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A Paley–Wiener type theorem for regular operators of bounded support

by

JÓZEF BURZYK (Katowice)

Abstract. The main result of this paper is the following Paley–Wiener type theorem: an entire function $F(z)$ is the Fourier transform of a Mikusiński operator with support in $[-\sigma, \sigma]$ if and only if $F(z)$ is of the Cartwright class, i.e. it is of exponential type σ and satisfies a certain integral inequality.

1. Introduction. In this paper the Mikusiński operators are understood in the sense of [4]. According to [4], $\mathscr{C}$ denotes the ring of continuous functions which vanish to the left of some point of the real line. The ring operations in $\mathscr{C}$ are the usual addition and the convolution as the product operation:

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau.$$

By the Titchmarsh theorem, $\mathscr{C}$ is a ring without zero divisors. The fractions with respect to this ring will be denoted by

$$\frac{f}{g}$$

where $f, g \in \mathscr{C}$ and $g \neq 0$, i.e., the fraction denotes the operator which is the class of all pairs (h, k) such that $h, k \in \mathscr{C}$, $k \neq 0$ and $f*k = g*h$. The space of operators is given the usual operations of addition and the convolution defined analogously to the product of usual fractions, i.e.,

$$\text{if} \quad x = \frac{f}{g},\ y = \frac{h}{k}, \quad \text{then} \quad x*y = \frac{f*h}{g*k}.$$

Traditionally the product of operators x and y as well as convolution of functions are both denoted by juxtaposition xy. In this paper the symbols

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$fg, f/g$ will denote the pointwise product and quotient of the functions f, g, respectively.

In general, the notion of support cannot be defined for all operators. T. K. Boehme in [3] introduces the so-called regular operators for which the support is defined. The class of regular operators is defined in terms of delta-sequences.

By a *delta-sequence* we shall mean a sequence $\{\delta_n\}$ of smooth functions with the following properties:

(i) There is a sequence of positive numbers ε_n converging to zero such that $\delta_n(t) = 0$ whenever $|t| \geqslant \varepsilon_n$ for $n \in N$.

(ii) $\int \delta_n = 1$ for $n \in N$.

(iii) There is a number M such that $\int |\delta_n| < M < \infty$ for $n \in N$.

The class of delta-sequences is closed under coordinatewise convolution.

Following T. K. Boehme ([3]) an operator x is said to be *regular* if it admits the so-called *regular representation*, i.e., there is a sequence $\{f_n\}$ in $\mathscr{C}$ and a delta-sequence $\{\delta_n\}$ such that

$$x = \frac{f_n}{\delta_n}$$

for $n \in N$.

Distributions vanishing to the left of some point of the real line are regular operators. There are regular operators which are not distributions. The set of regular operators is an algebra with respect to addition and multiplication of operators. Note that in Boehme's definition of a delta-sequence in [3], it is assumed that $\delta_n \geqslant 0$ for $n \in N$. Our class of delta-sequences is essentially wider in the sense that there are regular operators in our sense which are not regular in the Boehme sense.

A regular operator x is said to be *zero* on an open set U if for every compact set $K \subset U$ there is a regular representation $x = \dfrac{f_n}{\delta_n}$ for $n \in N$ such that $f_n(t) = 0$ for $t \in K$ and $n \in N$. By the *support* of a regular operator x we mean the smallest closed set outside which the operator equals zero, denoting it by $\operatorname{supp} x$.

The supports of a distribution vanishing to the left of some point of the real line in the distributional sense and in the above sense coincide. Notice that the support of every regular operator is bounded from the left.

A regular operator x is said to be of *bounded support* if there is a number $\sigma \geqslant 0$ such that $\operatorname{supp} x \subset [-\sigma, \sigma]$.

Distributions of bounded support are regular operators of bounded support. There are regular operators of bounded support which are not distributions of bounded support ([3]). The set of all regular operators of bounded support is an algebra with respect to the operations of addition and multiplication described above.

Now we are going to define the Fourier transform of a regular operator of bounded support. First we recall some basic definitions concerning entire functions of one complex variable.

An entire function $F(z)$ is said to be of *exponential type* σ if for each $\varepsilon > 0$ there is a constant C_ε such that

$$|F(z)| \leqslant C_\varepsilon e^{(\sigma+\varepsilon)|z|}$$

for $z \in \boldsymbol{C}$; $F(z)$ is said to be of *exponential type* if it is of type σ for some σ, $0 \leqslant \sigma < \infty$ ([2], Ch. 2).

In the sequel we shall frequently refer to the following integral inequality:

$$(*) \qquad \int_{-\infty}^{\infty} (1+t^2)^{-1} \log^+ |F(t)|\, dt < \infty,$$

where $\log^+ |F(t)| = \max(\log |F(t)|, 0)$. The class of all entire functions of exponential type satisfying condition $(*)$ is sometimes called the *Cartwright class*.

If f is a continuous function (distribution) of bounded support such that $\operatorname{supp} f \subset [-\sigma, \sigma]$ for some $\sigma \geqslant 0$ and $F(z)$ is the Fourier transform of $f(t)$, i.e.

$$\hat{f}(z) = F(z) = \int_{-\infty}^{\infty} e^{izt} f(t)\, dt,$$

then $F(z)$ is an entire function of exponential type σ and on the real axis it satisfies condition $(*)$. It is worth while to remark that there are entire functions of exponential type σ satisfying condition $(*)$ which are not the Fourier transforms of any distribution of bounded support. In this paper we show that this cannot happen for regular operators of bounded support.

