum_k \lambda_k \chi_{[k/N,(k+1)/N[} \quad \text{with } \lambda_k = f(k/N) - f((k+1)/N).$$

We have $\lambda_k \geqslant 0$, $\sum_k \lambda_k \leqslant 2A$. Hence

$$\|f - E^{\mathscr{B}} f\| \leqslant \Big\|\sum_k \lambda_k \chi_{[k/N,(k+1)/N[}\Big\|$$

$$\leqslant \sum_k \lambda_k \|\chi_{[k/N,(k+1)/N[}\| \leqslant 2A \|\chi_{[0,1/N[}\| < \varepsilon.$$

Thus there is a bounded subset M' of a finite-dimensional subset of L with Hausdorff distance to M less than ε. ■

We now give a "subsequence splitting lemma" useful for the study of stability in r.i. spaces. A similar lemma for Lorentz spaces $L_{p,q}$ is proved in [R]. See also [JMST] for a similar lemma, valid in "good Banach lattices" having a cotype (this last condition is wrongly omitted in [JMST]).

LEMMA 15. *Let $(f_n)_n$ be a bounded sequence in a r.i. space L not containing c_0. There exist a subsequence $(f_{n_k})_{k=1}^{\infty}$ and a decomposition $f_{n_k} = f_k' + f_k''$ such that:*

(i) *f_k', f_k'' are disjoint.*
(ii) *$(f_k')_k$ is an equinormable set in L.*
(iii) *$(f_k'')_k$ converges to 0 in measure.*

Proof. By Lemma 14 and a diagonal argument, we find a subsequence (denoted again by $(f_n)_n$ for simplicity) such that, for any $\varepsilon > 0$ and $B < \infty$, $(\chi_{[\varepsilon,B]} f_n^*)_n$ converges in L. The limit is $\chi_{[\varepsilon,B]}\varphi$, where *a priori* $\varphi \in L_0(\boldsymbol{R})$; but, by the Fatou property of L, we have $\varphi \in L$. Let $\varepsilon_k \searrow 0$, $B_k \nearrow \infty$. We choose a strictly increasing sequence $(n_k)_k$ such that

$$n \geqslant n_k \Rightarrow \|\chi_{[\varepsilon_k,B_k]}\varphi - \chi_{[\varepsilon_k,B_k]} f_n^*\|_L < 1/k.$$

Finally, we set $f_k' = \chi_{A_k} f_{n_k}$, $f_k'' = \chi_{A_k^c} f_{n_k}$, where

$$A_k = \{\omega \colon f_{n_k}^*(\varepsilon_k) \geqslant |f_{n_k}(\omega)| > f_{n_k}^*(B_k)\}. \ ■$$

Remark. It is easily seen that a sequence $(g_k)_k$ converging to zero in measure can be split (up to extraction) as the sum $g_k' + g_k''$ of a "peak sequence" (g_k') (with supports tending to 0 in measure) and a "flat sequence" (g_k'') (i.e. $\|g_k''\|_\infty \to 0$ as $k \to \infty$).

An application of this lemma and the use of the "stability in distribution" property (cf. [R], Prop. 1) enable us to find a criterion for the stability of r.i. spaces. We first give an equivalent formulation of "stability in distribution" and make explicit its proof (very shortened in [R]).

LEMMA 16. *Let $(f_n)_n$, $(g_m)_m$ be two equinormable sequences in the order continuous r.i. space L. Then*

$$\lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} (f_n + g_m)^* = \lim_{m,\mathscr{V}} \lim_{n,\mathscr{U}} (f_n + g_m)^*$$

(where the limits exist in the norm topology of L by Lemma 14).

Proof. It is sufficient to prove that the two limits appearing here have the same distribution (with respect to the Lebesgue measure).

