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Added in proof (January 1989). N. Makarov has informed me that the conjecture stated in § 4(d) is false but the question remains open if $\limsup_{n\to\infty}$ is replaced by $\liminf_{n\to\infty}$.

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On the Hausdorff dimension of some fractal sets

by

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Dedicated to the memory of M. Irwin

Abstract. We describe a method of estimating from below the Hausdorff dimension of some fractal sets. These include compact connected subsets of tori with nondense orbit under a hyperbolic toral automorphism, graphs of Weierstrass nowhere differentiable functions, e.g. $\sum_{n=0}^{\infty}\lambda^n \sin 2^n x$, $1/2<\lambda<1$, and also graphs of $\sum_{n=0}^{\infty}\lambda^n r_n$, r_n the nth Rademacher function. On the other hand, we prove that for λ^{-1} a Pisot–Vijayaraghavan number, the latter graph has Hausdorff dimension less than $2-\log\lambda^{-1}/\log 2$.

1. Introduction. This paper concerns the Hausdorff dimension and limit capacity of three types of related fractal sets. Our estimates of Hausdorff dimension from below rely on a fact formulated in § 2 as Lemma 1. Here it is as applied to the plane $\mathbf{R}^2$.

LEMMA 0. *Let K be a Borel subset of the x, y plane $\mathbf{R}^2$ whose projection to the x axis has positive 1-dimensional Lebesgue measure. Assume that there exist constants $C_1, C_2>0$, $0<\alpha<1$ such that for every horizontal interval $[x_1, x_2]\times\{y\}$ there exist a_1, a_2 with $x_1\leqslant a_1<a_2\leqslant x_2$ such that $a_2-a_1 = C_1(x_2-x_1)$ and the rectangle*

$$[a_1, a_2]\times[y-\tfrac{1}{2}C_2(x_2-x_1)^{\alpha}, y+\tfrac{1}{2}C_2(x_2-x_1)^{\alpha}]$$

is disjoint from K. Then the Hausdorff dimension $\mathrm{HD}(K)$ *satisfies*

$$\mathrm{HD}(K)\geqslant C(\alpha, C_1)>1 \tag{0}$$

where $C(\alpha, C_1)$ is a constant depending only on α and C_1.

We recall some definitions: For a metric space (X,ϱ), $A\subset X$, $r>0$ we denote by $N(A,r)$ the minimum number of balls in X with radii $\leqslant r$, needed to cover A. The *lower* and *upper capacities* of A are defined as

$$\underline{\mathrm{Cap}}\, A = \liminf_{r\to 0}\frac{\log N(A,r)}{-\log r},\qquad \overline{\mathrm{Cap}}\, A = \limsup_{r\to 0}\frac{\log N(A,r)}{-\log r}.$$

For every $s>0$ we define the (outer) *s-Hausdorff measure* of A by

$$\Lambda_s(A) = \lim_{r\to 0}\Big(\inf \sum_{j=0}^{\infty} (\operatorname{diam} U_j)^s\Big),$$

the infimum being taken over all countable covers $\{U_j\}$ of A by balls in X of radii $\leqslant r$.

The *Hausdorff dimension* $\mathrm{HD}(A)$ is the parameter s_0 such that $\Lambda_s(A) = \infty$ for $s<s_0$, $\Lambda_s(A)=0$ for $s>s_0$.

For a measure η on X we define the Hausdorff dimension of the measure η as $\mathrm{HD}(\eta) = \inf\{\mathrm{HD}(A)\colon \eta(X\setminus A)=0\}$.

In fact, in Lemmas 0 and 1 we shall estimate from below the Hausdorff dimension of a measure μ on K which projected to the x axis is the length measure (restricted to the image of K under the projection).

In the case where K is the graph of a continuous function $f\colon [0,1]\to \boldsymbol{R}$ and the assumptions of Lemma 0 are satisfied, it is easy to see that $\underline{\mathrm{Cap}}\, K \geqslant 2-\alpha$. So the question arises whether $\mathrm{HD}(K)\geqslant 2-\alpha$. To our knowledge there are only some partial results in this direction and we do not think that the answer is affirmative in such a generality (1).

In § 5 we concentrate on continuous functions satisfying the conditions

$$(1)\qquad \begin{gathered} |f(x_1)-f(x_2)| \leqslant C_3|x_1-x_2|^\alpha, \\ \sup\{|f(a_1)-f(a_2)|\colon x_1\leqslant a_1\leqslant a_2\leqslant x_2\}\geqslant C_4|x_1-x_2|^\alpha \end{gathered}$$

for some constants $C_3, C_4>0$, $0<\alpha<1$ and all $x_1, x_2\in[0,1]$.

In Theorem 4 we prove that these conditions imply the hypotheses of Lemma 0 for $K=\operatorname{graph} f$, hence $\mathrm{HD}(\operatorname{graph} f)>1$.

A famous example of a function satisfying (1) is the Weierstrass nowhere differentiable function

$$(2)\qquad \mathscr{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\beta^n x$$

for $\lambda^{-1}, \beta, \lambda\beta>1$. The property (1) for $\mathscr{W}$ was proved by Hardy [H]. (Here $\alpha=-\log\lambda/\log\beta$.) Thus $\mathrm{HD}(\operatorname{graph}\mathscr{W})>1$.

To our knowledge it is still not known whether $\mathrm{HD}(\operatorname{graph}\mathscr{W}) = 2-\alpha$.

Let us mention that (1) is true (so the conclusions apply) not only for $\mathscr{W}$ defined with the help of the cosine function, but also with an arbitrary C^1 almost periodic function provided $\mathscr{W}$ is not C^1 (see [KMY]). In § 5 we describe some other classes of functions satisfying (1).

Let us remark that Besicovitch and Ursell [BU] studied the series (2) with cosine replaced by $q(x)=\operatorname{dist}(x,\boldsymbol{Z})$ and the summation in (2) only over

(1) The answer is negative. See Note at the end of the paper.

a special set of integers. They proved then that $\mathrm{HD}(\operatorname{graph}\mathscr{W}) = 2-\alpha$. For this result and historical notes we refer the reader to the book [Fal]. Adapting the method of [BU] Mauldin and Williams [MW] recently gave an estimate of the Hausdorff dimension for a larger class of functions. Our Theorems 5, 5a, 5b are related to their result.

The second type of fractal set under discussion is a continuum K (a compact conected set) in a torus T^m which contains more than one point and which is "pre-invariant", i.e. the closure of its orbit under some hyperbolic toral automorphism f does not contain any coset of a toral subgroup invariant under a power of f.

The existence of such a set and a description from the point of view of the topological dimension of K and of $\operatorname{cl}\bigcup_{n\in\boldsymbol{Z}} f^n(K)$ was done in [Bo], [Han] and [P]. One reason for the importance of such sets is that they necessarily appear in the boundaries of cells of some Markov partitions (cf. [Bo] and [Be]).

In the case where K is a curve J. Franks [F] and R. Mañé [M] proved respectively that K is not C^2 and is not rectifiable. Finally, M. Irwin [I$_1$], [I$_2$] and the second author [U] exhibited its fractal nature by studying respectively the Hölder continuity exponent (estimated from above by a number <1) and capacity (estimated from below by a number >1). Examples were also constructed (in [U] including K's of higher topological dimensions) showing that the estimates are best possible.

In this paper (§ 3) we give estimates from below for the Hausdorff dimension of K by a number >1. The idea is as follows. Let K be contained in, say, a two-dimensional unstable manifold E^u with $f^n(K)$ disjoint from a family of uniformly scattered squares (in E^u) of a fixed size. Then K is disjoint from their f^n-preimages, which are little parallelograms stretched along the weaker expansion direction in E^u. We arrive at the situation of Lemmas 0 and 1. In § 3 we also prove that $\mathrm{HD}(K)>\dim_{\mathrm{top}}(K)$ for K of any topological dimension, under suitable assumptions.

In § 4 we estimate from below the capacity of K of arbitrary topological dimension. The estimate is best possible because of the examples mentioned above (this fills a gap in [U]).

The third type of "fractal" set considered in the paper (§ 6) is the graph of the function

$$(3)\qquad f_\lambda = \sum_{n=0}^{\infty} \lambda^n r_n \quad \text{on the interval } [0,1),$$

for $\frac{1}{2}<\lambda<1$, where r_n is the nth Rademacher function: $r_n(x) = R(2^n x)$ where $R(x)=-1$ or $+1$ for $x\in[0,\frac{1}{2})+\boldsymbol{Z}$ and $x\in[\frac{1}{2},1)+\boldsymbol{Z}$ respectively.

We shall call f_λ a *limit Rademacher function*. It seems to be easier to handle such a function than that given by the formula (2). That

$HD(\text{graph} f_\lambda) > 1$ follows again from Lemmas 0 and 1. However, we can rely on Erdős's result [E_1] that for almost every λ close enough to 1, the probability distribution ν given by the random variable f_λ is absolutely continuous with respect to the Lebesgue measure l_1. We deduce that $HD(\mu) = HD(\text{graph} f_\lambda) = 2-\alpha$, for $\alpha = -\log\lambda/\log 2$, and the measures μ and $\Lambda_{2-\alpha}$ on graph f_λ are equivalent. The case of some exceptional λ's, the so-called Pisot–Vijayaraghavan numbers, is very exciting. Then $\nu \perp l_1$ (see [E_2], [G_1] and [G_2]). In fact, $HD(\nu) < 1$ [AY]. We conclude that for the measure μ on graph f_λ, $HD(\mu) < 2-\alpha$. Moreover, we prove that $HD(\text{graph} f_\lambda) < 2-\alpha$.

Notation. If W is an affine k-dimensional space over the field $\boldsymbol{R}$ and if the corresponding linear space $\tilde{W}$ is equipped with a scalar product $\langle\cdot,\cdot\rangle$ and an orthonormal basis $(e_1, \ldots, e_k)$ in $\tilde{W}$ is chosen, then the triple $(W, \langle\cdot,\cdot\rangle, (e_1, \ldots, e_k))$ is called a *euclidean space with basis.*

Sometimes, for simplicity, we will omit in such triples the symbols of the scalar product or basis.

We will denote the Lebesgue measure on W by m_W.

A k-dimensional closed parallelepiped $R \subset W$ is said to be *canonical* if all its edges are parallel to vectors from the basis $(e_1, \ldots, e_k)$.

If $x \in W$ and $r > 0$, then $R(x, r)$ or $R_W(x, r)$ denotes the canonical cube with centre at x and edges of length r. If the centre is not specified we call such a cube an *(r, W)-cube.*

For $V \subset W$ an affine subspace of W, $\pi_V\colon W \to V$ denotes the orthogonal projection onto V.

If $(W_1, \langle\cdot,\cdot\rangle_1, (e_1, \ldots, e_k))$, $(W_2, \langle\cdot,\cdot\rangle_2, (e_{k+1}, \ldots, e_m))$ are two euclidean spaces with bases then the Cartesian product $W_1 \times W_2$ becomes in a natural way a euclidean space with the corresponding linear space $\tilde{W}_1 \oplus \tilde{W}_2$ and with the basis $(e_1, \ldots, e_m)$.

Every linear space with a scalar product and an orthonormal basis is regarded in an obvious way as a euclidean space with basis.

In estimating the Hausdorff dimension of an arbitrary Borel σ-finite measure θ on a euclidean space we shall always rely on the following well-known fact:

FROSTMAN'S LEMMA. *If for θ-a.e. x,*

$$\delta_1 \leqslant \liminf_{r\to 0} \frac{\log \theta(B(x, r))}{\log r} \leqslant \delta_2$$

then $\delta_1 \leqslant HD(\theta) \leqslant \delta_2$. The same is true if balls are replaced by canonical cubes.

This fact follows easily from the definition of Hausdorff dimension with the use of the Vitali or Besicovitch covering theorems (see e.g. [Fal], p. 19, Ex. 1.8, or [PUZ], proof of Th. 1). We have named it "Frostman's lemma" because it generalizes a fact whose discovery is attributed to Frostman (cf. [K]).

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§ 2. Basic lemmas

LEMMA 1. *Let $(W_1, (e_1, \ldots, e_m))$, $(W_2, (e_{m+1}, \ldots, e_{m+l}))$ be euclidean spaces with bases and let $W = W_1 \times W_2$. Assume that for a Borel set $K \subset W$ the following two conditions are satisfied:*

1° $m_{W_1}(\pi_{W_1}(K)) > 0$.

2° *There exist constants $C_1, C_2 > 0$ and $0 < \alpha < 1$ such that for every (b, W_1)-cube R (see notations in § 1) with $b > 0$ sufficiently small and for every $y \in W_2$ there exists a $(C_1 b, W_1)$-cube $R' \subset R$ such that the parallelepiped $R' \times R(y, C_2 b^\alpha)$ is disjoint from K.*

Then $HD(K) \geqslant C(\alpha, C_1, m) > \dim W_1 = m$. Here $C(\alpha, C_1, m)$ is a constant depending only on α, C_1 and m.