Assume that x is a regular operator of bounded support such that $\operatorname{supp} x \subset [-\sigma, \sigma]$ for some $\sigma \geqslant 0$ and

$$(1) \qquad x = \frac{f_n}{\delta_n}$$

for $n \in N$ is a regular representation for x, i.e. $\{f_n\}$ is a sequence in $\mathscr{C}$ and $\{\delta_n\}$ is a delta-sequence such that $\operatorname{supp} \delta_n \subset [-\varepsilon_n, \varepsilon_n]$ for $n \in N$, where $\varepsilon_n \to 0$. By the definition of the support of x for every interval $[c, d] \subset \boldsymbol{R} \setminus [-\sigma-\varepsilon_n, \sigma+\varepsilon_n]$ there is a sequence $\{g_m\}$ in $\mathscr{C}$ and a delta-sequence $\{\tilde{\delta}_n\}$ such that

$$(2) \qquad x = \frac{g_m}{\tilde{\delta}_m}$$

and $g_m = 0$ on $[c-\varepsilon_n, d+\varepsilon_n]$. By (1) and (2), we get

$$f_n * \tilde{\delta}_m = g_m * \delta_n$$

for $m, n \in N$. We note that $g_m * \delta_n = 0$ on $[c, d]$ for $m, n \in N$, and $f_n * \tilde{\delta}_m \to f_n$ almost uniformly (a.u.) as $m \to \infty$. Hence $f_n = 0$ on $[c, d]$ or, equivalently, $\operatorname{supp} f_n \subset [-\sigma-\varepsilon_n, \sigma+\varepsilon_n]$ for $n \in N$. Consequently, the Fourier transforms $\hat{f}_n(z)$ and $\hat{\delta}_n(z)$ are entire functions of exponential type $\sigma+\varepsilon_n$ and ε_n, respectively.

Moreover, by (1) and the definition of an operator, we have $f_n * \delta_m = f_m * \delta_n$ and, consequently,

$$\hat{f}_n(z)\hat{\delta}_m(z) = \hat{f}_m(z)\hat{\delta}_n(z)$$

for $m, n \in N$. Since $\hat{\delta}_m(z) \to 1$ a.u., we see that the orders of the zeros of $\hat{\delta}_n(z)$ are not greater than those of $\hat{f}_n(z)$. Thus the quotients $\hat{f}_n(z)/\hat{\delta}_n(z)$ for $n \in N$ are entire functions which are all equal. Their common value is denoted by $\hat{x}(z)$ and called the *Fourier transform* of x. Therefore we have

$$\hat{x}(z) = \hat{f}_n(z)/\hat{\delta}_n(z)$$

for $n \in N$. Since $\hat{\delta}_n(z) \to 1$ a.u., we see that $\hat{f}_n(z) \to \hat{x}(z)$ a.u. The definition of the Fourier transform of a regular operator of bounded support is consistent, i.e. it does not depend on the regular representation of the operator.

If x and y are regular operators of bounded support, then

$$(x*y)^{\wedge}(z) = \hat{x}(z)\hat{y}(z).$$

The main result of this paper is the following Paley–Wiener type theorem.

THEOREM 1. (a) *The Fourier transform of a regular operator with support in $[-\sigma, \sigma]$ for some $\sigma \geq 0$ is an entire function of exponential type σ and satisfies condition* (*).

(b) *Conversely, every entire function of exponential type σ which satisfies condition* (*) *is the Fourier transform of a regular operator with support in $[-\sigma, \sigma]$.*

The proof of Theorem 1 is given in Section 2.

In Section 3 we are concerned with various characterizations of regular operators. Section 4 is devoted to a certain type of convergence of sequences of regular operators.

2. Proof of Theorem 1. We start this proof with recalling some facts concerning entire functions. From Th. 6.3.6, p. 85 in [2] it follows that if $F(z)$ is an entire function of exponential type, then (*) holds iff

$$\int_{-\infty}^{\infty} (1+t^2)^{-1} \left|\log|F(t)|\right| dt < \infty.$$

Hence, by routine calculations, we get

LEMMA 1. *If $F(z)$, $G(z)$ and $H(z)$ are entire functions of exponential type such that*

$$H(z) = F(z)/G(z)$$

and $F(z)$ and $G(z)$ satisfy condition (*), *then so does $H(z)$.*

In the sequel we shall refer to the following

LEMMA 2. *If $F(z)$, $G_n(z)$ and $H_n(z)$ are entire functions, $G_n(z)$ is of exponential type $\sigma+\eta_n$, $H_n(z)$ is of exponential type ε_n, where $\eta_n > 0$, $\varepsilon_n > 0$, $\eta_n \to 0$ and $\varepsilon_n \to 0$, and*

$$F(z) = G_n(z)/H_n(z)$$

for $n \in N$, then $F(z)$ is of exponential type σ.