We can suppose the (f_n), (g_m) to be uniformly bounded in L and the measure space (Ω, m) to be finite. We are thus led to prove

$$\lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \int F(|f_n + g_m|)\, dm = \lim_{m,\mathscr{V}} \lim_{n,\mathscr{U}} \int F(|f_n + g_m|)\, dm \tag{5}$$

for all bounded continuous functions F. We approximate the function $G(s,t) = F(|s+t|)$ by a finite sum $\sum_i \varphi_i(s)\psi_i(t)$ where φ_i, ψ_i are bounded continuous. As for each i, $(\varphi_i(f_n))_n$, $(\psi_i(g_m))_m$ weakly converge in $L^2(\Omega)$, the commutation of the limits (5) becomes clear. ■

PROPOSITION 17. *A r.i. space L not containing c_0 is stable if and only if*

$$\lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \|\varphi \dotplus f_n \dotplus g_m\| = \lim_{m,\mathscr{V}} \lim_{n,\mathscr{U}} \|\varphi \dotplus f_n \dotplus g_m\|$$

whenever $\varphi \in L$, (f_n) and (g_m) are L_0-null sequences (and $\mathscr{U}$, $\mathscr{V}$ are nontrivial ultrafilters on $\boldsymbol{N}$).

(Here $\dotplus$ signifies disjoint sum: $\|f \dotplus g\| = \|f_1 + g_1\|$ where f_1, g_1 are disjoint and have respectively the same distribution as f, g.)

Proof. Let (f_n), (g_m) be bounded sequences in L. We may suppose to have decompositions $f_n = f_n' + f_n''$, $g_m = g_m' + g_m''$ as in the splitting lemma 15. Then

$$\lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \|f_n' + f_n'' + g_m' + g_m''\| = \lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \|(f_n' + g_m') \dotplus f_n'' \dotplus g_m''\|$$

$$= \lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \|(f_n' + g_m')^* \dotplus f_n'' \dotplus g_m''\|$$

$$= \lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} \|\varphi \dotplus f_n'' \dotplus g_m''\|$$

where $\varphi = \lim_{n,\mathscr{U}} \lim_{m,\mathscr{V}} (f_n' + g_m')^*$. Lemma 16 then clearly implies Prop. 17. ■

A particular case where Proposition 17 can be used is the one where $\mathscr{T}_{00}(L)$, the space of types defined by L_0-null sequences, is generated by only one type (which is necessarily an l^q-type in the terminology of [KM]). So are the L_q spaces; the $L_{p,q}$ and $L_{w,q}$ spaces (by [R]); and the duals $L_{w,q}^*$ (when they are reflexive).

Moreover, this method will be used in the sequel in order to establish the stability in two new cases: vector-valued function spaces and interpolation spaces between r.i. spaces.

If L is a lattice of functions on Ω we define $L(E)$ as the vector space of Bochner measurable functions $F\colon \Omega \to E$ such that the corresponding scalar functions $f(\cdot) = \|F(\cdot)\|_E$ belong to L, endowed with the natural norm $\|F\|_{L(E)} = \|f\|_L$.

Note that Prop. 17 shows that a r.i. space L (not containing c_0) is stable iff its sublattices generated by disjoint (L_0-null) sequences are stable. We will use a vectorial version of this fact to prove the following.

PROPOSITION 18. *If L is a stable r.i. space and E a stable Banach space then $L(E)$ is stable.*

Proof. Let $(F_n)_n$, $(G_k)_k$ be two bounded sequences in $L(E)$. We apply Lemma 15 to the bounded sequences $(f_n)_n$, $(g_k)_k$ in L where

$$f_n = \|F_n\|_E, \quad g_k = \|G_k\|_E,$$

and a simple disjointification argument to obtain

$$\lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \|F_n + G_k\|_E = \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \big\| \|F_n^0 + G_k^0\|_E \dotplus \|F_n'\|_E \dotplus \|G_k'\|_E \big\|_L$$

where $(\|F_n^0\|_E)_n$, $(\|G_k^0\|_E)_k$ are equinormable in L and $(\|F_n'\|_E)_n$, $(\|G_k'\|_E)_k$ are L_0-null sequences in L.