Proof. Consider a Borel measure μ on K such that $(\pi_1)_*(\mu) = m_{W_1}|_{\pi_1(K)}$. Here π_1 stands for π_{W_1}. For K compact μ exists by the Hahn–Banach and Riesz theorems. For an arbitrary Borel set the existence of μ easily follows from the von Neumann theorem on the existence of a measurable selector (see e.g. [KM], Ch. XIII, Sec. 1, Th. 3).

Fix an (r, W)-cube $Q = R \times R(y, r)$ with $r > 0$ not greater than b discussed in 2°.

As we have already mentioned in the introduction we shall estimate $HD(\mu)$. We shall do it relying on Frostman's lemma, by estimating $\mu(Q \cap K)$ from above.

For every integer $j \geqslant 0$ consider the partition $\mathscr{P}_j$ of R into equal $(r/M^j, W_1)$-cubes where $M = E(2/C_1)+1$ and $E(x)$ is the integer part of x. Let $\hat{\mathscr{P}}_j = \{P \in \mathscr{P}_j\colon P \times R(y, r) \cap K = \emptyset\}$. By the definition of M, $C_1 r/M^j \geqslant 2r/M^{j+1}$. Therefore it follows from 2° that, as long as $C_2(r/M^j)^\alpha \geqslant r$, every cube in $\mathscr{P}_j$ (in particular, in $\mathscr{P}_j \setminus \hat{\mathscr{P}}_j$) contains at least one cube in $\hat{\mathscr{P}}_{j+1}$. Consequently,

$$m_{W_1}\Big(R \setminus \bigcup_{i=0}^{j+1} \big(\bigcup \hat{\mathscr{P}}_i\big)\Big) \leqslant (1-M^{-m})\, m_{W_1}\Big(R \setminus \bigcup_{i=0}^{j} \big(\bigcup \hat{\mathscr{P}}_i\big)\Big). \tag{1}$$

Now let q be the largest integer for which $C_2(r/M^q)^\alpha \geqslant r$. In particular, this implies that $C_2(r/M^{q+1})^\alpha < r$ and consequently

$$q+1 \geqslant \frac{-(1-\alpha)\log r + \log C_2}{\alpha \log M}. \tag{2}$$

By the construction of $\hat{\mathscr{P}}_j$ we have

$$\pi_1(K \cap Q) \subset R \setminus \bigcup_{i=0}^{q+1} (\bigcup \hat{\mathscr{P}}_i).$$

So by the definition of μ and by (1),

$$\mu(K \cap Q) \leqslant m_{W_1}\big(\pi_1(K \cap Q)\big) \leqslant (1-M^{-m})^{q+1} m_{W_1}(R) = (1-M^{-m})^{q+1} r^m.$$

Thus in view of (2),

$$\log \mu(K \cap Q) \leqslant (q+1)\log(1-M^{-m}) + m \log r$$
$$\leqslant \left(m + \frac{\alpha-1}{\alpha}\frac{\log(1-M^{-m})}{\log M}\right)\log r - \frac{\log C_2 \log(1-M^{-m})}{\alpha \log M}.$$

By the Frostman lemma (§ 1) we conclude that

$$\mathrm{HD}(K) \geqslant \mathrm{HD}(\mu) \geqslant m + \frac{\alpha-1}{\alpha}\cdot\frac{\log(1-M^{-m})}{\log M} > m. \ \blacksquare \tag{3}$$

Remark 1. The estimate (3) is rather crude and in many cases it is possible to improve it. It may be that under some suitable geometric assumptions $C(\alpha, C_1, m)$ does not depend on C_1. On the other hand, the example of f_λ with λ^{-1} a Pisot–Vijayaraghavan number shows that the expected estimate by $m+1-\alpha$ is false in general.

Lemma 1 implies the following simple corollary:

LEMMA 2. *Let* $(W_1, (e_1, \ldots, e_m))$, $(W_2, (e_{m+1}, \ldots, e_{m+l}))$ *be euclidean spaces with bases. Let* $W = W_1 \times W_2$ *and* $V = \bigoplus_{j=1}^k \boldsymbol{R}e_{i_j}$ $(1 \leqslant i_j \leqslant m)$. *Assume that for a Borel set* $K \subset W$ *the following two conditions are satisfied:*

1° $m_{W_1}(\pi_{W_1}(K)) > 0$.

2° *There exist constants* $C_1, C_2 > 0$ *and* $0 < \alpha < 1$ *such that for every* $x \in W_1$, *every* $(b, x+V)$*-cube* R *with* b *sufficiently small, and every* $y \in W_2$ *there exists a* $(C_1 b, x+V)$*-cube* $R' \subset R$ *such that the parallelepiped* $R' \times R(y, C_2 b^\alpha)$ *is disjoint from* K.

Then $\mathrm{HD}(K) \geqslant C(\alpha, C_1, k) + (m-k) > m = \dim W_1$.

Proof. Partition W_1 into affine spaces parallel to V, apply Lemma 1 to each of them separately and use Fubini's theorem.

§ 3. Toral pre-invariant sets: Hausdorff dimension. In this section we apply Lemmas 1 and 2 to the case of continua with nondense orbits under hyperbolic toral automorphisms.

Let $f\colon T^k \to T^k$ be a hyperbolic toral automorphism, $\tilde{f}\colon \boldsymbol{R}^k \to \boldsymbol{R}^k$ its linear covering mapping and $\pi\colon \boldsymbol{R}^k \to T^k$ the corresponding projection. We also make the following technical assumptions:

1° All the eigenvalues of $\tilde{f}$ are real and positive.

2° There exists a linear basis $\{e_j\}_{j=1}^k$ of $\boldsymbol{R}^k$ consisting of eigenvectors of $\tilde{f}$.

In fact, weaker assumptions are sufficient. In every case the reader will easily guess them.

Let $\lambda_1 \leqslant \ldots \leqslant \lambda_s < 1 < \lambda_{s+1} \leqslant \ldots \leqslant \lambda_{s+u}$ be the eigenvalues of $\tilde{f}$, let $E^s = \bigoplus_{i=1}^s \boldsymbol{R}e_i$, $E^u = \bigoplus_{i=1}^u \boldsymbol{R}e_{s+i}$ be the stable and unstable subspaces of $\boldsymbol{R}^k$ and set

$$f_1 = \tilde{f}|_{E^u}\colon E^u \to E^u, \quad f_2 = \tilde{f}|_{E^s}\colon E^s \to E^s.$$

There exists a unique scalar product $\langle\cdot,\cdot\rangle$ on $\boldsymbol{R}^k$ with respect to which the basis $(e_1, \ldots, e_k)$ is orthonormal and the triple $(\boldsymbol{R}^k, \langle\cdot,\cdot\rangle, (e_1, \ldots, e_k))$ becomes a euclidean space with basis. We write $\pi_s = \pi_{E^s}\colon \boldsymbol{R}^k \to E^s$ and $\pi_u = \pi_{E^u}\colon \boldsymbol{R}^k \to E^u$. In this section we will consider the norm and metric on $\boldsymbol{R}^k$ induced by this scalar product. In particular, the torus $T^k = \boldsymbol{R}^k/\boldsymbol{Z}^k$ will be considered with the projection of this metric.

The following lemma permits us to consider some problems about the dynamics of f by looking at the mappings f_1 and f_2 separately.

LEMMA 3. *If* $K \subset \boldsymbol{R}^k$ *is a compact set, then* $\mathrm{cl}\bigcup_{m=-\infty}^{\infty} f^m(\pi(K))$ *contains a coset of a toral subgroup invariant under* f *if and only if the same is true of* $\mathrm{cl}\bigcup_{m=0}^{\infty} f^m(\pi \circ \pi_u(K))$ *or* $\mathrm{cl}\bigcup_{m=0}^{\infty} f^{-m}(\pi \circ \pi_s(K))$.

Proof. Recall some standard definitions. For $g\colon X \to X$ a homeomorphism of a metric space X and $Y \subset X$ we define

$$\omega(Y, g) = \{x \in X\colon \text{there exist a sequence of points } x_n \in Y \text{ and integers } k_n \to \infty \text{ as } n \to \infty, \text{ such that } g^{k_n}(x_n) \to x\},$$
$$\alpha(Y, g) = \omega(Y, g^{-1}).$$

Observe that

$$\omega(\omega(Y, g)) \subset \omega(Y, g), \quad \alpha(\alpha(Y, g)) \subset \alpha(Y, g). \tag{0}$$

If $\mathrm{cl}\bigcup_{m=-\infty}^{\infty} f^m(\pi(K))$ contains a coset $a+S$ of a toral subgroup S invariant under f then by the Baire theorem at least one of the intersections

$$\omega(\pi(K), f) \cap (a+S), \quad \alpha(\pi(K), f) \cap (a+S), \quad \{f^m(\pi(K)) \cap (a+S)\}_{m=-\infty}^{\infty}$$

contains a nonempty set U open in $a+S$.

Suppose first that for some $m \in \mathbf{Z}$, $f^m(\pi(K)) \cap (a+S) \supset U$. Since $f|_S\colon S \to S$ is also a hyperbolic toral automorphism, it is topologically mixing, so $\sup_{x\in S} \operatorname{dist}(x, f^n(U-a))$ tends to 0 as $n \to \pm\infty$. Hence

(1) $\omega(\pi(K), f) \supset \omega(U, f) \supset b+S$ for every $b \in \omega(\{a\}, f)$.

Similarly

(1′) $\alpha(\pi(K), f) \supset b+S$ for every $b \in \alpha(\{a\}, f)$.

If $\omega(\pi(K), f) \cap (a+S) \supset U$ then (1) holds by the same argument and by (0). Similarly, if $\alpha(\pi(K), f) \cap (a+S) \supset U$ then (1′) holds.

We conclude that $\operatorname{cl} \bigcup_{m=-\infty}^{\infty} f^m(\pi(K))$ contains a coset of a toral f-invariant subgroup iff $\omega(\pi(K), f)$ or $\alpha(\pi(K), f)$ does. But since K is bounded, by the hyperbolic effect we have

$$\omega(\pi(K), f) = \omega(\pi \circ \pi_u(K), f), \quad \alpha(\pi(K), f) = \alpha(\pi \circ \pi_s(K), f).$$

Since $\alpha(\pi \circ \pi_u(K), f) = \omega(\pi \circ \pi_s(K), f) = \{\pi(0)\}$, $\omega(\pi \circ \pi_u(K), f)$ contains a coset of a toral f-invariant subgroup iff $\operatorname{cl} \bigcup_{m=0}^{\infty} f^m(\pi \circ \pi_u(K))$ does and similarly for $\pi_s(K)$. The lemma is proved. ■

Notation. For any set K in $\mathbf{R}^k$ or in T^k we denote its orbit $\bigcup_{j=-\infty}^{\infty} \tilde{f}^j(K)$, or $\bigcup_{j=-\infty}^{\infty} f^j(K)$ respectively, by $O(K)$.

DEFINITION 1. A set $K \subset T^k$ is called *pre-invariant* if $\operatorname{cl} O(K)$ does not contain any coset of a toral subgroup invariant under a power of f.

Under our special assumptions on the eigenvalues this is equivalent to the formally weaker condition that $\operatorname{cl} O(K)$ contains no coset of a toral subgroup invariant under f. Observe that in the case of an irreducible f (i.e. if there is no nontrivial toral subgroup invariant under a power of f) $K \subset T^k$ is pre-invariant iff $O(K)$ is not dense in T^k.

A set $K \subset \mathbf{R}^k$ is called *pre-invariant* if its projection $\pi(K) \subset T^k$ is pre-invariant.

Now we shall state a simple but important geometric lemma. It was essentially proved in [U], p. 41, but for the reader's convenience we shall prove it here again. (In [U] the proof was a little bit obscure.)