Proof. Let ε and β be positive numbers such that $\sigma\beta < \varepsilon$ and let n be an integer such that

$$(\sigma+2\eta_n)(1+\beta)+4\varepsilon_n\beta^{-1}(1+\beta) < \sigma+\varepsilon. \tag{3}$$

Under the conditions of the lemma there are constants A and B such that

$$|G_n(z)| < Ae^{(\sigma+2\eta_n)|z|}, \quad |H_n(z)| < Be^{2\varepsilon_n|z|}$$

for $z \in C$. By the Hörmander inequality ([5], p. 154, Lemma 3.1), we get

$$|F(z)| = |G_n(z)/H_n(z)| \leq Ae^{(\sigma+2\eta_n)(1+\beta)|z|}(Be^{2\varepsilon_n(1+\beta)|z|})^{2/\beta}.$$

In view of (3), we have $|F(z)| < C_\varepsilon e^{(\sigma+\varepsilon)|z|}$ for $z \in C$, where $C_\varepsilon = AB^{2/\beta}$. This means that F is of type σ, which was to be proved.

Now we can prove the first part of Theorem 1.

Proof of Theorem 1(a). Assume that x is a regular operator such that $\operatorname{supp} x \subset [-\sigma, \sigma]$ for some $\sigma \geq 0$ and assume that $x = \frac{f_n}{\delta_n}$ for $n \in N$ is a regular representation for x such that $\operatorname{supp}\delta_n \subset [-\varepsilon_n, \varepsilon_n]$ for $n \in N$, where $\varepsilon_n \to 0$. Then, by the definition of the Fourier transform, we have

$$\hat{x}(z) = \hat{f}_n(z)/\hat{\delta}_n(z)$$

for $n \in N$ and $\hat{x}(z)$ is an entire function. For every $n \in N$, $\hat{f}_n$ and $\hat{\delta}_n$ are entire functions of exponential type $\sigma+\varepsilon_n$ and ε_n, respectively. Hence, by Lemma 2, $\hat{x}$ is an entire function of exponential type σ. Moreover, the functions $\hat{f}_n$ and $\hat{\delta}_n$ satisfy condition (*). Thus, by Lemma 1, $\hat{x}$ satisfies condition (*). This completes the proof of the first part of Theorem 1.

The proof of the second part is preceded by some definitions, remarks and lemmas.

First, denote traditionally by $\mathscr{D}$ the space of Schwartz's test functions and, moreover, put

$$\mathscr{Z} = \{\hat{\varphi}\colon\ \varphi \in \mathscr{D}\}, \qquad \mathscr{D}_\varepsilon = \{\varphi \in \mathscr{D}\colon\ \operatorname{supp}\varphi \subset [-\varepsilon, \varepsilon]\}, \qquad \mathscr{Z}_\varepsilon = \{\hat{\varphi}\colon\ \varphi \in \mathscr{D}_\varepsilon\}.$$

By the classical Paley–Wiener theorem $F \in \mathscr{Z}_\varepsilon$ iff F is an entire function of exponential type ε and $F(t)$ is a rapidly decreasing function on the real line.

To prove the second part of Theorem 1 it is enough to show that if F is an entire function of exponential type σ and satisfies condition $(*)$, then there is a delta-sequence $\{\delta_n\}$ such that $F\hat{\delta}_n \in \mathscr{Z}$ for $n \in N$. In fact, assume that $\{\delta_n\}$ is a delta-sequence such that $F\hat{\delta}_n \in \mathscr{Z}$ and $\operatorname{supp}\delta_n \subset [-\varepsilon_n, \varepsilon_n]$ for $n \in N$, where $\varepsilon_n \to 0$. Then $F\hat{\delta}_n \in \mathscr{Z}_{\sigma+\varepsilon_n}$ for $n \in N$ or, equivalently, there is a sequence $\{f_n\}$ in $\mathscr{D}$ such that $\operatorname{supp} f_n \subset [-\sigma-\varepsilon_n, \sigma+\varepsilon_n]$ and $F\hat{\delta}_n = \hat{f}_n$ for $n \in N$. Hence $\hat{f}_n\hat{\delta}_m = \hat{f}_m\hat{\delta}_n$ and, consequently, $f_n * \delta_m = f_m * \delta_n$ for $m, n \in N$. Letting

$$x = \frac{f_n}{\delta_n}$$

for $n \in N$, we see that $\hat{x} = F$ and $\operatorname{supp} x \subset [-\sigma, \sigma]$.

The main tool in proving the existence of a delta-sequence with the above property is the following

THEOREM 2 (Beurling–Malliavin [1]). *If F is an entire function of exponential type and satisfies condition $(*)$, then for every $\varepsilon > 0$ there is a nonzero measure f such that $\operatorname{supp} f \subset [-\varepsilon, \varepsilon]$ and $F\hat{f} \in \mathscr{Z}$.*

As a matter of fact, we shall use the following stronger form of the Beurling–Malliavin theorem.

THEOREM 3. *If F is an entire function of exponential type and satisfies condition $(*)$, then for every $\varepsilon > 0$ and for every $z_0 \in C$ there exists $\varphi \in \mathscr{D}_\varepsilon$ such that $F\hat{\varphi} \in \mathscr{Z}$, $\hat{\varphi}(z_0) \neq 0$ and $\int\varphi = 1$.*

Proof. By the Beurling–Malliavin theorem, we have $F\hat{f} \in \mathscr{Z}$ for some nonzero measure f such that $\operatorname{supp} f \subset [-\varepsilon/2, \varepsilon/2]$. Choose $\psi \in \mathscr{D}_{\varepsilon/2}$ and put $\varphi = f * \psi$. Clearly, $\varphi \in \mathscr{D}_\varepsilon$, $\hat{\varphi}$ is of exponential type ε, $F\hat{\varphi} = F\hat{f}/\hat{\varphi}$ and $F\hat{\varphi} \in \mathscr{Z}$.