The above expression is equal to

$$\lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \big\| \varphi \dotplus \|F_n'\|_E \dotplus \|G_k'\|_E \big\|_L$$

where $\varphi = \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \|F_n^0 + G_k^0\|_E^*$. As E is stable, the vectorial analog of Lemma 16 is true ([R], Prop. 1) and says that $\varphi = \lim_{k,\mathscr{V}} \lim_{n,\mathscr{U}} \|F_n^0 + G_k^0\|_E^*$. Proposition 18 is now clear. ∎

III. Stability of real interpolation spaces between r.i. spaces

Now we pass to the stability of real interpolation spaces between r.i. spaces. We begin by recalling some usual notation. Let (A_0, A_1) be an interpolation pair of Banach spaces. i.e. these spaces are given as subspaces of a common Hausdorff topological vector space so that the corresponding embeddings are continuous. Let X be a Banach space with an unconditional basis $(\varepsilon_m)_{-\infty}^{\infty}$, throughout fixed, and $0 < \theta < 1$. The space $(A_0, A_1)_{\theta,X}$ is defined to be the Banach space of all elements $f \in A_0 + A_1$ such that $\sum_{m=-\infty}^{\infty} j_m(f)\varepsilon_m \in X$ where

$$j_m(f) = \inf\{e^{-\theta m}\|f_0\|_0 + e^{(1-\theta)m}\|f_1\|_1;\ f = f_0 + f_1,\ f_0 \in A_0,\ f_1 \in A_1\}$$

($\|\cdot\|_0$, $\|\cdot\|_1$ denote respectively the norms of A_0, A_1) and normed by

$$\|f\|_{\theta,X} = \Big\| \sum_{m=-\infty}^{\infty} j_m(f)\varepsilon_m \Big\|_X.$$

When $X = l_p$, this space is isomorphic to the classical real interpolation space $(A_0, A_1)_{\theta,p}$ introduced by Lions and Peetre (see [BL] and [Be]). The norm we use has been introduced in a paper by Davis, Figiel, Johnson and Pełczyński ([DFJP]) and it is equivalent to the classical one if the basis $(\varepsilon_m)_m$ is translation invariant.

The stability of $(l_p, l_q)_{\theta,X}$, $1 \leqslant p, q < \infty$, for X stable was proved in [B]. The theorem we will show here extends that case as well as the one corresponding to Lorentz spaces $L_{p,q}$ which occurs in [R].

In order to prove the stability of $(A_0, A_1)_{\theta,X}$ we realize that $(A_0, A_1)_{\theta,X}$ is the diagonal subspace of $\bigoplus_{m,X}(A_0 + A_1, j_m)$ and so, by using Proposition 4 of [B], it is sufficient to establish the stability of each norm j_m, $m \in \mathbf{Z}$, whenever X is stable.

We will be concerned with the case where A_0 and A_1 are r.i. spaces. Since there are no differences in proving the stability of any j_m, $m \in \mathbf{Z}$, we will consider the case $m = 0$. We simplify the notation in the following way. If $f \in A_0 + A_1$, let

$$\|f\|_* = \inf\{\|f_0\|_0 + \|f_1\|_1;\ f = f_0 + f_1,\ f_i \in A_i,\ i = 0, 1\}.$$

The space $(A_0 + A_1, \|\cdot\|_*)$ is a r.i. space and we study the case $\Omega = [0, 1]$ in the first place.

PROPOSITION 19. *Let A_0, A_1 be r.i. spaces on $[0, 1]$. Suppose that A_0 is separable and stable. If*

(i) $$\lim_{R\to\infty} \sup_{\|f\|_1 \leqslant 1} \|f\chi_{\{|f|>R\}}\|_0 = 0,$$

then $(A_0 + A_1, \|\cdot\|_*)$ is stable. Moreover, if X is a stable Banach space with an unconditional basis and $0 < \theta < 1$, then $(A_0, A_1)_{\theta,X}$ is stable.

Proof. Condition (i) is equivalent to the equiintegrability of the unit ball of A_1 in A_0, and so it implies that A_1 embeds in A_0. Note that we need not suppose the stability of A_1. In this case $\|\cdot\|_*$ is equivalent to $\|\cdot\|_0$.

Let φ be a fixed function in A_0 and $(f_n)_n$, $(g_k)_k$ two peak sequences in A_0. We have to show that

$$\lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \|\varphi \dotplus f_n \dotplus g_k\|_* = \lim_{k,\mathscr{V}} \lim_{n,\mathscr{U}} \|\varphi \dotplus f_n \dotplus g_k\|_*.$$

Let $n, k \in \mathbf{N}$ and $\varepsilon > 0$ fixed. There exist $h_i \in A_i$ $(i = 0, 1)$ such that $h_0 + h_1 = \varphi \dotplus f_n \dotplus g_k$ and

$$\|h_0\|_0 + \|h_1\|_1 < \|\varphi \dotplus f_n \dotplus g_k\|_* + \varepsilon.$$