LEMMA 4. *For every pre-invariant set $K \subset \mathbf{R}^k$ and an f-invariant linear subspace $V \subset \mathbf{R}^k$, $V \neq \{0\}$, there exist $r, l > 0$ such that for every $x \in \mathbf{R}^k$ there exists $\bar{x} \in V$ for which $\|\bar{x}\| \leqslant l$ and*

(2) $$O(K) \cap R(x+\bar{x}, r) = \emptyset.$$

Proof. Since $\operatorname{cl} \pi(V)$ is a toral subgroup invariant under f, for every $x \in \mathbf{R}^k$ there exist $\bar{\bar{x}} \in V$ and $r(x) > 0$ such that

$$O(\pi(K)) \cap \pi(R(x+\bar{\bar{x}}, r(x))) = \emptyset.$$

Choose $x_1, \ldots, x_m \in \mathbf{R}^k$ for which $\bigcup_{i=1}^{m} B(\pi(x_i), \frac{1}{2} r(x_i)) = T^m$. Therefore, for every $x \in \mathbf{R}^k$ there exists $1 \leqslant i \leqslant m$ such that $\operatorname{dist}(\pi(x), \pi(x_i)) < \frac{1}{2} r(x_i)$ and consequently $\operatorname{dist}(\pi(x+\bar{\bar{x}}_i), \pi(x_i+\bar{\bar{x}}_i)) < \frac{1}{2} r(x_i)$. Thus taking $\bar{x} = \bar{\bar{x}}_i$ and $r = \frac{1}{2} \min(r(x_1), \ldots, r(x_m))$ we get

$$O(\pi(K)) \cap \pi(R(x+\bar{x}, r)) = \emptyset$$

for every $x \in \mathbf{R}^k$, hence (2).

Finally, take $l = \max(\|\bar{\bar{x}}_1\|, \ldots, \|\bar{\bar{x}}_m\|)$. ■

Remark 2. Of course this lemma holds for every hyperbolic toral automorphism without our special assumptions on the eigenvalues.

LEMMA 5. *Given a sequence $s+1 \leqslant i_1 < \ldots < i_m \leqslant k$, let $\lambda = \lambda_{i_m}$ and define*

$$W_1 = \oplus \{\mathbf{R}e_{i_j}\colon \lambda_{i_j} = \lambda\}, \quad W_2 = \oplus \{\mathbf{R}e_{i_j}\colon \lambda_{i_j} < \lambda\}.$$

If $K \subset E^u$ is a pre-invariant subset, then for every $a \in E^u$, assumption 2° of Lemma 1 is satisfied for the euclidean space $W = (a+W_1) \times (a+W_2)$ and for the set $K \cap W$.

The constants C_1, C_2 depend on l and r from the preceding lemma, but not on the point a, and $\alpha = \log\eta/\log\lambda$ where $\eta = \max\{\lambda_{i_j}\colon \lambda_{i_j} < \lambda\}$.

Proof. Let r, l be given by Lemma 4 applied to the pre-invariant set K and the f-invariant linear subspace W_1. We can assume that $l \geqslant r/2$. Fix an $(\varepsilon, a+W_1)$-cube $R = R(z, \varepsilon)$. Fix $y \in a+W_2$. Let $q \geqslant 0$ be the least integer for which

(3) $$\lambda^q \varepsilon/2 \geqslant 2l.$$

(In fact, $q > 0$ if ε is small enough.) This in particular implies

(4) $$\lambda^{q-1} \varepsilon/2 < 2l.$$

Denote the point $f_1^q((z, y)) + \overline{f_1^q((z, y))}$ by t_q (see Lemma 4 for the meaning of the bar here). We have $\overline{f_1^q((z, y))} \in W_1$ and $\|\overline{f_1^q((z, y))}\| \leqslant l$.

By Lemma 4 we have for $f_1^q(a) = a_q$,

(5) $$f_1^q(K) \cap R_{a_q+W}(t_q, r) = \emptyset.$$

We have $R_{a_q+W}(t_q, r) = R_{a_q+W_1}(t_q, r) \times R_{a_q+W_2}(t_q, r)$ and by (3) and $l \geqslant r/2$,

(6) $$f_1^q(R \times \{y\}) \supset R_{a_q+W_1}(t_q, r).$$

Let $Q = f_1^{-q}(R_{a_q+W}(t_q, r)) = \hat{R}' \times R''$. By (6), $\hat{R}' \subset R$ and by (4) the edge of $\hat{R}'$ is not shorter than $r\varepsilon/(4\lambda l)$, so $\hat{R}'$ contains an $(r\varepsilon/(4\lambda l), a+W_1)$-cube R'.

The edges of R'' are not shorter than

$$\eta^{-q} r = r(\lambda^{-q})^{\log\eta/\log\lambda} \geqslant r\left(\frac{\varepsilon}{4\lambda l}\right)^{\alpha} = r(4\lambda l)^{-\alpha}\varepsilon^{\alpha}.$$

By (5), $(R' \times R'') \cap K = \emptyset$ so condition 2° of Lemma 1 is satisfied for $C_1 = r/(4\lambda l)$ and $C_2 = r(4\lambda l)^{-\alpha}$. ∎

Remark 3. Of course, this lemma remains true for $K \subset E^s$.

Definition 2. If $K \subset \boldsymbol{R}^k$ then we denote by $W(K)$ the linear $\tilde{f}$-invariant subspace of $\boldsymbol{R}^k$ of minimal dimension such that for some $x \in \boldsymbol{R}^k$, $K \subset x + W(K)$. Of course, $W(K)$ is unique.

Theorem 1. *If all the eigenvalues of $\tilde{f}\colon \boldsymbol{R}^k \to \boldsymbol{R}^k$ are distinct, then every pre-invariant nontrivial continuum $K \subset T^k$ (a compact connected set containing more than one point) has Hausdorff dimension greater than* 1.

Proof. Let $\tilde{K} \subset \boldsymbol{R}^k$ be a continuum such that $\pi(\tilde{K}) = K$. By Lemma 3, $\pi_u(\tilde{K})$ and $\pi_s(\tilde{K})$ are pre-invariant. Since $\tilde{K}$ is a nontrivial continuum, so is either $\pi_u(\tilde{K})$ or $\pi_s(\tilde{K})$, say $\pi_u(\tilde{K})$. Hence $W = W(\pi_u(\tilde{K})) = \bigoplus_{j=1}^{m} \boldsymbol{R}e_{i_j}$ where $m \geqslant 1$, $\{i_j\}_{j=1}^{m}$ is an increasing sequence, $s+1 \leqslant i_j \leqslant k$ and $\pi_u(\tilde{K}) \subset x + W$. In fact, $\dim W \geqslant 2$, for otherwise $\pi_u(\tilde{K})$ would be an interval in the line $x + \boldsymbol{R}e_{i_1}$ and so would not be pre-invariant. Take

$$W_1 = x + \boldsymbol{R}e_{i_m}, \qquad W_2 = x + \bigoplus_{j=1}^{m-1} \boldsymbol{R}e_{i_j}.$$

As $\pi_{W_1}(\tilde{K})$ is not one point, it contains an interval. Hence $m_{W_1}(\tilde{K}) > 0$. Because of Lemma 5 the assumptions of Lemma 1 are satisfied for the set $\pi_u(\tilde{K}) \cap (W_1 \times W_2)$. Hence

$$\mathrm{HD}(K) = \mathrm{HD}(\tilde{K}) \geqslant \mathrm{HD}(\pi_u(\tilde{K})) > \dim W_1 = 1. \ ∎$$

Theorem 2. *If $K \subset E^u$ is a compact pre-invariant set such that $\dim W(K) = 1 + \dim_{\mathrm{top}}(K)$, $W(K) = \bigoplus_{j=1}^{p} \boldsymbol{R}e_{i_j}$ where $\{i_j\}_{j=1}^{p}$ is an increasing sequence, $s+1 \leqslant i_j \leqslant k$ and $\lambda_{i_p} > \lambda_{i_1}$, then $\mathrm{HD}(\pi(K)) > \dim_{\mathrm{top}}(K)$.*

Proof. Let $K \subset x + W(K)$. Using the notation of Lemma 2 (§ 2) take

$$W_1 = x + \bigoplus_{j=2}^{p} \boldsymbol{R}e_{i_j}, \qquad W_2 = x + \boldsymbol{R}e_{i_1}, \qquad V = \bigoplus \{\boldsymbol{R}e_{i_j}\colon \lambda_{i_j} = \lambda_{i_p}\}.$$

It follows from Lemma 5 (with V playing the role of W_1 there) that assumption 2° of Lemma 2 is satisfied. Therefore to use Lemma 2 we only have to show that

(7) $$m_{W_1}(\pi_{W_1}(K)) > 0.$$

If (7) were false then $\dim_{\mathrm{top}}(\pi_1(K)) < \dim W_1 = \dim_{\mathrm{top}}(K)$ $(\pi_1 = \pi_{W_1})$; see [Eng]. Since $\pi_1|_K\colon K \to \pi_1(K)$ is a closed map, the theorem on dimension lowering mappings (see [Eng], Th. 1.10.4) would imply that there exists $y \in \pi_1(K)$ such that $\dim_{\mathrm{top}}(K \cap \pi_1^{-1}(y)) \geqslant 1$ and hence that $K \cap (y + \boldsymbol{R}e_{i_1})$ is an open nonempty interval. This contradicts the assumption that K is a pre-invariant set. ∎

Of course, an analogous result for $K \subset E^s$ is also true.

Remark 4. Analogous results are true for algebraic expanding maps on T^k with assumptions on the eigenvalues as at the beginning of this section.

§ 4. Toral pre-invariant sets: Capacity. In this section we shall generalize Theorem 1 from [U] (the estimate of capacity from below) to higher dimensions. Let us start with another definition of capacity, valid for bounded subsets of a euclidean space W. For every $\varepsilon > 0$ let P_ε be a partition of W into canonical cubes with edges of length ε. For any bounded subset A of W let $P_\varepsilon(A)$ denote the family of those elements of P_ε which have nonempty intersection with A. We have

Lemma 6.

$$\underline{\mathrm{Cap}}\, A = \liminf_{\varepsilon \to 0} \frac{\log \mathrm{Card}\, P_\varepsilon(A)}{-\log \varepsilon}, \qquad \overline{\mathrm{Cap}}\, A = \limsup_{\varepsilon \to 0} \frac{\log \mathrm{Card}\, P_\varepsilon(A)}{-\log \varepsilon}.$$

Proof. This is straightforward from the definitions of lower and upper capacities (§ 1). ∎

The following simple lemma about capacity, analogous to Lemma 1, holds:

Lemma 7. *Let $(W_1, (e_1, \ldots, e_m))$, $(W_2, (e_{m+1}, \ldots, e_{m+l}))$ be euclidean spaces with bases and let $W = W_1 \times W_2$. Assume that $K \subset W$ is the graph of a continuous function $\varphi\colon Q \to W_2$ where Q is a canonical (r, W_1)-cube. Assume that there exist constants $C > 0$, $0 < \alpha < 1$ such that for every (b, W_1)-cube R with $b > 0$ sufficiently small and for every $y \in W_2$ there exists $x \in Q$ such that the cube $\{x\} \times R(y, Cb^\alpha)$ is disjoint from K. Then*

(1) $$\underline{\mathrm{Cap}}\, K \geqslant m + 1 - \alpha.$$

Proof. Set $\pi_1 = \pi_{W_1}$, $\pi_2 = \pi_{W_2}$. For an arbitrary small $\varepsilon > 0$, for every (ε, W_1)-cube $R \subset Q$ and for every $(x, y) \in \pi_1^{-1}(R) \cap K$ there is $(x', y') \in (\pi_1^{-1}(R) \cap K) \setminus \pi_2^{-1}(R(y, C\varepsilon^\alpha))$. By the continuity of φ there exists a continuous curve γ in $K \cap \pi_1^{-1}(R)$ joining (x, y) with (x', y'). So to cover γ, in particular to cover $\pi_1^{-1}(R) \cap K$ with (ε, W)-cubes we need at least $C\varepsilon^\alpha$ of them.

To estimate $\underline{\mathrm{Cap}}\ K$ we shall use the definition from Lemma 6. Assume that R belongs to the partition $\pi_1(P_\varepsilon)$ and the cubes covering γ discussed above belong to P_ε. Since we may assume they are contained in $\pi_1^{-1}(R)$, they do not repeat for different R's ($R \in \pi_1(P_\varepsilon)$). So

$$\mathrm{Card}\, P_\varepsilon(K) \geqslant C\varepsilon^{\alpha-1}(r/\varepsilon-2)^m \geqslant 2^{-m} Cr^m \varepsilon^{\alpha-1-m}$$

for ε small enough. (We subtracted 2 in the brackets since we can consider only those $R \in \pi_1(P_\varepsilon)$ which are fully contained in Q.) So

$$-\log \mathrm{Card}\, P_\varepsilon(K)/\log\varepsilon \geqslant m+1-\alpha-\log(2^{-m}Cr^m)/\log\varepsilon,$$

which for $\varepsilon \to 0$ gives (1). ∎

As a corollary we obtain the main result of this section (corresponding to Theorem 2 in § 3):

THEOREM 3. *If $K \subset E^u$ is a pre-invariant set, if $W(K) = W_1 \times W_2$ with*

$$W_1 = \bigoplus_{j=1}^{m} \boldsymbol{R} e_{i_j}, \quad W_2 = \bigoplus_{j=1}^{l} \boldsymbol{R} e_{s_j},$$

where $\{i_j\}_{j=1}^m$, $\{s_j\}_{j=1}^l$ are increasing sequences, $s+1 \leqslant i_j$, $s_j \leqslant k$ and $i_m > s_l$, and if K can be expressed as the graph of a continuous function $\varphi\colon Q \to W_2$, with Q a canonical cube in W_1, then

$$\underline{\mathrm{Cap}}\ K \geqslant m+1-\log\lambda_{s_l}/\log\lambda_{i_m}.$$

Proof. If $\lambda_{s_l} = \lambda_{i_m}$ then the theorem is obvious. If not, let $V = \bigoplus\{\boldsymbol{R}e_{i_j}\colon \lambda_{i_j} = \lambda_{i_m}\}$. By Lemma 5 (with V playing the role of W_1 there) the assumptions of Lemma 7 are satisfied and our theorem follows. ∎

Remark 5. An analogous theorem is true if we replace E^u by E^s.