Suppose that z_0 and 0 are zeros of $\hat{\varphi}(z)$ of orders p and q, respectively, where $p, q \geqslant 0$. Define

$$H(z) = \hat{\varphi}(z)/(z^q(z-z_0)^p).$$

Obviously, $H(0)H(z_0) \neq 0$. Letting

$$H_1(z) = H(0)^{-1}H(z),$$

we have $H_1(0) = 1$, $H_1(z_0) \neq 0$, $FH_1 \in \mathscr{Z}$ and H_1 is of exponential type ε. By the Paley–Wiener theorem for functions of class $\mathscr{D}$, there exists a function φ_0 in $\mathscr{D}_\varepsilon$ such that $\hat{\varphi}_0 = H_1$. Hence $\hat{\varphi}_0(z_0) \neq 0$, $F\hat{\varphi}_0 \in \mathscr{Z}$ and $\hat{\varphi}_0(0) = 1$. The last equality is equivalent to $\int\varphi_0 = 1$. The proof of Theorem 3 is complete.

We denote by B_σ the class of all entire functions F of exponential type σ satisfying condition $(*)$. We define on B_σ the norms

$$p_\varepsilon(F) = \sup\{|F(z)|\exp(-\varepsilon|z|)\colon\ z \in C\}$$

for $\varepsilon > \sigma$ and the quasi-norm

$$q(F) = \int_{-\infty}^{\infty}(1+t^2)^{-1}\log(1+|F(t)|)\,dt.$$

It is easy to see that $F \in B_\sigma$ iff $p_\varepsilon(F) < \infty$ for every $\varepsilon > \sigma$ and $q(F) < \infty$. One can show that B_σ endowed with the topology generated by the norms p_ε $(\varepsilon > \sigma)$ and the quasi-norm q is a complete metrizable space. Moreover, $F_n \to 0$ in B_σ iff $p_\varepsilon(F_n) \to 0$ for every $\varepsilon > \sigma$ and

$$\int_{-\infty}^{\infty}(1+t^2)^{-1}\log^+|F_n(t)|\,dt \to 0.$$

LEMMA 3. *An entire function F belongs to B_0 iff there is a delta-sequence $\{\delta_n\}$ such that $\operatorname{supp}\delta_n \subset [-\varepsilon_n, \varepsilon_n]$ and $F\hat{\delta}_n \in \mathscr{Z}_{\varepsilon_n}$ for $n \in N$, where $\varepsilon_n \to 0$. In other words, $F \in B_0$ iff F is the Fourier transform of a regular operator whose support consists of the single point 0.*

Proof. Assume that $F\hat{\delta}_n \in \mathscr{Z}_{\varepsilon_n}$ with $\operatorname{supp}\delta_n \subset [-\varepsilon_n, \varepsilon_n]$ for $n \in N$ and $\varepsilon_n \to 0$. This means that there are functions $\varphi_n \in \mathscr{D}_{\varepsilon_n}$ such that $F\hat{\delta}_n = \hat{\varphi}_n$ for $n \in N$. Put

$$x = \frac{\varphi_n}{\delta_n}$$

for $n \in N$ and note that x is a regular operator such that $\operatorname{supp} x = \{0\}$, and $\hat{x} = F$. By the first part of Theorem 1, it follows that $F \in B_0$.

To prove the converse assume that $F \in \mathscr{B}_0$ and introduce the auxiliary functions

$$G(z) = 1 + F(z)\overline{F(\bar{z})}, \qquad G_n(z) = G(nz), \qquad n \in N.$$

Since $F \in B_0$, we have $G \in B_0$ and $G_n \in B_0$ for $n \in N$. Since B_0 is a complete linear metric space, there exists a sequence of positive numbers λ_n such that

$$\sum_{n=1}^{\infty} \lambda_n G_n = H \in B_0.$$

By Theorem 3, there is a function $\varphi \in \mathscr{D}$ such that $\operatorname{supp}\varphi \subset [-1, 1]$, $\int \varphi = 1$ and $H\hat{\varphi} \in \mathscr{Z}$, in particular $\hat{\varphi}$ is a rapidly decreasing function on the real line. It is clear that the sequence $\{\delta_n\}$ defined by $\delta_n(t) = n\varphi(nt)$ is a delta-sequence and $F\hat{\delta}_n$ is a function of exponential type $1/n$. Since $G(t) = 1+|F(t)|^2 \geqslant 0$ for $t \in \boldsymbol{R}$, we have

$$|\lambda_n F(t)\hat{\delta}_n(t)| \leqslant \lambda_n G(t)\left|\hat{\varphi}\left(\frac{t}{n}\right)\right| = \lambda_n G_n\left(\frac{t}{n}\right)\left|\hat{\varphi}\left(\frac{t}{n}\right)\right| \leqslant H\left(\frac{t}{n}\right)\left|\hat{\varphi}\left(\frac{t}{n}\right)\right|,$$

which means that $F\hat{\delta}_n$ is a rapidly decreasing function on the real line for $n \in \boldsymbol{N}$. Hence, by the Paley–Wiener theorem, $F\hat{\delta}_n \in \mathscr{Z}_{1/n}$, which completes the proof of Lemma 3.