Now consider

$$\tilde{h}_0 = h_0 \chi_{\mathrm{supp}\varphi} + (h_0+h_1)\chi_{\mathrm{supp}(f_n \dot+ g_k)},$$
$$\tilde{h}_1 = h_1 \chi_{\mathrm{supp}\varphi}.$$

We have $\tilde{h}_i \in A_i$ $(i=0, 1)$, $\tilde{h}_0+\tilde{h}_1 = \varphi \dot+ f_n \dot+ g_k$, $\|\tilde{h}_1\|_1 \leqslant \|h_1\|_1$ and

$$|\,\|h_0\|_0 - \|\tilde{h}_0\|_0| \leqslant \|h_1 \chi_{\mathrm{supp}(f_n+g_k)}\|_0.$$

Because of condition (i), for k and n large enough, we may assume $|\,\|h_0\|_0-\|\tilde{h}_0\|_0| < \varepsilon$ (see the comments after the definition of equinormability).

Hence, if we set $\tilde{l}_0 = \tilde{h}_0-(f_n \dot+ g_k) = \tilde{h}_0 \chi_{\mathrm{supp}\varphi}$ we have $\tilde{l}_0+\tilde{h}_1 = \varphi$ and

$$\|\tilde{l}_0 \dot+ (f_n \dot+ g_k)\|_0 + \|\tilde{h}_1\|_1 < \|\varphi \dot+ f_n \dot+ g_k\|_* + 2\varepsilon$$

for k and n large enough. Thus

$$\lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \|\varphi \dot+ f_n \dot+ g_k\|_* = \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\|l_0 \dot+ f_n \dot+ g_k\|_0 + \|l_1\|_1)$$

and the same is true for the reverse limits.

We are going to finish the proof by proving that

(6) $$\lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\|l_0 \dot+ f_n \dot+ g_k\|_0 + \|l_1\|_1) = \inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\tau(l_0)+\|l_1\|_1)$$

where τ is the type on A_0 defined by

$$\tau(l_0) = \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \|l_0 \dot+ f_n \dot+ g_k\|_0 = \lim_{k,\mathscr{V}} \lim_{n,\mathscr{U}} \|l_0 \dot+ f_n \dot+ g_k\|.$$

The inequality $\leqslant$ is trivial. Conversely, we may suppose that φ is a simple function, then $\varphi = \sum_{j=1}^{N} a_j \chi_{B_j}$ (the B_j's pairwise disjoint). Consider the sub-σ-algebra $\mathscr{B}$ on $[0, 1]$ generated by the sets B_j, $1 \leqslant j \leqslant N$, and the Borel sets in $[0, 1] \setminus (\bigcup_{j=1}^{N} B_j)$.

It is a well-known fact that the conditional expectation operator $E^{\mathscr{B}}$ is a norm one projection on every r.i. space. If $\varphi = l_0+l_1$, $l_i \in A_i$, then $\varphi = E^{\mathscr{B}}(l_0)+E^{\mathscr{B}}(l_1)$, and

$$\|E^{\mathscr{B}}(l_0) \dot+ f_n \dot+ g_k\|_0 \leqslant \|l_0 \dot+ f_n \dot+ g_k\|_0, \quad \|E^{\mathscr{B}}(l_1)\|_1 \leqslant \|l_1\|_1.$$

Consequently, when considering

$$\inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\|l_0 \dot+ f_n \dot+ g_k\|_0 + \|l_1\|_1)$$

we may suppose that the l_i's are simple functions supported on B_j's.

Given $\varepsilon > 0$, for each pair of $n, k \in N$ we can choose a simple function $l_i^{n,k}$, $i = 0, 1$, so that $\varphi = l_0^{n,k}+l_1^{n,k}$ and

$$\|l_0^{n,k} \dot+ f_n \dot+ g_k\|_0 + \|l_1^{n,k}\|_1 < \inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\|l_0 \dot+ f_n \dot+ g_k\|_0 + \|l_1\|_1) + \varepsilon.$$