Remark 6. In the proof of Lemma 7 we used Darboux's property for φ restricted to subcubes of Q. It is an interesting question whether this assumption can be omitted, i.e. whether Theorem 3 holds for φ an arbitrary function of Baire class.

Remark 7. The estimate given in Theorem 3 is best possible. Namely, improving the construction from the proof of Th. 2(ii) in [U] or from Th. 6 in [I$_2$] one can easily obtain in $W_1 \times W_2$ (assume $\dim W_2 = 1$) a set K which is the graph of $\varphi\colon W_1 \to W_2$ where φ is Hölder continuous with exponent exactly $\alpha = \log\lambda_{s_l}/\log\lambda_{i_m}$. This implies $\overline{\mathrm{Cap}}\, K \leqslant m+1-\alpha$. (One can even ensure $\dim_{\mathrm{top}}(\mathrm{cl}\bigcup_{n=-\infty}^{\infty} f^n(K)) = \dim_{\mathrm{top}}(K) = m$.)

Remark 8. A result analogous to Theorem 3 holds for f an algebraic expanding mapping of T^k.

§ 5. Graphs of real, continuous, nowhere differentiable functions. Let us start with the following general result:

THEOREM 4. *Suppose that $0 < \alpha < 1$, I is a compact nontrivial interval and $\varphi\colon I \to \boldsymbol{R}$ is a continuous function which satisfies the following two conditions:*

1° There exists a constant $C_3 > 0$ such that for all $x, y \in I$

$$|\varphi(x)-\varphi(y)| \leqslant C_3|x-y|^\alpha.$$

2° There exists $C_4 > 0$ such that for every subinterval $J \subset I$

$$\sup_{x,y\in J}|\varphi(x)-\varphi(y)| \geqslant C_4|J|^\alpha$$

where $|J|$ denotes the length of J.

Then

$$\mathrm{HD}(\mathrm{graph}\,\varphi) \geqslant D(\alpha, C_4/C_3) > 1 \tag{1}$$

where $D(\cdot,\cdot)$ is a constant depending only on α and C_4/C_3.

Proof. We want to use Lemma 1 for the set $K = \mathrm{graph}\,\varphi$. To this end we take for W the plane $\boldsymbol{R}^2$ containing K, with the standard scalar product and basis. For W_1, W_2 we take the lines $\boldsymbol{R}$ containing the domain and range of φ respectively.

We will check condition 2° of Lemma 1. Let J be a subinterval of I of length b and $y \in W_2$. By 2° above there exists $x \in J$ such that $|\varphi(x)-y| \geqslant \frac{1}{2}C_4 b^\alpha$. We look for the maximal possible d such that for every $z \in [x-d, x+d] \cap J$, $|\varphi(z)-y| > \frac{1}{2}C_2 b^\alpha$ (of course, we look for C_2 as well). For this it is enough to have

$$|\varphi(z)-\varphi(x)| < \tfrac{1}{2}C_4 b^\alpha - \tfrac{1}{2}C_2 b^\alpha.$$

But by 1° we have $|\varphi(z)-\varphi(x)| \leqslant C_3 d^\alpha$. So it is enough to have

$$C_3 d^\alpha < \tfrac{1}{2}C_4 b^\alpha - \tfrac{1}{2}C_2 b^\alpha.$$

For this it is enough to take an arbitrary d satisfying $C_3 d^\alpha < \frac{1}{2}C_4 b^\alpha$ provided we decide to fix $C_2 > 0$ sufficiently small. Hence we can take for d any number less than $(C_4/(2C_3))^{1/\alpha} b$. We conclude that (1) is true with

$$D(\alpha, C_4/C_3) \geqslant C\big(\alpha, (C_4/(2C_3))^{1/\alpha}\big). \quad ∎ \tag{2}$$

Remark 9. Observe that to have $\underline{\mathrm{Cap}}(\mathrm{graph}\,\varphi) \geqslant 2-\alpha$ it is sufficient to assume 2° only (cf. Lemma 7, § 4).

If we assume 1° only we easily obtain $\overline{\mathrm{Cap}}(\mathrm{graph}\,\varphi) \leqslant 2-\alpha$. In fact, we get more:

PROPOSITION 1. *If* $\varphi\colon I\to R$ *is a Hölder continuous function, i.e. for all* $x, y\in I$, $|\varphi(x)-\varphi(y)|\leqslant C_3|x-y|^{\alpha}$, *then for the Hausdorff measure* $\Lambda_{2-\alpha}$ *we have* $\Lambda_{2-\alpha}(\operatorname{graph}\varphi)<\infty$, *and moreover for every Borel set* $E\subset\operatorname{graph}\varphi$, $\Lambda_{2-\alpha}(E)\leqslant 2C_3\,\mu(E)$.

Proof. Cover the set $\pi_{W_1}(E)$ with intervals J_i, $i=1,2,\ldots$, of lengths r_i such that $\sum r_i\leqslant\sigma m_{W_1}(\pi_{W_1}(E))=\sigma\mu(E)$ for $\sigma>1$ arbitrarily close to 1. (Here m_{W_1} is the Lebesgue measure on $I\subset W_1$.) Every set $(J_i\times R)\cap\operatorname{graph}\varphi$ has height at most $C_3 r_i^{\alpha}$ so it can be covered with $C_3 r_i^{\alpha-1}$ balls of diameters $\sqrt{2}r_i$. Thus

$$\Lambda_{2-\alpha}(E)\leqslant\sum_i C_3 r_i^{\alpha-1}(\sqrt{2}r_i)^{2-\alpha}\leqslant 2C_3\sum_i r_i\leqslant 2C_3\,\sigma\mu(E).\ \blacksquare$$

Applications of Theorem 4 to nowhere differentiable Weierstrass functions were already described in the introduction. We shall close this section with theorems about larger and other classes of functions satisfying the assumptions of Theorem 4.

But first let us make a remark about the higher dimensional case.

Remark 10. Theorem 4 also holds for $\varphi\colon W_1\to W_2$, with W_1, W_2 of arbitrary dimensions $m\geqslant 1$, $l\geqslant 1$. I and J are then canonical cubes in W_1 and it is convenient to write the inequalities in assumptions 1° and 2° in the form

$$C_3|J|^{\alpha}\geqslant\sup_{x,y\in J}\|\varphi(x)-\varphi(y)\|\geqslant C_4|J|^{\alpha}.$$

(Here $|J|$ is the length of the edge of J.)

The result is

$$\operatorname{HD}(\operatorname{graph}\varphi)\geqslant D(\alpha, C_4/C_3, m)>m,$$

$$D(\alpha, t, m)\geqslant C(\alpha, (t/2)^{1/\alpha}, m),$$

analogously to (2).

In the case $\dim W_2=1$ this result can be applied to the Kaplan–Mallet-Paret–Yorke attracting fractal torus (see [KMY]). This example will also be discussed in Remark 14 at the end of the paper.

Let us return to graphs of real functions.

THEOREM 5. *Fix* $0<\alpha<1$ *and let* $q\colon R\to R$ *be a Lipschitz continuous function of period* 1. *Assume that there exists a compact interval* $\Delta=[a,b]\subset R$ *such that* $q|_{\Delta}$ *is monotone and* $q(a)\neq q(b)$. *Then for* $\beta>1$ *sufficiently large the function* $f\colon R\to R$,

$$f(x)=\sum_{n=0}^{\infty}\beta^{-\alpha n}q(\beta^n x+\theta_n),$$

where $\theta_0,\theta_1,\ldots$ *are arbitrary phases, satisfies conditions* 1°, 2° *of Theorem* 4 *and in consequence* $\operatorname{HD}(\operatorname{graph} f)>1$.

Proof. Checking 1° is easy so we leave it to the reader. We shall prove 2°.

Multiplying q by -1 if necessary, we can suppose that $q|_{\Delta}$ is increasing (not necessarily strictly). Assume also that β satisfies

$$\beta|\Delta|\geqslant 1+|\Delta| \tag{3}$$

($|\Delta|$ denotes the length of Δ).

In order to prove 2° we shall show the following claim:

If n *and* $k\geqslant 0$ *are arbitrary integers and* $\varphi_0,\varphi_1,\ldots,\varphi_k\in R$ *then there exists an interval* $J_k\subset[n,n+2]$ *(which depends also on* $\varphi_0,\ldots,\varphi_k$*) of length* $\beta^{-k}|\Delta|$ *such that for every* $0\leqslant j\leqslant k$, q *restricted to* $\beta^j J_k+\varphi_j$ *is increasing and* $\beta^k J_k+\varphi_k=\Delta\ (\mathrm{mod}\ 1)$, *i.e. there is* $m\in Z$ *such that* $\beta^k J_k+\varphi_k=m+\Delta$.

We fix $n\in Z$ and proceed inductively with respect to k. As q is of period 1 we find an interval $\Delta'=\Delta\ (\mathrm{mod}\ 1)$ contained in $[n,n+2]+\varphi_0$ and therefore we can take $\Delta'-\varphi_0$ for J_0.

Suppose now that J_k has already been constructed. By (3), $|\beta^{k+1}J_k+\varphi_{k+1}|=\beta^{k+1}|J_k|=\beta|\Delta|\geqslant 1+|\Delta|$ and so there exists an interval $\tilde{\Delta}=\Delta\ (\mathrm{mod}\ 1)$ contained in $\beta^{k+1}J_k+\varphi_{k+1}$. Since $q|_{\Delta}$ is increasing, we can take

$$J_{k+1}=\beta^{-(k+1)}(\tilde{\Delta}-\varphi_{k+1})\subset J_k\subset[n,n+2].$$

Thus the claim is proved.

Now, take β so large that

$$\frac{M}{\beta^{\alpha}-1}\leqslant\tfrac{1}{4}(q(b)-q(a)) \tag{4}$$

where $M=\sup_{x_1,x_2\in[0,1]}|q(x_1)-q(x_2)|$ and let $k=k(\beta)$ be the smallest integer $\geqslant 1$ for which

$$\frac{L|\Delta|}{\beta^{1-\alpha}-1}\beta^{-\alpha(s+k(\beta)/\alpha)}\leqslant\tfrac{1}{4}\beta^{-\alpha(s+k(\beta))}(q(b)-q(a)) \tag{5}$$

where L is the Lipschitz constant of q.

Let $J\subset[0,1]$ be a closed interval and $s\geqslant 0$ the smallest integer for which

$$\beta^s|J|\geqslant 4. \tag{6}$$

In particular, this definition implies that

$$\beta^{s-1}|J|<4. \tag{7}$$

Hence there exists an integer n such that $[n, n+2] \subset \beta^s J$. Let $[x, y] = \beta^{-s} J_k$ where $J_k \subset [n, n+2]$ is the interval given by our claim with $\varphi_0 = \theta_s$, $\varphi_1 = \theta_{s+1}, \ldots, \varphi_k = \theta_{s+k}$. So $[x, y] \subset J$ and we shall estimate $|f(y)-f(x)|$ from below.

To simplify the writing set $A_m = \beta^{-\alpha m}|q(\beta^m y+\theta_m)-q(\beta^m x+\theta_m)|$. Since $|J_k| = \beta^{-k}|\Delta|$ we have by (5)

$$\sum_{m=0}^{s-1} A_m \leqslant L\beta^{-(s+k)}|\Delta| \sum_{m=0}^{s-1} \beta^{m-\alpha m} = L|\Delta|\,\beta^{-(s+k)}\frac{\beta^{(1-\alpha)s}-1}{\beta^{1-\alpha}-1} \tag{8}$$
$$\leqslant \frac{L|\Delta|}{\beta^{1-\alpha}-1}\beta^{-\alpha(s+k/\alpha)} \leqslant \tfrac{1}{4}\beta^{-\alpha(s+k)}\big(q(b)-q(a)\big).$$

By (4) we get

$$\sum_{m=s+k+1}^{\infty} A_m \leqslant M \sum_{m=s+k+1}^{\infty} \beta^{-\alpha m} = M\beta^{-\alpha(s+k+1)}/(1-\beta^{-\alpha}) \tag{9}$$
$$= M\beta^{-\alpha(s+k)}/(\beta^{\alpha}-1) \leqslant \tfrac{1}{4}\beta^{-\alpha(s+k)}\big(q(b)-q(a)\big).$$

In view of the claim and the definition of $[x, y]$ we have

$$\sum_{m=s}^{s+k} A_m \geqslant \beta^{-\alpha(s+k)}\big(q(b)-q(a)\big). \tag{10}$$

Now (8), (9) and (10) give

$$f(y)-f(x) = \sum_{m=0}^{\infty} A_m \geqslant \tfrac{1}{2}\big(q(b)-q(a)\big)\beta^{-\alpha(s+k)}$$

and by (7)

$$f(y)-f(x) \geqslant \frac{q(b)-q(a)}{2(4\beta)^{\alpha}\beta^{\alpha k}}|J|^{\alpha},$$

which finishes the proof. ∎

Changing this proof a little, it is easy to see that the following theorems are also true.