We precede the next lemma by some notation and definitions. If f is a function on $\boldsymbol{R}$ and $\lambda \in \boldsymbol{R}$, then $\tau_\lambda f$ and $\check{f}$ are defined as

$$(\tau_\lambda f)(t) = f(t-\lambda), \quad \check{f}(t) = f(-t).$$

If x is an operator, then $\tau_\lambda x$ is an operator such that

$$\tau_\lambda x = \frac{\tau_\lambda f}{g} \quad \text{if} \quad x = \frac{f}{g}.$$

Note that if x and y are operators, then

$$(4) \qquad \tau_\lambda(x*y) = (\tau_\lambda x)*y = x*(\tau_\lambda y).$$

A subspace H of $\mathscr{D}$ is said to be *shift-invariant* if $f \in H$ implies $\tau_\lambda f \in H$ for $\lambda \in \boldsymbol{R}$.

LEMMA 4. *If H is a shift-invariant subspace of $\mathscr{D}$, then the following conditions are equivalent:*

(a) *There is a delta-sequence $\{\delta_n\}$ such that $\delta_n \in H$ for $n \in N$.*

(b) *For every $\varepsilon > 0$ and for every $z_0 \in \boldsymbol{C}$, there is a function $\varphi \in H_\varepsilon = H \cap \mathscr{D}_\varepsilon$ such that $\hat{\varphi}(z_0) \neq 0$.*

Proof. Since $\hat{\delta}_n \to 1$ a.u., (a) implies (b). To prove the converse it is enough to show under condition (b) that for every $\varepsilon > 0$ the closure of H_ε in the norm contains $\mathscr{D}_\varepsilon$, i.e., $\mathscr{D}_\varepsilon \subset \mathrm{cl}_L H$. Suppose that $\mathrm{cl}_L H \not\supset \mathscr{D}_\varepsilon$ for some $\varepsilon > 0$. Then there exists a nonzero bounded measurable function u such that $\operatorname{supp} u \subset [-\varepsilon, \varepsilon]$ and

$$(5) \qquad \int_{-\varepsilon}^{\varepsilon} u(\tau)\varphi(\tau)\,d\tau = 0$$

for every $\varphi \in H_\varepsilon$. Note that

$$(\check{u}*\varphi)(t) = \int_{-\varepsilon}^{\varepsilon} u(s)(\tau_{-t}\varphi)(s)\,ds.$$

Let η be an arbitrary positive number such that $\eta < \varepsilon/2$ and let φ be an arbitrary function in H_η. Then, for every $t \in [-\varepsilon+\eta, \varepsilon-\eta]$, we have $\tau_{-t}\varphi \in H_\varepsilon$, because H is a shift-invariant subspace of $\mathscr{D}$. Hence, by (5), we get

$$\operatorname{supp}\check{u}*\varphi \subset [-\eta-\varepsilon, -\varepsilon+\eta] \cup [-\eta+\varepsilon, \varepsilon+\eta].$$

Let φ_1 and φ_2 be functions, associated with φ, such that $\operatorname{supp}\varphi_i \subset [-\eta, \eta]$ for $i = 1, 2$ and

$$(6) \qquad \check{u}*\varphi = \tau_{-\varepsilon}\varphi_1 + \tau_\varepsilon\varphi_2.$$

For arbitrary $\varphi, \psi \in H_\eta$, we have $(\check{u}*\varphi)*\psi = (\check{u}*\psi)*\varphi$, so

$$(\tau_{-\varepsilon}\varphi_1)*\psi + (\tau_\varepsilon\varphi_2)*\psi = (\tau_{-\varepsilon}\psi_1)*\varphi + (\tau_\varepsilon\psi_2)*\varphi$$

and

$$[\operatorname{supp}(\tau_{-\varepsilon}\varphi_1)*\psi] \cap [\operatorname{supp}(\tau_\varepsilon\psi_2)*\varphi] = \emptyset.$$

Therefore we obtain $\varphi_i*\psi = \psi_i*\varphi$ for $i = 1, 2$ and $\varphi, \psi \in H_\eta$ whenever $0 < \eta < \varepsilon/2$. Consequently,

$$(7) \qquad \hat{\varphi}_i\hat{\psi} = \hat{\psi}_i\hat{\varphi}$$

for $i = 1, 2$ and $\varphi, \psi \in H_\eta$. We assert that the orders of the zeros of $\hat{\varphi}$ are not greater than those of φ_i for $i = 1, 2$. In fact, if $z_0 \in \boldsymbol{C}$ and (b) holds, then there is a function $\psi \in H_\eta$ such that $\hat{\psi}(z_0) \neq 0$ and (7) holds for $i = 1, 2$. This implies our assertion. Therefore, for every $\varphi \in H_\eta$ and $i = 1, 2$, the quotient

