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On the law of iterated logarithm for Bloch functions *

by

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Abstract. We present a proof of the law of iterated logarithm for Bloch holomorphic functions on the unit disc D by approximating the sequence of sums of trigonometric polynomials which are convolutions of a Bloch function with Fejér type kernels by a martingale on ∂D.

§ **1. Introduction.** A holomorphic function b on the unit disc $D \subset C$ is called a *Bloch function* if

(1.1) $$\|b\|_{\mathscr{B}} \equiv |b(0)|+\sup_{z\in D}(1-|z|^2)\,|b'(z)| < \infty .$$

Denote the class of all Bloch functions by $\mathscr{B}$.

The following theorem was recently proved by N.G. Makarov in [M].

THEOREM 1 (Makarov). *There exists a universal constant $C_M > 0$ such that if $b \in \mathscr{B}$ then*

(1.2) $$\limsup_{t\to 1-} |b(tz)|/\sqrt{\log\left(\frac{1}{1-t}\right)\log\log\log\left(\frac{1}{1-t}\right)} \leqslant C_M\,\|b\|_{\mathscr{B}}$$

for almost all $z \in \partial D$.

For every holomorphic univalent function f on D with $f'(0) = 1$, the function $\log f'$ is a Bloch function with $\|\log f'\|_{\mathscr{B}} \leqslant 6$ (see [H], L. 17.4.1). So (1.2) yields for almost every $z \in \partial D$

$$|f'(tz)| \leqslant \exp\left((6C_M+o(1))\sqrt{\log\left(\frac{1}{1-t}\right)\log\log\log\left(\frac{1}{1-t}\right)}\right) \quad \text{as } t\to 1- .$$

This provides information about the harmonic measures on the boundary of $f(D)$ (see [M]).

* This is a considerably revised version of the paper with the same title published as a preprint of the University of Warwick, January 1986.

The aim of this paper is to show that Theorem 1 is equivalent to the upper class part of the law of iterated logarithm (LIL) for partial sums of some weakly dependent random variables and to explain how a standard procedure [PhS] reduces it to the LIL for martingales [S]. By the way we obtain the estimate(¹)

$$C_M \leqslant 16/\sqrt{\log 2}.$$

Let us recall after Makarov that an important example of the Bloch function, easy to cope with, is the lacunary series $\sum_{n=0}^{\infty} z^{2^n}$. The random variables to be considered in this case are just $\operatorname{Re} z^{2^n}$, $\operatorname{Im} z^{2^n}$ on ∂D. (In fact, to improve the estimate of C_M the consideration of $\operatorname{Re}\sum_{n=0}^{m} \alpha z^{2^n}$ for an arbitrary α, $|\alpha| = 1$, is useful.)

We will base on Makarov's description of the Bloch class in terms of convolutions with the polynomial kernels W_n, $n \geqslant 0$, where $W_0(z) = 1+z$ and for $n > 0$, W_n is defined by

$$\hat{W}_n(2^n) = 1,$$

$$\hat{W}_n \equiv 0 \quad \text{outside } (2^{n-1}, 2^{n+1}),$$

$$\hat{W}_n \text{ is linear on } [2^{n-1}, 2^n] \text{ and on } [2^n, 2^{n+1}]$$

($\hat{f}(k)$ denotes the nth Fourier coefficient of the function f).

Makarov's characterization of the Bloch class is as follows: A holomorphic function b on D is a Bloch function if and only if

$$\|b\|'_{\mathscr{B}} \equiv \sup_{n \geqslant 0} \|b * W_n\|_\infty < \infty.$$

In the Appendix we shall prove the "only if" part and give the estimate

$$\sup_{b \in \mathscr{B}} \|b\|'_{\mathscr{B}}/\|b\|_{\mathscr{B}} \leqslant 8.$$

In several places we shall apply S. Bernstein's inequality (see [Z], Ch. X, Th. 3.13, 3.16): for every trigonometric polynomial $S(z) = \sum_{j=-n}^{n} c_n z^n$ on $S^1 = \partial D$, if $1 \leqslant p \leqslant \infty$ then

$$\|S'\|_p \leqslant n\|S\|_p.$$

We shall also use a kind of an opposite inequality in L^∞ following from [Z], Ch. V, Th. 1.5.

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(¹) Better estimates have appeared recently: $C_M \leqslant 2$, see [B], $C_M \leqslant 1$, see [Po]. On the other hand, $C_M > 0.685$, [Po], (2.17).

§ 2. Equivalence of LIL's. For a given $b \in \mathscr{B}$ set $b_n = b * W_n$. We shall consider the sequences of random variables $\operatorname{Re} b_n$ and $\operatorname{Im} b_n$ on ∂D. They are uniformly bounded and

$$\int_{\partial D} \operatorname{Re} b_n \, d\mu = \int_{\partial D} \operatorname{Im} b_n \, d\mu = 0 \quad \text{for } n > 0$$

(μ is the normalized length measure on ∂D).