THEOREM 5a. *Fix $0<\alpha<1$ and let $q\colon \mathbf{R}\to\mathbf{R}$ be a Lipschitz continuous function of period* 1. *If $q|_{[0,1/2]}$ is monotone and $q(0)\neq q(1/2)$, then for every integer $p \geqslant 2$ the function*

$$f(x) = \sum_{n=0}^{\infty} p^{-\alpha n} q(p^n x)$$

satisfies conditions 1°, 2° *of Theorem* 4.

THEOREM 5b. *If $0<\alpha<1$, $q\colon \mathbf{R}\to\mathbf{R}$ is Lipschitz continuous of period* 1 *and q is monotone on the interval $[0, 1/4]$, $q(0)\neq q(1/4)$ and $q(0) = q(1/2)$, then the assertion of Theorem* 5a *remains true.*

The point here is that under the hypotheses of Theorems 5a and 5b the sets J_k can be so chosen that the series (9) vanishes.

In the case of Theorem 5a this observation remains true even if we add noninteger phases $\theta_0, \theta_1, \ldots$, as in Theorem 5. Then, however, we need to assume that $p \geqslant 3$ to be able to construct the sets J_k (cf. (3)).

Observe that Theorems 5a and 5b work for the classical examples:

$$q(x) = \cos 2\pi x, \quad q(x) = \operatorname{dist}(x, \mathbf{Z}+1/2),$$
$$q(x) = \sin 2\pi x, \quad q(x) = \operatorname{dist}(x, \mathbf{Z})$$

(Th. 5a for the first two examples, Th. 5b for the latter ones).

Let us mention that the essence of our proof of Theorem 5 is the same as in [KMY] (in particular, a weak dependence of the random variables $q(\beta^i x)$ and $q(\beta^j x)$ for $i \neq j$). Nevertheless our proof is purely geometric. We need not use Fourier analysis because our function q is periodic rather than almost periodic as in [KMY]. Unfortunately, we do not succeed with β close to 1.

There is also a relation with [BU] and [MW], Th. 8, as we have already mentioned in the introduction. Mauldin and Williams assume more about q and β than we do. Then they get the estimate

$$\operatorname{HD}(\operatorname{graph} f) \geqslant 2-\alpha-(C/\log\beta)$$

for a constant $C>0$, which approaches $2-\alpha$ as $\beta\to\infty$.

Our approach through Theorem 4 and Lemma 1 is much more general, so we get a weaker estimate.

§ 6. Graphs of limit Rademacher functions. As we did not know how to cope with the conjecture $\operatorname{HD}(\operatorname{graph} \mathscr{W}) = 2-\alpha$ for $\mathscr{W}$ a Weierstrass nowhere differentiable function given by, say, the formula (2) from the introduction, we decided to study a function which seems simpler, with $\cos x$ replaced by $q(x)$ where $q(x) = 1$ for $x\in[0, \frac{1}{2})$, $q(x) = -1$ for $x \in [\frac{1}{2}, 1)$. We obtain the function given by the formula (3) from the introduction:

$$f_\lambda(x) = \sum_{n=0}^{\infty} \lambda^n r_n. \tag{1}$$

Here r_n is the nth Rademacher function, $r_n(x) = q(2^n x \pmod 1)$, and λ is arbitrary such that $\frac{1}{2}<\lambda<1$.

We shall call such a function a *limit Rademacher function*.

For any integer $k \geqslant 0$ we write $f_{\lambda,k} = \sum_{n=0}^{k-1}\lambda^n r_n$. We still write $\alpha = -\log\lambda/\log 2$.

PROPOSITION 2.

$$\underline{\mathrm{Cap}}\,(\mathrm{graph}\,f_\lambda) = \overline{\mathrm{Cap}}\,(\mathrm{graph}\,f_\lambda) = 2-\alpha.$$

Moreover (as in Prop. 1 in § 5), for every Borel set $E \subset \mathrm{graph}\,f_\lambda$,

$$\Lambda_{2-\alpha}(E) \leqslant \frac{4}{1-\lambda}\mu(E).$$

Proof. Consider an interval $J = J_{k,l} = [l2^{-k}, (l+1)2^{-k})$, $k \geqslant 0$, $0 \leqslant l \leqslant 2^k-1$. By (1),

$$\sup_{x\in J} f_\lambda(x) - \inf_{x\in J} f_\lambda(x) \leqslant \sum_{n=k}^{\infty}\lambda^n - \sum_{n=k}^{\infty}(-\lambda^n) = \frac{2}{1-\lambda}(2^{-k})^\alpha,$$

hence $\overline{\mathrm{Cap}}\,(\mathrm{graph}\,f_\lambda) \leqslant 2-\alpha$.

Since for any $\lambda \in (\frac{1}{2}, 1)$ and $k \geqslant 0$,

$$\lambda^k - \sum_{n=k+1}^{\infty}\lambda^n \leqslant -\lambda^k + \sum_{n=k+1}^{\infty}\lambda^n,$$

every limit Rademacher function f_λ has Darboux's property on every interval $J_{k,l}$ (this is in fact well known, see e.g. [E1]). Consequently, the proof of the inequality $\underline{\mathrm{Cap}}\,(\mathrm{graph}\,f_\lambda) \geqslant 2-\alpha$ from Lemma 7 (§ 4) applies.

The estimate for $\Lambda_{2-\alpha}(E)$ can be proved similarly to Prop. 1 in § 5. ■

Now we pass to estimates of the Hausdorff dimension.

THEOREM 6. *For every $\frac{1}{2} < \lambda < 1$, $\mathrm{HD}(\mathrm{graph}\,f_\lambda) > 1$.*

Proof. We want to use Lemma 1. As in the proof of Theorem 4 we consider $K = \mathrm{graph}\,f_\lambda$ in the (x, y) euclidean plane $\boldsymbol{R}^2 = W = W_1 \times W_2$, where $f_\lambda\colon [0, 1) \to W_2$, $[0, 1) \subset W_1$.

Fix an integer N such that

$$\lambda^N < \tfrac{1}{2} \quad (\text{say } N = E(\alpha^{-1})+1,\ \alpha = -\log\lambda/\log 2). \tag{2}$$

Take in W_1 an arbitrary interval $J_{k,l} = [l2^{-k}, (l+1)2^{-k})$ where $k \geqslant 0$, $0 \leqslant l \leqslant 2^k-1$. In $J_{k,l}$ there exist two subintervals $J_{k+N,s}$ and $J_{k+N,s'}$ on which $r_n = +1$ for every $n = k, \ldots, k+N-1$ and $r_n = -1$ for every $n = k, \ldots, k+N-1$, respectively. So for any $x \in J_{k+N,s}$, $x' \in J_{k+N,s'}$,

$$|f_{\lambda,k+N}(x) - f_{\lambda,k+N}(x')| = 2\sum_{n=k}^{k+N-1}\lambda^n = 2\lambda^k\frac{1-\lambda^N}{1-\lambda},$$

hence

$$|f_\lambda(x) - f_\lambda(x')| \geqslant 2\lambda^k\frac{1-\lambda^N}{1-\lambda} - \sum_{n=k+N}^{\infty}\lambda^n|r_n(x)-r_n(x')|$$
$$\geqslant 2\lambda^k\left(\frac{1-\lambda^N}{1-\lambda} - \lambda^N\frac{1}{1-\lambda}\right) = 2\lambda^k\frac{1-2\lambda^N}{1-\lambda} > 0 \quad \text{by (2)}.$$

So for every $y \in W_2$ and for either x or x', say for x,

$$|f_\lambda(x) - y| \geqslant \lambda^k\frac{1-2\lambda^N}{1-\lambda} = (2^{-k})^\alpha\frac{1-2\lambda^N}{1-\lambda}.$$

An arbitrary interval $J \subset W_1$ contains some $J_{k,l}$ such that $|J| > \frac{1}{2}|J_{k,l}|$ ($|\cdot|$ means length). All this gives condition 2° of Lemma 1 with constants α,

$$C_1 = 2^{-N-1} = 2^{-(E(\alpha^{-1})+2)} \tag{3}$$

(since $|J_{k+N,s}|/|J| = (|J_{k+N,s}|/|J_{k,l}|)\cdot(|J_{k,l}|/|J|) \geqslant 2^{-N}\cdot 2^{-1}$) and $C_2 = 2^{1-\alpha}(1-2\lambda^N) \times (1-\lambda)^{-1}$. ■

Now we plan to prove that for almost every λ, $\mathrm{HD}(\mathrm{graph}\,f_\lambda) = 2-\alpha$ and that for some λ's, $\mathrm{HD}(\mu) < 2-\alpha$. We shall rely on Erdős's results so we need to be able to compare the measure $\mu = \mu_\lambda$, the lift of the length measure m_{W_1} to the graph of f_λ, with the probability distribution on W_2 given by f_λ.

Consider now the mapping $F_\lambda\colon [0, 1) \times \boldsymbol{R} \to [0, 1) \times \boldsymbol{R}$ defined as follows:

$$F_\lambda(x, y) = \big(2x\,(\mathrm{mod}\,1),\ \lambda^{-1}(y - q(x))\big).$$

Observe that $\mathrm{graph}\,f_\lambda$ is invariant for F_λ, i.e. $F_\lambda(\mathrm{graph}\,f_\lambda) = F_\lambda^{-1}(\mathrm{graph}\,f_\lambda) = \mathrm{graph}\,f_\lambda$. The projection $\pi_1 = \pi_{W_1}\colon \boldsymbol{R}^2 \to \boldsymbol{R} = W_1$ restricted to $\mathrm{graph}\,f_\lambda$ is a measure preserving conjugacy between F_λ on $\mathrm{graph}\,f_\lambda$ with measure μ_λ and the "baker's transformation" $b\colon x \to 2x\,(\mathrm{mod}\,1)$ on the interval $[0, 1)$ with measure m_{W_1}. Both dynamical systems are ergodic, with measure-theoretic Jacobian equal to $\log 2$ and entropy also equal to $\log 2$ (maximal possible for invariant measures).

The probability distribution ν_λ on $\boldsymbol{R} = W_2$ is defined to be the factor measure of μ_λ, $\nu_\lambda = (\pi_2)_*(\mu_\lambda)$, for $\pi_2 = \pi_{W_2}\colon \boldsymbol{R}^2 \to \boldsymbol{R} = W_2$ (unfortunately, the mapping $F_\lambda|_{\mathrm{graph}\,f_\lambda}$ is not constant on the fibers of π_2 so its factor does not exist).

Often we shall omit the index λ in f, F, μ and ν if λ is fixed.

In this section we shall compare $\theta = \mu$ or ν with the Hausdorff (in particular, Lebesgue) measures Λ_s. We say a Borel measure θ is *absolutely continuous* with respect to Λ_s if for every Borel set A, $\Lambda_s(A) = 0$ implies $\theta(A)$

$= 0$. We use the notation $\theta \ll \Lambda_s$. If $\theta \ll \Lambda_s$ is not true we write $\theta \not\ll \Lambda_s$. We say θ is *singular* with respect to Λ_s if there exists a set A of full θ-measure such that $\Lambda_s(A) = 0$. We use the notation $\theta \perp \Lambda_s$.

Let us state a technical lemma relying on the dynamics of F and the Frostman lemma from the introduction.

LEMMA 8. 1° *For μ-almost every $z = (x, y) \in \operatorname{graph} f$,*

$$L(z) \equiv \liminf_{k\to\infty} \frac{\log \mu(Q_k(z))}{-k\log 2} = \mathrm{HD}(\mu) \tag{4}$$

where $Q_k(z) = J_{k,l}(x) \times B(y, 2^{-k-1})$ and $J_{k,l}$ is the interval of the form $[l2^{-k}, (l+1)2^{-k})$ containing x.