$$(8) \qquad \varphi_i/\varphi$$

is an entire function which, by (7), does not depend on $\varphi \in H_\eta$ whenever $0 < \eta < \varepsilon/2$. Denote the common value of the quotients above by F_i for $i = 1, 2$. By Lemma 2 (with $\sigma = 0$), F_i is an entire function of exponential type 0 for $i = 1, 2$. Moreover, φ_i and φ satisfy condition (*). By Lemma 1, F_i satisfies condition (*), so $F_i \in B_0$ for $i = 1, 2$. Consequently, by Lemma 3, there are regular operators x_i such that $\operatorname{supp} x_i \subset \{0\}$, $\hat{x}_i = F_i$ and, by (8), $x_i*\varphi = \varphi_i$ for $i = 1, 2$ and $\varphi \in H_\eta$ whenever $0 < \eta < \varepsilon/2$. Hence, by (6) and (4) we get

$$\check{u}*\varphi = \tau_{-\varepsilon}(x_1*\varphi) + \tau_\varepsilon(x_2*\varphi) = (\tau_{-\varepsilon}x_1 + \tau_\varepsilon x_2)*\varphi.$$

This implies that

$$\check{u} = \tau_{-\varepsilon}x_1 + \tau_\varepsilon x_2.$$

In view of this equality $u = 0$ on $[-\varepsilon, \varepsilon]$, which contradicts our assumption and thus proves Lemma 4.

Now we can prove the second part of Theorem 1.

Proof of Theorem 1(b). Assume that $F \in B_\sigma$, i.e. F is an entire function of exponential type σ and satisfies condition (*). We put

$$H = \{\varphi \in \mathscr{D}\colon F\hat{\varphi} \in Z\}$$

and note that H is a shift-invariant subspace of $\mathscr{D}$. Let ε be a positive number and let z_0 be a fixed complex number. By Theorem 3, there is a $\varphi \in \mathscr{D}_\varepsilon$ such that $F\hat{\varphi} \in \mathscr{Z}$ and $\hat{\varphi}(z_0) \neq 0$. This implies that H satisfies condition (b) of Lemma 4. Therefore there is a delta-sequence $\{\delta_n\}$ in H. This means that $F\hat{\delta}_n \in \mathscr{Z}$ for $n \in N$, which was to be proved.

This completes the proof of Theorem 1.

3. Characterizations of regular operators. Let us now formulate several consequences of the results of the preceding section. The following corollary is a simple consequence of Theorem 1.

COROLLARY 1. *A Mikusiński operator x is a regular operator of bounded support iff it has a representation*

$$x = \frac{\varphi}{\psi},$$

where $\varphi, \psi \in \mathscr{D}$ and the quotient $\hat{\varphi}/\hat{\psi}$ is an entire function.

Proof. Obviously, if x is a regular operator of bounded support, then it has a required representation. Assume now that a Mikusiński operator x has a representation with the required properties. Then the Fourier transforms $\hat{\varphi}$ and $\hat{\psi}$ are of exponential type, and $F = \hat{\varphi}/\hat{\psi}$ is an entire function of exponential type. Moreover, $\hat{\varphi}$ and $\hat{\psi}$ satisfy condition (*). Hence, by Lemma 1, F satisfies condition (*). Consequently, by Theorem 1 (b), there exists a regular operator y of bounded support such that $\hat{y} = F$. Using basic definitions it can easily be shown that $x = y$. This completes the proof of Corollary 1.

We adopt the following notation. For fixed $z \in C$ we put $E_z(t) = e^{zt}$ for $t \in R$, and

$$E_z^+(t) = \begin{cases} e^{zt} & \text{for } t \geqslant 0, \\ 0 & \text{for } t < 0, \end{cases} \qquad E_z^-(t) = \begin{cases} 0 & \text{for } t > 0, \\ e^{zt} & \text{for } t \leqslant 0. \end{cases}$$

We have

$$E_z(t) = \frac{1}{s-z},$$

where s is the differential operator ([6], p. 35).

Observe that if $\varphi \in \mathscr{D}_\varepsilon$, $\hat{\varphi}(z_0) = 0$ and

$$\psi = E^{+}_{-iz_0} * \varphi \quad \text{or, equivalently,} \quad \psi = \frac{1}{s+iz_0}\varphi,$$

then $\psi \in \mathscr{D}_\varepsilon$. Moreover, if k is the order of the zero of $\hat{\varphi}$ at z_0, then the order of the zero of $\hat{\psi}$ at z_0 is $k-1$.

A simple consequence of Lemma 4 is the following

COROLLARY 2. *Let H be a subspace of $\mathscr{D}$ such that the following conditions are fulfilled:*

1° *H is a shift-invariant subspace.*
2° *$H_\varepsilon = H \cap \mathscr{D}_\varepsilon \neq \{0\}$ for every $\varepsilon > 0$.*
3° *$\varphi \in H$, $\hat{\varphi}(z_0) = 0$ and $\psi = E^{+}_{-iz_0} * \varphi$ imply $\psi \in H$.*

Then H contains a delta-sequence.

Proof. Assume that $\varepsilon > 0$, $z_0 \in C$. If there is a $\varphi \in H_\varepsilon$ such that $\hat{\varphi}(z_0) \neq 0$, then, by Lemma 4, H contains a delta-sequence.