Since all the functions $\{l_i^{n,k};\ i = 0, 1,\ n, k \in N\}$ belong to a finite-dimensional bounded set we may assume that the limits

$$\bar{l}_0 = \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} l_0^{n,k} \quad \text{and} \quad \bar{l}_1 = \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} l_1^{n,k}$$

exist. Thus $\bar{l}_0+\bar{l}_1 = \varphi$ and

$$\begin{aligned} \tau(\bar{l}_0)+\|\bar{l}_1\|_1 &= \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \|\bar{l}_0 \dot+ f_n \dot+ g_k\|_0 + \|\bar{l}_1\|_1 \\ &= \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} (\|l_0^{n,k} \dot+ f_n \dot+ g_k\|_0 + \|l_1^{n,k}\|_1) \\ &\leqslant \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\|l_0 \dot+ f_n \dot+ g_k\|_0 + \|l_1\|_1) + \varepsilon. \end{aligned}$$

Eventually we get

$$\inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} \tau(l_0)+\|l_1\|_1 \leqslant \lim_{n,\mathscr{U}} \lim_{k,\mathscr{V}} \inf_{\substack{l_0+l_1=\varphi \\ l_i \in A_i}} (\|l_0 \dot+ f_n \dot+ g_k\|_0 + \|l_1\|_1)$$

and this ends the proof of formula (6) and of the proposition. ∎

The main tool in the above proposition is the fact that peak sequences in A_1 are null sequences in A_0 and this is deduced from condition (i). The use of the conditional expectation operator implies that the "inf" appearing in the formulas is taken over a "finite number" of decompositions $\varphi = l_0+l_1$.

For the corresponding theorem in the sequence case we need an inverse condition, namely flat sequences in A_0 are null sequences in A_1.

PROPOSITION 20. *Let A_0, A_1 be r.i. spaces on N. Suppose now that A_1 is separable and stable. If*

(ii) $$\lim_{\delta \to 0} \sup_{\|g\|_0 \leqslant 1} \|g \chi_{\{|g| < \delta\}}\|_1 = 0,$$

then $(A_0+A_1, \|\cdot\|_*)$ is stable. Moreover, if X is a stable Banach space with an unconditional basis and $0 < \theta < 1$, then $(A_0, A_1)_{\theta,X}$ is stable.

Proof. In this case we have $A_0 \subseteq A_1$. The proof goes as in the preceding case. The only difference is to pass the flat part from the norm $\|\cdot\|_0$ to the norm $\|\cdot\|_1$ by using condition (ii). ∎

The case $\Omega = [0, \infty)$ is the union of the two previous ones. Now we need the stability of A_0 and A_1 as well as conditions (i) and (ii). We also have to make sure that $(A_0+A_1, \|\cdot\|_*)$ does not contain c_0.

We will sketch the proof of this last result for the sake of completeness.

LEMMA 21. *If A_0, A_1 are r.i. spaces on $[0, \infty)$ which both do not contain c_0, then $(A_0+A_1, \|\cdot\|_*)$ has the same property.*

Sketch of proof. If c_0 would embed in $(A_0+A_1, \|\cdot\|_*)$, there would be a sequence $(s_n)_n$ of disjoint simple functions in A_0+A_1 whose closed span would be isomorphic to c_0 (cf. [MN], Cor. I.8). Put

$$s_n = \sum_{i=K_n}^{K_{n+1}} a_i \chi_{B_i}$$

where $(K_n)_n$ is an increasing sequence of integers and the sets $(B_i)_i$ are pairwise disjoint.

Let E_0, E_1, E be the closed linear spans of the functions $(\chi_{A_i})_i$ in A_0, A_1, A_0+A_1, respectively. By using the conditional expectation operator with respect to the σ-algebra generated by the subsets $(B_i)_i$, it is easy to prove that

$$\|f\|_* = \inf_{\substack{f=f_0+f_1 \\ f_i\in E_i}} (\|f_0\|_0+\|f_1\|_1) \quad \text{if } f\in E.$$

Thus $E \cong E_0 \oplus_1 E_1/E_0 \cap E_1$. Since E_0, E_1 have unconditional bases they are dual spaces; then E would also be a dual space, and this contradicts the fact of containing a copy of c_0. ■

We now obtain, similarly to Props. 19 & 20:

PROPOSITION 22. *Let A_0, A_1 be separable and stable r.i. spaces on $[0, \infty)$. Suppose that conditions* (i) *and* (ii) *appearing in the preceding propositions are satisfied. Then $(A_0+A_1, \|\cdot\|_*)$ is stable. Moreover, if X is a stable Banach space with an unconditional basis and $0<\theta<1$, then $(A_0, A_1)_{\theta,X}$ is stable.*

As an application of Proposition 19 we can give the following:

COROLLARY 23. *Every stable Banach space with 1-unconditional basis can be $(1+\varepsilon)$-embedded (as a complemented subspace) into a stable r.i. function space on $[0, 1]$.*

In [B] it is shown that a stable Banach space X with 1-unconditional basis can be embedded into a stable space with symmetric basis, in fact an interpolation space $[l_p, l_q]_{\theta,X}$.

Here we use basically the same construction, but with the interpolation space $[L_p, L_q]_{\theta,X}$. Such embeddings were considered in [JMST], §10. ■

Acknowledgements. We thank Professor B. Maurey for his suggestions concerning the main results of §III.

References

[A] J. Arazy, *Basic sequences, embeddings and the uniqueness of the symmetric structure in unitary sequence spaces*, J. Funct. Anal. 40 (1981), 302–340.

[B] J. Bastero, *Embedding unconditional stable Banach spaces into symmetric stable Banach spaces*, Israel J. Math. 53 (3) (1986), 373–380.

[BM] J. Bastero and J. M. Mira, *Stability of vector valued Banach sequence spaces*, Bull. Polish Acad. Sci. Math. 34 (1–2) (1986), 47–53.

[Be] B. Beauzamy, *Espaces d'interpolation réels; topologie et géométrie*, Lecture Notes in Math. 666, Springer, 1978.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.

[C] M. A. Canela, *Stable quotients of l_p*, Arch. Math. (Basel) 44 (1985), 446–450.

[DCK] D. Dacunha-Castelle et J.-L. Krivine, *Application des ultraproduits à l'étude des espaces et algèbres de Banach*, Studia Math. 41 (1972), 315–334.

[DFJP] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. 17 (1974), 311–327.

[Ga] D. J. H. Garling, *Stable Banach spaces, random measures and Orlicz function spaces*, in: Lecture Notes in Math. 928, Springer, 1982, 121–175.

[G] S. Guerre, *Espaces quotient et stabilité*, C. R. Acad. Sci. Paris Sér. I 300 (14) (1985), 485–487.

[G2] —, *Types et suites symétriques dans L^p*, Israel J. Math. 53 (2) (1986), 191–208.

[G3] —, *Sur les espaces stables universels*, Studia Math. 81 (1985), 221–229.

[GK] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs 18, Amer. Math. Soc.

[JMST] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. 19 (217) (1979).

[KM] J.-L. Krivine et B. Maurey, *Espaces de Banach stables*, Israel J. Math. 39 (1981), 273–295.

[LR] M. Lévy et Y. Raynaud, *Ultrapuissances des espaces $L^p(L^q)$*, C. R. Acad. Sci. Paris Sér. I 299 (3) (1984), 81–84.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I: Sequence Spaces; II: Function Spaces*, Springer, 1977 and 1979.

[MC] C. A. McCarthy, *c_p*, Israel J. Math. 5 (1967), 249–271.

[MN] P. Meyer-Nieberg, *Zur schwachen Kompaktheit in Banachverbänden*, Math. Z. 134 (1973), 303–315.

[R] Y. Raynaud, *Deux nouveaux exemples d'espaces stables*, C. R. Acad. Sci. Paris Sér. I 292 (1981), 715–717.

[R2] —, *Sur les sous-espaces de $L^p(L^q)$*, Sém. de Géométrie des Espaces de Banach, Univ. Paris 6 et 7, 1984–85.

[R3] —, *Stabilité des espaces C_E*, Sém. de Géométrie des Espaces de Banach, Univ. Paris 7, 1982–83.

DEPTO DE TEORÍA DE FUNCIONES
FACULTAD DE CIENCIAS
UNIVERSIDAD DE ZARAGOZA
50 009 Zaragoza, Spain

ÉQUIPE D'ANALYSE
UNIVERSITÉ PARIS VI
Tour 46, 4ème étage
4, Pl. Jussieu, 75252 Paris Cedex 05, France

Received September 24, 1987 (2364)