2° *For ν-a.e. $y \in W_2$,*

$$\liminf_{r\to 0} \frac{\log \nu(B(y, r))}{\log r} = \mathrm{HD}(\nu). \tag{5}$$

1° *and* 2° *remain true if* lim inf *is replaced by* lim sup.

Proof. Observe that $L(z)$ is a.e. constant. Indeed,

$$\mu(Q_{k-1}(F(z))) \geq \mu(J_{k-1,s}(\pi_1(F(z))) \times B(\pi_2(F(z)), 2^{-k-1}\lambda^{-1})) = 2\mu(Q_k(z)).$$

We conclude that for every $z \in \operatorname{graph} f$, $L(F(z)) \leq L(z)$. In view of the ergodicity of F this implies L is constant almost everywhere.

If we considered the balls $B(z, 2^{-k})$ instead of the squares $Q_k(z)$ we could deduce the equality (4) from the Frostman lemma. Of course, for every z,

$$L(z) \geq \liminf_{k\to\infty} \frac{\log \mu(B(z, 2^{-k}))}{-k\log 2} \tag{6}$$

since $Q_k(z) \subset B(z, 2^{-k+1})$. So $L(z) \geq \mathrm{HD}(\mu)$.

To prove $L(z) \leq \mathrm{HD}(\mu)$ for a.e. z we have two methods:

1. We can simply prove the inequality opposite to (6) for a.e. z, using a trick which is standard in smooth dynamics with singularities. For arbitrary $\beta > 0$ and every $k \geq 1$ let

$$A_{\beta,k} = \bigcup \{[l2^{-k} - 2^{-(1+\beta)k}, l2^{-k} + 2^{-(1+\beta)k}]: l = 0, 1, \ldots, 2^k\} \cap [0, 1).$$

We have $m_{W_1}(A_{\beta,k}) = 2^{-\beta k+1}$, so by Borel–Cantelli's lemma m_{W_1}-almost every $x \in W_1$ does not belong to any $A_{\beta,k}$ for k large enough. Hence for μ-a.e. z and k large enough $Q_k(z) \supset B(z, 2^{-(1+\beta)k})$, so

$$L(z) \leq (1+\beta) \liminf_{k\to\infty} \frac{\log \mu(B(z, 2^{-(1+\beta)k}))}{-(1+\beta)k\log 2}. \tag{7}$$

As $\beta > 0$ can be chosen arbitrarily small we obtain what we wanted.

2. By Egorov's Theorem there exists a Borel set $S \subset \operatorname{graph} f_\lambda$ with $\mu(S) > 0$ such that

$$\liminf \frac{\log \mu(Q_k(z))}{-k\log 2}$$

converges to $L(z) = L$ uniformly, in particular for every $\varepsilon > 0$ there exists k_0 such that $(\log \mu(Q_k(z)))/(-k\log 2) \geq L - \varepsilon$ for all $z \in S$, $k \geq k_0$. We write this in the form $\mu(Q_k(z)) \leq 2^{-k(L-\varepsilon)}$.

Consider $\tilde{\mu}$, the restriction of μ to S. Denote by Q' the square $Q_k(x + 2^{-k}, y)$ or $Q_k(x - 2^{-k}, y)$ for $z = (x, y)$ according as $x \in [l2^{-k}, (l+1)2^{-k})$ is closer to $(l+1)2^{-k}$ or $l2^{-k}$. Then $B(z, 2^{-k-1}) \subset Q_k(z) \cup Q'$. Suppose $z \in S$, so $\tilde{\mu}(Q_k(z)) \leq 2^{-k(L-\varepsilon)}$. We have either $\tilde{\mu}(Q') = 0$ if $Q' \cap S = \emptyset$, or $\tilde{\mu}(Q') \leq \tilde{\mu}(Q_{k-1}(z')) \leq 2^{-(k-1)(L-\varepsilon)}$ if there exists $z' \in Q' \cap S$. In both cases

$$\tilde{\mu}(B(z, 2^{-k-1})) \leq 2 \cdot 2^{-(k-1)(L-\varepsilon)}$$

so

$$\frac{\log \tilde{\mu}(B(z, 2^{-k-1})) - \log 2}{-(k-1)\log 2} \geq L - \varepsilon$$

and we conclude, as from (7), that $L \leq \mathrm{HD}(\tilde{\mu})$ (of course, $\mathrm{HD}(\tilde{\mu}) \leq \mathrm{HD}(\mu)$).

In the proof of 2° we rely on the inequality $\nu(B(\pi_2 F(z), r)) \leq 2\nu(B(y, \lambda r))$ which follows from

$$\mu(\pi_2^{-1} B(\pi_2 F(z), r)) = 2\mu([0, \tfrac{1}{2}) \times B(y, \lambda r)) \leq 2\mu(\pi_2^{-1} B(y, \lambda r)) \quad \text{if } x \in [0, \tfrac{1}{2}).$$

(If $x \in [\frac{1}{2}, 1)$, we replace $[0, \frac{1}{2})$ by $[\frac{1}{2}, 1)$ in the above.) The remainder of the argument is the same as in the proof of 1° except that we do not need the considerations under 1 or 2 above.

The case of lim sup is handled similarly. ∎

Remark 11. Similarly to 2° we can easily prove the well-known fact [JW] that if $\nu \not\ll m_{W_2}$ then $\nu \perp m_{W_2}$. Indeed, if there exists S with $\nu(S) > 0$ and $m_{W_2}(S) = 0$ then for ν-almost every $y \in S$

$$\limsup_{r\to 0} \nu(B(y, r))/r = \infty. \tag{8}$$

Otherwise, for a set $\hat{S} \subset S$ with $\nu(\hat{S}) > 0$ we have $\nu(B(y, r)) < Cr$ for all $y \in \hat{S}$ and $r \geqslant r_0$ for some constants r_0, $C > 0$, hence $m_{W_2}(\hat{S}) > 0$ by the definition of the (outer) Lebesgue measure.

As in the proof of Lemma 8, by the ergodicity of $F|_{\operatorname{graph} f}$, we obtain (8) for ν-a.e. y, and then Besicovitch's covering theorem yields $\nu \perp m_{W_2}$.

THEOREM 7. *For every* $\lambda \in (\frac{1}{2}, 1)$ *we have*:

1° $\mathrm{HD}(\mu_\lambda) \geqslant 1 + \mathrm{HD}(\nu_\lambda)(1-\alpha)$.

2° *If for* ν_λ*-a.e.* $y \in W_2$,

$$\limsup_{r \to 0} \frac{\log \nu_\lambda(B(y, r))}{\log r} = t$$

then $\mathrm{HD}(\mu_\lambda) \leqslant 1 + t(1-\alpha)$.

Remark 12. It appears to follow from [LYg] that $\mathrm{HD}(\mu_\lambda) = 1 + \mathrm{HD}(\nu_\lambda)(1-\alpha)$. However, it is not necessary to rely on the complicated machinery of [LYg] to obtain the main results of this section. (We shall develop this remark at the end of this paper.)

Proof of Theorem 7. For all $z \in \boldsymbol{R}^2$ and $k \geqslant 0$ we have

$$\mu(Q_k(z)) = 2^{-k} \mu(F^k(Q_k(z))) = 2^{-k} \nu(B(\pi_2 F^k(z), 2^{-k-1} \lambda^{-k})), \tag{9}$$

since $F^k(Q_k(z))$ is a full horizontal strip from $x = 0$ to 1. So

$$\frac{\log \mu(Q_k(z))}{-k \log 2} = 1 + \frac{\log \nu(B(\pi_2 F^k(z), 2^{-k-1} \lambda^{-k}))}{-k \log 2}. \tag{10}$$

By Egorov's Theorem there exists a compact set $S \subset W_2$, $\nu(S) > 0$, with the uniform behaviour of liminf and limsup, i.e. such that there exists a continuous function $\varepsilon(r)$: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$, $\varepsilon(0) = 0$, for which

$$\mathrm{HD}(\nu) - \varepsilon(r) \leqslant \frac{\log \nu(B(y, r))}{\log r} \leqslant t + \varepsilon(r)$$

for all $r > 0$, $y \in S$.

By the Birkhoff Ergodic Theorem for F, for μ-a.e. $z \in \operatorname{graph} f_\lambda$ there exists a sequence of integers $n_j \to \infty$ satisfying

$$n_{j+1}/n_j \to 1 \quad \text{and} \quad F^{n_j}(z) \in \pi_2^{-1}(S) \quad \text{for } j = 1, 2, \ldots$$

So for every $k \geqslant 0$, if $n_j \leqslant k \leqslant n_{j+1}$ we have applying (10)

$$\begin{aligned} \frac{\log \mu(Q_k(z))}{-k \log 2} &\geqslant \frac{\log \mu(Q_{n_j}(z))}{-n_j \log 2} \cdot \frac{n_j}{n_{j+1}} \\ &= \left(1 + \frac{\log \nu(B(\pi_2 F^{n_j}(z), 2^{-n_j-1} \lambda^{-n_j}))}{\log(2^{-n_j-1} \lambda^{-n_j})} \cdot \frac{\log(2^{-n_j-1} \lambda^{-n_j})}{-n_j \log 2}\right) \frac{n_j}{n_{j+1}} \\ &= (1 + X_j Y_j) Z_j. \end{aligned}$$

We have $X_j \geqslant \mathrm{HD}(\nu) - \varepsilon(2^{-n_j-1} \lambda^{-n_j})$, also Y_j tends to $1-\alpha$ and Z_j tends to 1 as $k \to \infty$. This proves 1°.

To prove 2° we proceed similarly. We write

$$\frac{\log \mu(Q_k(z))}{-k \log 2} \leqslant \frac{\log \mu(Q_{n_{j+1}}(z))}{-n_{j+1} \log 2} \cdot \frac{n_{j+1}}{n_j} = (1 + X_{j+1} Y_{j+1}) Z_j^{-1}$$

and use $X_{j+1} \leqslant t + \varepsilon(\cdot)$. ∎

COROLLARY 1. $\mathrm{HD}(\operatorname{graph} f_\lambda) \geqslant 1 + \mathrm{HD}(\nu_\lambda)(1-\alpha)$.

COROLLARY 2. *If* $\nu_\lambda \ll m_{W_2}$ *then*

$$\mathrm{HD}(\mu_\lambda) = \mathrm{HD}(\operatorname{graph} f_\lambda) = \underline{\mathrm{Cap}}(\operatorname{graph} f_\lambda) = \overline{\mathrm{Cap}}(\operatorname{graph} f_\lambda) = 2-\alpha.$$

Proof. The inequalities $\mathrm{HD}(\mu_\lambda) \leqslant \mathrm{HD}(\operatorname{graph} f_\lambda) \leqslant \underline{\mathrm{Cap}}(\operatorname{graph} f_\lambda) \leqslant \overline{\mathrm{Cap}}(\operatorname{graph} f_\lambda) \leqslant 2-\alpha$ follow from Proposition 2. Since $\nu_\lambda \ll m_{W_2}$, $\mathrm{HD}(\nu_\lambda) = 1$. Therefore by Theorem 7, $\mathrm{HD}(\mu_\lambda) \geqslant 2-\alpha$. ∎

Remark 13. If we assume additionally

$$d\nu_\lambda/dm_{W_2} \in L^\infty(m_{W_2}) \tag{11}$$

then $\mu_\lambda(Q_k(z)) \leqslant C 2^{-(2-\alpha)k}$ for a constant $C > 0$ and all z, k (see (9)). We conclude that $\mu_\lambda \ll \Lambda_{2-\alpha}$. So in view of Proposition 2, μ_λ is equivalent to $\Lambda_{2-\alpha}$ on $\operatorname{graph} f_\lambda$.

If (11) does not hold then with the use of (9) and Besicovitch's covering theorem it is not hard to prove $\mu_\lambda \perp \Lambda_{2-\alpha}$. It is quite possible that $\nu_\lambda \ll m_{W_2}$ implies (11) automatically.

Now for every $a \in (\frac{1}{2}, 1)$ let $Z(a)$ denote the set of those $\lambda \in (a, 1)$ for which $\nu_\lambda \not\ll m_{W_2}$. P. Erdős proved in [E₁] that there exists $a_0 \in (\frac{1}{2}, 1)$ such that $l_1(Z(a_0)) = 0$ (l_1 denotes the Lebesgue measure). In fact, his method gives $\lim_{a \to 1} \mathrm{HD}(Z(a)) = 0$. Thus we get:

COROLLARY 3. *For every* $a \in (a_0, 1)$ *there exists a set* $Z(a) \subset (a, 1)$ *such that for every* $\lambda \in (a, 1) \setminus Z(a)$,

$$\mathrm{HD}(\operatorname{graph} f_\lambda) = \mathrm{HD}(\mu_\lambda) = 2-\alpha, \qquad \lim_{a \to 1} \mathrm{HD}(Z(a)) = 0.$$

Erdős's theorem, and hence Corollary 3, remain true if $Z(a)$ denotes the larger set consisting of those $\lambda \in (0, 1)$ for which $d\nu_\lambda/dm_{W_2}$ is not of a given degree of smoothness (or does not exist). Under this convention $\lambda \in (a, 1) \setminus Z(a)$ implies, by Remark 13, that μ_λ is equivalent to $\Lambda_{2-\alpha}$ on $\operatorname{graph} f_\lambda$.