Now suppose that $\hat{\varphi}(z_0) = 0$ for each $\varphi \in H_\varepsilon$ and choose φ for which the order of z_0 is minimal, say k. Since $\varphi \in \mathscr{D}_\varepsilon$, we have $\psi \in \mathscr{D}_\varepsilon$, by the preceding remark, and thus $\psi \in H_\varepsilon$, in view of condition 3°. Moreover, the order of z_0 for $\hat{\psi}$ is $k-1$. This contradiction completes the proof.

The following corollary gives us a useful characterization of regular operators:

COROLLARY 3. *An operator x is regular iff there are continuous functions f_n and g_n such that $\operatorname{supp} g_n \subset [-\varepsilon_n, \varepsilon_n]$ with $\varepsilon_n \to 0$ and $x = \dfrac{f_n}{g_n}$ for $n \in N$.*

Proof. Evidently, if x is a regular operator, then it has a required representation. To prove the converse we put

$$H = \{\varphi \in \mathscr{D}\colon x * \varphi \in \mathscr{C}\}.$$

In view of (4), H is a shift-invariant subspace of $\mathscr{D}$. Hence, by Corollary 2, H contains a delta-sequence $\{\delta_n\}$, i.e. $x * \delta_n \in \mathscr{C}$ for $n \in N$. Hence, $x = \dfrac{x * \delta_n}{\delta_n}$ for $n \in N$, i.e. x is a regular operator, as desired. The proof is complete.

An immediate consequence of Corollary 3 is the following

COROLLARY 4. *If x is a regular operator and $\operatorname{supp} x = \{0\}$, then $\dfrac{1}{x}$ is a regular operator.*

4. Convergence of sequences of regular operators. We start this section with introducing a convergence of sequences of regular operators.

We say that a sequence $\{x_n\}$ of regular operators *converges* to a regular operator x and write $x_n \xrightarrow{\mathscr{R}} x$ if there is a delta-sequence $\{\delta_k\}$ such that

$$x_n = \frac{f_{nk}}{\delta_k}, \qquad x = \frac{f_k}{\delta_k}$$

for $n, k \in \boldsymbol{N}$ and $f_{nk} \to f_k$ in $\mathscr{C}$, i.e. $f_{nk} \to f_k$ a.u. and there is a number α such that $f_{nk}(t) = 0$ for $t \in (-\infty, \alpha)$ and $n \in \boldsymbol{N}$.

The above convergence is stronger than the following *type* I *convergence*: $x_n \to x$ (type I) if there are representations

$$x_n = \frac{f_n}{\varphi}, \qquad x = \frac{f}{\varphi},$$

where $f_n, f, \varphi \in \mathscr{C}$ and $f_n \to f$ a.u. (see [6], p. 155).

THEOREM 4. *A sequence $\{x_n\}$ of regular operators converges to a regular operator x iff there are functions f_{nk}, f_k and φ_k in $\mathscr{C}$ such that* $\operatorname{supp}\varphi_k \subset [-\varepsilon_k, \varepsilon_k]$ *with* $\varepsilon_k \to 0$,

$$x_n = \frac{f_{nk}}{\varphi_k}, \qquad x = \frac{f_k}{\varphi_k}$$

for $n, k \in \boldsymbol{N}$ and $f_{nk} \to f_k$ a.u. in $\mathscr{C}$ as $n \to \infty$ for $k \in \boldsymbol{N}$.

Proof. Evidently, if $x_n \to x$, then there are functions satisfying the above conditions. To prove the converse we put

$$H = \{\varphi \in \mathscr{D}\colon\ x_n * \varphi \to x * \varphi \ \text{ in } \mathscr{C}\}$$

and assume the conditions are satisfied. By (4), H is shift-invariant. In view of the conditions, the assumptions of Corollary 2 are fulfilled. Hence there is a delta-sequence $\{\delta_n\}$ in H. This means that $x_n * \delta_k \to x * \delta_k$ in $\mathscr{C}$ as $n \to \infty$ for $k \in \boldsymbol{N}$, which completes the proof.

THEOREM 5. *If $\{\alpha_n\}$ is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \alpha_n^{-1} < \infty,$$

then the series

$$\sum_{n=1}^{\infty} \lambda_n E_{\alpha_n}^{+}(t)$$

converges whenever the series

$$\sum_{n=1}^{\infty} \lambda_n \alpha_n^{-1}$$

converges. In particular, the sequence $\{E_{\alpha_n}^{+}(t)\}$ converges to zero.

Proof. Consider

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{i\alpha_n}\right).$$

By routine calculations it can be checked that $F(z)$ is of exponential type zero and $z_n = i\alpha_n$ $(n \in N)$ are the only zeros of F. Moreover, $|F(t)| \geqslant 1$ for $t \in \boldsymbol{R}$ and $F(0) = 1$.

By Theorem 6.3.14 in [2], p. 86, the improper integral

$$\int_0^{\infty} t^{-2} \log|F(t)\,F(-t)|\,dt$$

converges. The integral exists as the Lebesgue integral, because $|F(t)\,F(-t)| \geqslant 1$. Hence we get

$$\int_0^{\infty} t^{-2}\log|F(t)|\,dt + \int_{-\infty}^{0} t^{-2}\log|F(t)|\,dt < \infty.$$

Consequently, we have

$$\int_{-\infty}^{\infty} (1+t^2)^{-1}\log^{+}|F(t)|\,dt < \infty.$$

Therefore, by Theorem 1(b), there is a regular operator x such that $\operatorname{supp} x \subset \{0\}$ and $\hat{x} = F$. Let $\{\varphi_k\}$ be a sequence in $\mathscr{C}$ and let $\{\delta_k\}$ be a delta-sequence such that $x = \dfrac{\varphi_k}{\delta_k}$, $\operatorname{supp}\varphi_k \subset [-\sigma_k, \sigma_k]$ and $\operatorname{supp}\delta_k \subset [-\varepsilon_k, \varepsilon_k]$ for $k \in \boldsymbol{N}$ with $\sigma_k \to 0$ and $\varepsilon_k \to 0$. We see that

$$E_{\alpha_n}^{+} * \varphi_k = E_{\alpha_n} * \varphi_k - E_{\alpha_n}^{-} * \varphi_k \tag{9}$$

for $n, k \in \boldsymbol{N}$. Hence we get

$$E_{\alpha_n} * \varphi_k = E_{\alpha_n} * \int_{\boldsymbol{R}} e^{-\alpha_n \tau}\varphi_k(\tau)\,d\tau = E_{\alpha_n}\hat{\varphi}_k(i\alpha_n)$$

$$= F(i\alpha_n)\,\hat{\delta}_k(i\alpha_n)\,E_{\alpha_n} = 0$$

for $F(i\alpha_n) = 0$. Moreover, we can write

$$\|E_{\alpha_n}^{-} * \varphi_k\|_T \leqslant \|E_{\alpha_n}^{-1}\|_T\,\|\varphi_k\|_T \leqslant \alpha_n^{-1} M_k,$$

where $M_k = (E_{\alpha_n}(T) - 1)\|\varphi_k\|_T$. The last inequality implies that the series

$$\sum_{n=1}^{\infty} E_{\alpha_n}^{-} * \varphi_k$$

converges in $L_{\text{loc}}(-\infty, \infty)$ if $\sum_n \lambda_n \alpha_n^{-1} < \infty$. Consequently, by (9), the series

$$\sum_{n=1}^{\infty} E_{\alpha_n}^+ * \varphi_k$$

converges in $L_{\text{loc}}(-\infty, \infty)$ if $\sum_n \lambda_n \alpha_n^{-1} < \infty$, which proves the theorem.

In [7] it is shown that $E_{\alpha_n}^+ \to 0$ with respect to type I convergence.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
ODDZIAŁ W KATOWICACH
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
KATOWICE BRANCH
Wieczorka 8, 40-013 Katowice, Poland

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On vector-valued Fourier multiplier theorems*

by

FRANK ZIMMERMANN (Kiel)

Abstract. The classical Fourier multiplier theorems of Littlewood–Paley, Marcinkiewicz, and Mikhlin are generalized to the vector-valued setting in d dimensions. A direct and a tensor product approach yield slightly different results. While the direct approach works in general UMD-spaces, the tensor product technique requires some unconditional structure and it is shown that the latter results fail for the Schatten classes S_p with $p \neq 2$.

0. Introduction and results. Let X be a complex Banach space. We first consider the d-dimensional periodic case. For $1 \leqslant p < \infty$ let $\tilde{L}_p$ resp. $\tilde{L}_p(X)$ be the usual Lebesgue space on $[0, 1]^d$ with the normalized Lebesgue measure. A sequence $a = (a_\varkappa)_{\varkappa \in \mathbf{Z}^d} \subseteq \mathbf{C}$ is said to be an $\tilde{L}_p(X)$-*Fourier multiplier* if there is a constant $C < \infty$ such that

(0.1) For all finitely nonzero sequences $(x_\varkappa)_{\varkappa \in \mathbf{Z}^d} \subseteq X$

$$\Big\| \sum_{\varkappa \in \mathbf{Z}^d} a_\varkappa x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} \Big\|_{\tilde{L}_p(X)} \leqslant C \Big\| \sum_{\varkappa \in \mathbf{Z}^d} x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} \Big\|_{\tilde{L}_p(X)}.$$

The set of all $\tilde{L}_p(X)$-Fourier multipliers will be denoted by $\tilde{M}_p(X)$ and the smallest constant C such that (0.1) holds by $\|a\|_{\tilde{M}_p(X)}$. For an $\tilde{L}_p(X)$-Fourier multiplier a the operator

$$\sum_{\varkappa \in \mathbf{Z}^d} x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} \mapsto \sum_{\varkappa \in \mathbf{Z}^d} a_\varkappa x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle}$$

extends uniquely to an operator on $\tilde{L}_p(X)$ which will be denoted by T_a.

To state the Littlewood–Paley theorem we need a decomposition of $\mathbf{Z}^d$. Actually, we will work with two different ones:

1) A coarse decomposition arising as differences of dyadic cuboids: $D_0 = \{0\}$ and for $n = dr + j$, $r \in \mathbf{N}_0$, $j \in \{1, \ldots, d\}$ let

$$D_n = \{\varkappa = (\varkappa_1, \ldots, \varkappa_d) \in \mathbf{Z}^d \mid |\varkappa_i| < 2^{r+1} \text{ for } i \in \{1, \ldots, j-1\},$$
$$2^r \leqslant |\varkappa_j| < 2^{r+1}, |\varkappa_i| < 2^r \text{ for } i \in \{j+1, \ldots, d\}\}.$$

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