Erdős's method of studying ν_λ is to consider the characteristic function

(Fourier transform) of ν_λ which is $\varphi(x,\lambda)=\prod_{n=0}^{\infty}\cos\lambda^n x$. For fixed λ, the existence and continuity of the mth derivative of the distribution ν_λ follows from $\varphi(x,\lambda)=o(|x|^{m+1})$ as $x\to\infty$, which roughly speaking is true if the numbers from the sequence $\lambda,\lambda^2,\lambda^3,\ldots$ are often far from the integers.

A diametrically opposite situation occurs if $\varphi(x,\lambda)$ does not converge to 0 as $x\to\infty$. This happens if and only if λ^{-1} is a so-called Pisot–Vijayaraghavan number (PV-number). (The "if" part was observed by Erdős [E_2], the "only if" part by Pisot and Salem, see e.g. [C], Ch. VIII, 1, Th. II.)

We recall the definition: $a>1$ is a *PV-number* if it is an algebraic integer and all its conjugates have absolute values <1.

Erdős's conclusion was that if λ^{-1} is a PV-number, then $\nu_\lambda\perp m_{W_2}$. We can prove more:

THEOREM 8. *If λ^{-1} is a PV-number* ($\frac{1}{2}<\lambda<1$) *then* $\mathrm{HD}(\nu_\lambda)<1$ *and* $\mathrm{HD}(\mu_\lambda)<2-\alpha$. *Moreover,* $\mathrm{HD}(\operatorname{graph} f_\lambda)<2-\alpha$.

The inequality $\mathrm{HD}(\nu_\lambda)<1$ is not new. Namely, relying on Garsia's papers [G_1] and [G_2], Alexander and Yorke proved in [AY] that $\bar{R}(\nu_\lambda)<1$ where $\bar{R}$ is the upper Rényi dimension. (Alexander and Yorke used the term "information dimension" and the notation $\bar{d}_{\mathrm{info}}$.) It follows from [Yg], Proposition 4.3, part (2), that $\mathrm{HD}(\nu)\leqslant\bar{R}(\nu)$, hence $\mathrm{HD}(\nu_\lambda)<1$. (In fact, the Volume Lemma yields $\mathrm{HD}(\nu)=\bar{R}(\nu)=\underline{R}(\nu)$, cf. Remark 14 and [Yg], Theorem 4.4.) By Theorem 7 this implies $\mathrm{HD}(\mu_\lambda)<2-\alpha$.

Our aim is to prove $\mathrm{HD}(\operatorname{graph} f_\lambda)<2-\alpha$. However, for the reader's convenience and to prepare some notation for that proof we first prove $\mathrm{HD}(\nu_\lambda)<1$. We rely on Garsia's ideas, paying special attention to the places not clearly written in [G_1]. Then we use the Shannon–McMillan–Breiman Theorem which allows us to omit much more complicated arguments from [AY] and [Yg].

Proof of Theorem 8([2]). By $\nu_\lambda\not\ll m_{W_2}$, for every sequence $A_k\to 0$ there exists $\gamma>0$ such that for every $\varepsilon>0$ it is possible to find an integer k and a set S_k of integers such that

(12) $$\# S_k\leqslant \varepsilon/A_k\quad (\# S_k \text{ is the number of elements in } S_k),$$

(13) $$\sum_{l\in S_k}\nu_\lambda\big((lA_k,(l+1)A_k)\big)\geqslant\gamma.$$

As λ^{-1} is a PV-number, the distance between any two values of $f_{\lambda,k}=\sum_{n=0}^{k-1}\lambda^n r_n$ is at least $C\lambda^k$ where C is a constant depending only on λ (see e.g. [G_1], Lemma 1.6). This forces an abundance of equalities $P(\lambda)=\tilde{P}(\lambda)$

([2]) Another proof of this theorem, including a formula for $\mathrm{HD}(\operatorname{graph} f_\lambda)$ can be deduced from [McM]. See Note at the end of the paper.

for P, $\tilde{P}$ two different polynomials of degree $k-1$, with coefficients ± 1. This also forces the number σ_k of all possible values $P(\lambda)$ for such polynomials (i.e. values of $f_{\lambda,k}$) to satisfy

(14) $$\sigma_k\leqslant C_5\lambda^{-k}$$

while the number of different polynomials is 2^k. By considerations from Proposition 2 we have also

(15) $$\sigma_k\geqslant C_6\lambda^{-k}.$$

Here C_5, C_6 are positive constants depending only on λ.

For $A_k=C\lambda^k$ there is at most one value of $f_{\lambda,k}$ in every interval $[lC\lambda^k,(l+1)C\lambda^k)$. Let $\mathscr{D}_k$ be the partition of $[0,1)\subset W_1$ into unions of intervals such that for every $D\in\mathscr{D}_k$, $f_{\lambda,k}|_D$ is constant and if $D,D'\in\mathscr{D}_k$, $D\neq D'$, then $f_{\lambda,k}|_D\neq f_{\lambda,k}|_{D'}$.

Recall that $|f_{\lambda,k}-f_\lambda|\leqslant\lambda^k/(1-\lambda)=\big(1/C(1-\lambda)\big)C\lambda^k$. Set
$$T=E\big(1/(C(1-\lambda))\big)+1,\qquad S_k'=S_k+\{-T,-T+1,\ldots,T\}.$$
We deduce that
$$\sum_{l\in S_k}\nu_\lambda\big((lA_k,(l+1)A_k)\big)\leqslant\sum_{l\in S_k'}\nu_{\lambda,k}\big((lA_k,(l+1)A_k)\big)$$
where $\nu_{\lambda,k}$ denotes the probability distribution of $f_{\lambda,k}$. (In [G_1], Lemma 2.5, ν_λ is already replaced by $\nu_{\lambda,k}$ in (13). We do not understand why this was possible at that stage of the proof.)

Let
$$\hat{\mathscr{D}}_k=\{D\in\mathscr{D}_k\colon f_{\lambda,k}(D)\subset\bigcup_{l\in S_k'}[lC\lambda^k,(l+1)C\lambda^k)\}.$$

Then $\#\hat{\mathscr{D}}_k\leqslant(2T+1)\varepsilon/(C\lambda^k)$ but $m_{W_1}(\bigcup\hat{\mathscr{D}}_k)\geqslant\gamma$. So by (15), $\#\hat{\mathscr{D}}_k/\sigma_k\to 0$ as $\varepsilon\to 0$, $k\to\infty$, and we conclude that Garsia's property ([G_1], p. 430) is satisfied, i.e.
$$G(k)=m(\bigcup\hat{\mathscr{D}}_k)\log\frac{\#\hat{\mathscr{D}}_k}{\sigma_k m(\bigcup\hat{\mathscr{D}}_k)}+\big(1-m(\bigcup\hat{\mathscr{D}}_k)\big)\log\frac{\sigma_k-\#\hat{\mathscr{D}}_k}{\sigma_k\big(1-m(\bigcup\hat{\mathscr{D}}_k)\big)}\to-\infty.$$
(Here $m=m_{W_1}$.)

This implies that for the measure-theoretic entropy we have

(16) $$h_m(\mathscr{D}_k,b^k)-\log\sigma_k\to-\infty\quad\text{as } k\to\infty.$$

(Recall that b is the baker's transformation on $[0,1)$.) Indeed, since the partitions $\mathscr{D}_k$ and $b^{-kn}(\mathscr{D}_k)$ are independent for every $n>0$, we have
$$h_m(\mathscr{D}_k,b^k)=\sum_{D\in\mathscr{D}_k}\big(-m(D)\log m(D)\big).$$

So (omitting the index k to simplify notation)

$$h_m(\mathcal{S}, b^k) - \log\sigma = \Big(\big(\sum_{D\in\hat{\mathcal{S}}} -m(D)\log m(D)\big) - m(\bigcup\hat{\mathcal{S}})\log\sigma\Big)$$
$$+\Big(\big(\sum_{D\notin\hat{\mathcal{D}}} -m(D)\log m(D)\big) - m(\bigcup(\mathcal{D}\setminus\hat{\mathcal{D}}))\log\sigma\Big) = \mathrm{I}+\mathrm{II}.$$

Write $m(\bigcup\hat{\mathcal{D}}) = p$ and estimate

$$\mathrm{I} = p\Big(\Big(\sum_{D\in\hat{\mathcal{D}}} -\frac{m(D)}{p}\log\frac{m(D)}{p}\Big) - \log p - \log\sigma\Big)$$
$$\leqslant p(\log \#\hat{\mathcal{D}} - \log\sigma - \log p),$$

which is the first summand in $G(k)$. We can make the similar estimate for II replacing $\hat{\mathcal{S}}$ by $\mathcal{S}\setminus\hat{\mathcal{S}}$. We conclude that $\mathrm{I}+\mathrm{II} \leqslant G(k)$. This and $G(k)\to -\infty$ give (16).

Fix k such that $h_m(\mathcal{D}_k, b^k) - \log\sigma_k < \log\frac{1}{2}C_5^{-1}$. By (14), $h_m(\mathcal{D}_k, b^k) \leqslant \log\frac{1}{2}\lambda^{-k}$. By the Shannon–McMillan–Breiman Theorem applied to the partition $\mathcal{D}_k$ and the map b^k, for m-a.e. $x\in W_1$,

$$\lim_{n\to\infty}\frac{\log m(D_k^n(x))}{-n} = h_m(\mathcal{S}_k, b^k)$$

where $D_k^n(x)$ is the atom from the partition

$$\bigvee_{i=0}^{n-1} b^{-ki}(\mathcal{D}_k) = \{D_0\cap b^{-k}(D_1)\cap\ldots\cap b^{-k(n-1)}(D_{n-1})\colon D_i\in\mathcal{D}_k\},$$

containing x. Hence

$$m(D_k^n(x)) \geqslant 2^n\lambda^{kn}t^n$$

for $t<1$ arbitrarily close to 1 and every n sufficiently large (in dependence on t).

We clearly have

$$f_\lambda(D_k^n(x)) \subset B\Big(f_{\lambda,nk}(x), \lambda^{kn}\frac{1}{1-\lambda}\Big).$$

So, for ν-a.e. y, every $t<1$ and n large enough,

$$\nu\Big(B\Big(y, \lambda^{kn}\frac{1}{1-\lambda}\Big)\Big) \geqslant 2^n\lambda^{kn}t^n. \tag{17}$$

We conclude that

$$\mathrm{HD}(\nu_\lambda) \leqslant \liminf_{r\to 0}\frac{\log\nu_\lambda(B(y,r))}{\log r} \leqslant 1 - \frac{\log 2}{-k\log\lambda} < 1.$$

(It is easy to see that in fact

$$\mathrm{HD}(\nu_\lambda) \leqslant \lim_{k\to\infty}\frac{h_m(\mathcal{S}_k, b^k)}{k\log\lambda^{-1}}.$$

In [AY] even the equality is proved (for Rényi's dimension).) By Theorem 7 we obtain

$$\mathrm{HD}(\mu_\lambda) < 2-\alpha.$$

Proof of the estimate $\mathrm{HD}(\mathrm{graph}\, f_\lambda) < 2-\alpha$. Keep k fixed as before. As the partitions $b^{-kn}(\mathcal{S}_k)$, for $n = 0, 1, \ldots$, are independent, the random variables $I_n(x) = -\log m_{W_1}(D_k^1(b^{kn}(x)))$ are independent. Moreover, they are bounded and have the same probability distribution with expectation value $h_m(\mathcal{S}_k, b^k)$. Therefore considerations involving the "exponential form" of Chebyshev's inequality imply for an arbitrary $\varepsilon>0$ the existence of $\delta>0$ such that for n large enough and for

$$\Omega_n = \{x\in[0,1)\colon \sum_{j=0}^{n-1} I_j(x) \geqslant (h_m(\mathcal{S}_k, b^k)+\varepsilon)n\}$$

we have $m_{W_1}(\Omega_n) \leqslant \exp(-\delta n)$ (see e.g. [L], Sec. 2.11, the inequality (7)).

For $x\in[0,1)\setminus\Omega_n$ we have

$$\nu_\lambda\big(B(f_\lambda(x), \lambda^{kn}/(1-\lambda))\big) \geqslant \exp\Big(-\sum_{j=0}^{n-1} I_j(x)\Big) \tag{18}$$
$$\geqslant \exp\big(-(h_m(\mathcal{S}_k, b^k)+\varepsilon)n\big) \geqslant 2^n\lambda^{kn}\exp(-\varepsilon n)$$

(compare this with (17)).

Consider the family of intervals

$$\mathcal{B} = \{B(f_\lambda(x), \lambda^{kn}/(1-\lambda))\colon x\in[0,1)\setminus\Omega_n\}.$$

Clearly it is possible to choose a subfamily $\mathcal{B}'$ such that $\bigcup\mathcal{B}' = \bigcup\mathcal{B}$ and every point is covered by intervals from $\mathcal{B}'$ at most twice. Then by (18)

$$\#\mathcal{B}' \leqslant 2\cdot 2^{-n}\lambda^{-kn}\exp(\varepsilon n).$$

Let $r = E(nk\alpha/(1-\alpha))+1$. Then all the sets $Q_\tau = (F_\lambda^{-r})_\tau(\pi_2^{-1}(B))$, where $B\in\mathcal{B}'$ and $(F_\lambda^{-1})_\tau$ runs over all branches of F_λ^{-r}, are almost squares (precisely: rectangles with ratios of side lengths bounded by $2/(1-\lambda)$).

We obtain the set $F_\lambda^{-r}(\mathrm{graph}\, f_\lambda|_{[0,1)\setminus\Omega_n})$ covered by a family $\mathcal{K}$ consisting of $2^r\cdot 2\cdot 2^{-n}\lambda^{-kn}\exp(\varepsilon n)$ squares of side length $(2/(1-\lambda))2^{-r}$ (slightly larger than the rectangles Q_τ).

Since b preserves the measure m_{W_1} we have

$$\mu_\lambda(F_\lambda^{-r}(\mathrm{graph}\, f_\lambda|_{\Omega_n})) = m_{W_1}(b^{-r}(\Omega_n)) \leqslant \exp(-\delta n).$$

Ω_n is the union of some intervals of the form $[l2^{-nk}, (l+1)2^{-nk})$. By applying

b^{-r} we obtain a covering $\mathscr{L}$ of the set $b^{-r}(\Omega_n)$ by intervals of the form $[l2^{-t}, (l+1)2^{-t})$ where $t = nk+r$ such that $\sum_{I\in\mathscr{L}} m_{W_1}(I) \leqslant \exp(-\delta n)$.

For every $I \in \mathscr{L}$ there exists a covering of graph$(f_\lambda|_I)$ by a family $\mathscr{K}_I$ of squares of side length $m_{W_1}(I)$ such that $\# \mathscr{K}_I \leqslant (2/(1-\lambda))(m_{W_1}(I))^{\alpha-1}$ (cf. Proposition 2).

We now estimate the s-Hausdorff measure of graph f_λ for an arbitrary $s > 0$:

$$\begin{aligned}\Lambda_s(\operatorname{graph} f_\lambda) &\leqslant \lim_{n\to\infty} \sum \{(\operatorname{diam} Q)^s\colon\ Q \in \mathscr{K} \cup (\bigcup \{\mathscr{K}_I\colon\ I \in \mathscr{L}\})\} \\ &= \lim_{n\to\infty} \big[2^r \cdot 2 \cdot 2^{-n} \lambda^{-kn} \exp(\varepsilon n)\big((2/(1-\lambda))2^{-r}\big)^s \\ &\quad + (\exp(-n\delta)/2^{-t})(2/(1-\lambda))2^{-t(\alpha-1)}2^{-ts}\big].\end{aligned}$$

This is 0 for s sufficiently close to $2-\alpha$ (for $s = 2-\alpha$ it is bounded for every n by const$\cdot(2^{-n}\exp(\varepsilon n) + \exp(-n\delta))$). ∎

Remark 14 (continuation of Remark 12). Let us make the assumption that Ledrappier and Young's theory [LYg] is applicable in our case, for the map F. The map F has singular lines on which it is not continuous, but in considerations involving Pesin's theory such singularities are usually irrelevant. Also, F is not 1-1 so one should consider the natural extension (inverse limit). However, we prefer to be careful and not to write that the theory [LYg] is just applicable, since we have not checked all details of it.

One assumption is that the Volume Lemma holds ([LYg], Prop. 7.3.1). Namely, for μ_λ-a.e. z,

$$\liminf_{r\to 0} \frac{\log \mu_\lambda(B(z, r))}{\log r} = \limsup_{r\to 0} \frac{\log \mu_\lambda(B(z, r))}{\log r} \quad (= \mathrm{HD}(\mu_\lambda)).$$

This easily implies (see the proof of Lemma 8 and Theorem 7) that for ν_λ-a.e. y,

$$\liminf_{r\to 0} \frac{\log \nu_\lambda(B(y, r))}{\log r} = \limsup_{r\to 0} \frac{\log \nu_\lambda(B(y, r))}{\log r} = \mathrm{HD}(\nu_\lambda),$$

hence

$$\mathrm{HD}(\mu_\lambda) = 1 + \mathrm{HD}(\nu_\lambda)(1-\alpha), \tag{19}$$

which simplifies the statement of Theorem 7.

The Volume Lemma relies on the existence of a system of conditional measures $\mu_\lambda^{\mathrm{uu}}(z)$ of μ_λ on the strong unstable manifolds $W^{\mathrm{uu}}(z)$, which are horizontal lines here. The general formula [LYg], p. 545, takes the form

$$h_{\mu_\lambda}(F) = \delta^{\mathrm{uu}} \log 2 + (\mathrm{HD}(\mu_\lambda) - \delta^{\mathrm{uu}}) \log 1/\lambda. \tag{20}$$

Here $\delta^{\mathrm{uu}} = \mathrm{HD}(\mu_\lambda^{\mathrm{uu}}(z))$ for μ_λ-a.e. z, $\log 2$ is the characteristic Lyapunov exponent, and $\delta^{\mathrm{uu}} \log 2$ is the conditional entropy relative to a partition subordinate to the foliation into $W^{\mathrm{uu}}(z)$. The factors of the second summand in (20) have the same meaning for the "transverse" dynamics. We compute from (20)

$$\delta^{\mathrm{uu}} = (1 - \mathrm{HD}(\mu_\lambda)\alpha)/(1-\alpha), \tag{21}$$

so by (19),

$$\delta^{\mathrm{uu}} = 1 - \alpha \mathrm{HD}(\nu_\lambda), \qquad \alpha = -\log\lambda/\log 2. \tag{22}$$

The conclusion from Corollary 3 is that for most parameters λ (most in the sense defined in Corollary 3) and for μ_λ-a.e. z, $\mathrm{HD}(\mu_\lambda^{\mathrm{uu}}(z)) = 1-\alpha$ and the "transverse Hausdorff dimension" is 1. However, for λ^{-1} a PV-number both dimensions are different. (Caution: "transverse" does not mean: on an invariant transverse foliation, which consists of vertical lines here. Indeed, graph f_λ intersects every vertical line at one point only.)

Let us finish the paper with the remark that the map $F(x, y) = (2x \pmod 1, \lambda^{-1}(y - \cos x))$ preserves the graph of the Weierstrass function $\mathscr{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos 2^n x$. If [LYg] is applicable (one has to consider the inverse limit; W_λ^{uu}, $\mu_\lambda^{\mathrm{uu}}$ depend on backward trajectories) we also get the formulas (20), (21), but there is no way to compare μ_λ with ν_λ. One should consider, instead of W_2 with ν_λ, some space "transverse" to the sets W_λ^{uu}.

The case to which [LYg] is applicable literally is a smooth map $F\colon T^k \times \boldsymbol{R} \to T^k \times \boldsymbol{R}$, with T^k a k-dimensional torus, given by $F(x, y) = (Ax, \lambda^{-1}(y - q(x))$ where A is a hyperbolic toral automorphism, $\lambda^{-1} > 1$ and less than $\max|\beta|$, β running over all eigenvalues of A, q smooth. Then graph f_λ for $f_\lambda = \sum_{n=0}^{\infty} \lambda^n q(A^n x)$ is an F-invariant repeller. For F^{-1} it is the attracting fractal torus discussed in [KMY]. (We suppose that q is such that f_λ is not C^1-smooth.) If A has only one eigenvalue β with $|\beta| > 1$, then we have an analog of (20), and for $\delta^{\mathrm{uu}} = \mathrm{HD}(\mu_\lambda^{\mathrm{uu}}(z))$ and $\delta^{\mathrm{u}} = \mathrm{HD}(\mu_\lambda^{\mathrm{u}}(z))$ for μ_λ-a.e. z, we get the formula

$$\delta^{\mathrm{uu}} = (1 - \delta^{\mathrm{u}}\alpha)/(1-\alpha) \quad \text{with} \quad \alpha = -\log\lambda/\log\beta$$

(μ_λ^{u} denotes the conditional measures of μ_λ on unstable manifolds).

In particular, we obtain the equivalence of the two conjectures: $\delta^{\mathrm{u}} = 2-\alpha$ and $\delta^{\mathrm{uu}} = 1-\alpha$.

Note (added after completing the paper). Only after having finished the paper we became acquainted with C. McMullen's significant paper [McM] and its relationship with our results.

A slight modification of his construction gives a continuous function $f\colon [0, 1] \to \boldsymbol{R}$ satisfying Hölder's conditions (1) in § 1 (1°, 2° in § 5) with an arbitrary exponent α of the form $\log m/\log n$ ($n > m > 1$ are integers), such that HD(graph f) is an arbitrary number between 1 and $2-\alpha$.

One defines graph f as $\bigcap_{k=1}^{\infty} P_k$ where the sets P_k are defined inductively. Consider a set R consisting of pairs of integers (i, j) with $0 \leq i < n$, $0 \leq j < m$ satisfying the following conditions: for every i there exists precisely one $j = j_i$ such that $(i, j) \in R$, $j_0 = 0$, $j_1 = 1$, $j_{n-2} = m-2$, $j_{n-1} = m-1$; finally, for every $i = 1, \ldots, n-3$, $j_i \leq j_{i+1} \leq j_{i+2}$ or $j_i \geq j_{i+1} \geq j_{i+2}$ and if $j_i = j_{i+1} = j$ then $j_{i-1} < j > j_{i+2}$ or $j_{i-1} > j < j_{i+2}$ (i.e. every extremum is double).

Define $R_1 = R$ and by induction the sets $R_k \subset \{0, 1, \ldots, n^k-1\} \times \{0, 1, \ldots, m^k-1\}$. Having defined R_k we say $(u, v) \in R_{k+1}$ iff

$$(u, v) = \Big(\sum_{r=0}^{k} x_{k-r} n^r, \sum_{r=0}^{k} y_{k-r} m^r\Big), \quad \text{where}$$

$$\Big(\sum_{r=0}^{k-1} x_{k-r-1} n^r, \sum_{r=0}^{k-1} y_{k-r-1} m^r\Big) = (s, t) \in R_k,$$

and *either* $(x_k, y_k) \in R$ in the case $t_{s-1} < t_s$ or $t_s < t_{s+1}$, *or* $(n-1-x_k, y_k) \in R$ in the case $t_{s-1} > t_s$ or $t_s > t_{s+1}$ (a_b denotes the only a such that $(b, a) \in R_k$. This uniqueness must be checked along the induction).

Now define

$$P_k = \bigcup_{(s,t)\in R_k} [sn^{-k}, (s+1)n^{-k}] \times [tm^{-k}, (t+1)m^{-k}].$$

By McMullen's theorem the Hausdorff dimension of the resulting graph $f = \bigcap_{k=1}^{\infty} P_k$ is precisely

$$\log_m\Big(\sum_{j=0}^{m-1} d_j^{\log m/\log n}\Big)$$

where d_j denotes the number of i such that $(i, j) \in R$. This gives a dense set of possible Hausdorff dimensions. To get an arbitrary Hausdorff dimension one should keep modifying the pattern (the set R) along the inductive construction.

McMullen's method provides also another proof of Theorem 8 and allows one to write a precise formula for HD(graph f_λ). Namely,

$$\mathrm{HD}(\mathrm{graph}\, f_\lambda) = \lim_{k\to\infty} \frac{G_k^{(\alpha)}}{k \log \lambda^{-1}}$$

where $G_k^{(\alpha)} = \log \sum_{j=1}^{\sigma_k} (d_j^{(k)})^\alpha$, $\alpha = -\log\lambda/\log 2$. The function $\sum_{t=0}^{k-1} \lambda^t r_t$ takes σ_k different values $a_{k,1}, \ldots, a_{k,\sigma_k}$ and $d_j^{(k)}$ denotes the number of intervals of the form $[s2^{-k}, (s+1)2^{-k}]$, $s = 0, \ldots, 2^k-1$, where $\sum_{t=0}^{k-1} \lambda^t r_t = a_{k,j}$.

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