Proposition 1. *For any Bloch function b, (1.2) is equivalent to the upper class part of LIL for the sequences $\operatorname{Re} b_n$ and $\operatorname{Im} b_n$; more exactly, for every $z \in \partial D$*

$$\limsup_{n\to\infty} \Big|\sum_{j=0}^{n} b_j(z)\Big| / \sqrt{n \log\log n} \tag{2.1}$$

$$= \sqrt{\log 2} \limsup_{t \to 1-} |b(tz)| \Big/ \sqrt{\log\left(\frac{1}{1-t}\right) \log\log\log\left(\frac{1}{1-t}\right)}.$$

Proof. We shall estimate the quantity

$$\Delta_n(tz) = \sum_{j=0}^{n} b_j(z) - \sum_{j=0}^{\infty} b_j(tz)$$

for $t_{n-1} \leqslant t \leqslant t_n$ where $t_n = 2^{-(2^{-n})}$, $z \in \partial D$, n large enough.

Assume that $\|b\|'_{\mathscr{B}} \leqslant 1$. We have $b_j = z^{2^{j-1}} \tilde{b}_j$ for some polynomials $\tilde{b}_j$. In consequence $\|\tilde{b}_j\|_\infty = \|b_j\|_\infty \leqslant 1$ on ∂D, hence by the maximum principle $\|\tilde{b}_j\|_\infty \leqslant 1$ on D, so $|b_j(tz)| \leqslant t^{2^{j-1}}$. We obtain for $t \leqslant t_n$

$$\Big|\sum_{j=n+1}^{\infty} b_j(tz)\Big| \leqslant \sum_{j=n}^{\infty} 2^{-(2^{-n}2^j)} < 1.$$

Now let us estimate $\left|\sum_{j=0}^{n} (b_j(z) - b_j(tz))\right|$. By S. Bernstein's inequality and the maximum principle we have

$$\|b_j'\|_\infty \leqslant 2^{j+1} \|b_j\|_\infty \leqslant 2^{j+1} \quad \text{on } D.$$

So for $t \geqslant t_{n-1}$

$$\Big|\sum_{j=0}^{n} (b_j(z) - b_j(tz))\Big| \leqslant |1-t| \sum_{j=0}^{n} 2^{j+1} < 2^{-n+1} 2^{n+2} = 8.$$

Therefore

$$\Delta_n(tz) < 9 \quad \text{for } t_{n-1} \leqslant t \leqslant t_n.$$

Using for such t the estimate $2^{-n-1} < 1-t < 2^{-n+1}$ we easily obtain (2.1). ∎

§ 3. LIL for $\operatorname{Re} b_n$ $(\operatorname{Im} b_n)$. We shall follow the way in which Philipp and Stout coped with the lacunary trigonometric series (in [PhS]) approximating a subsequence of the sequence of partial sums by a martingale and estimating from above the conditional expectations of squares of the martingale difference sequence.

For a Bloch function b with $\|b\|'_{\mathscr{B}} \leqslant 1$ consider the sequence of random variables

$$\xi_k = \operatorname{Re} b_{3k} \quad (k > 0)$$

(similarly one considers the sequences $\operatorname{Re} b_{3k+1}$, $\operatorname{Re} b_{3k+2}$). Let $\mathscr{F}_k$ be the σ-field on $S^1 = \partial D$ generated by the arcs

$$U_{\nu,k} = \{\exp 2\pi i\omega\colon\ \omega \in [\nu 2^{-r(k)}, (\nu+1)2^{-r(k)})\}$$

for $\nu = 0, 1, \ldots, 2^{r(k)}-1$ where $r(k) = 3k+2+2\log k/\log 2$.

For every l with $0 \leqslant l < k$, S. Bernstein's inequality yields the following:

$$\|\xi_{k-l} - E(\xi_{k-l} | \mathscr{F}_k)\|_\infty \leqslant 2\pi\mu(U_{\nu,k}) \|\xi'_{k-l}\|_\infty \leqslant 2\pi 2^{-r(k)} 2^{3(k-l)-1} = \pi k^{-2} 2^{-3l}. \tag{3.1}$$

Now we shall estimate $\|E(\xi_m | \mathscr{F}_n)\|_\infty$ (to apply it for m much exceeding n). Clearly

$$\|E(\xi_m | \mathscr{F}_n)\|_\infty \leqslant \max_\nu \big(2\pi\mu(U_{\nu,n})\big)^{-1} \int_{U_{\nu,n}} \xi_m(\omega)\, d(2\pi\mu)(\tau) \leqslant (2\pi)^{-1} 2^{r(n)} \cdot 2\|\tilde{\xi}_m\|_\infty.$$

We consider here the function $\tilde{\xi}_m$ on ∂D such that $d\tilde{\xi}_m/d\tau = \xi_m$ (the real derivative in the direction tangent to ∂D). If

$$\xi_m = \operatorname{Re} b_{3m} = \operatorname{Re}\Big(\sum_j c_{j,3m} z^j\Big) = \sum_j \operatorname{Re} c_{j,3m} \cos 2\pi jx - \sum_j \operatorname{Im} c_{j,3m} \sin 2\pi jx,$$

for $z = e^{2\pi i x}$, we take

$$\tilde{\xi}_m(e^{2\pi i x}) = \sum_j j^{-1} \operatorname{Re} c_{j,3m} \sin 2\pi jx + \sum_j j^{-1} \operatorname{Im} c_{j,3m} \cos 2\pi jx = \operatorname{Im}\Big(\sum_j j^{-1} c_{j,3m} z^j\Big).$$

Denote $\sum_j j^{-1} c_{j,3m} z^j$ by $\tilde{b}_{3m}$. As this is a polynomial vanishing to order $t = 2^{3m-1}$ at 0, we have

$$\|\tilde{b}_{3m}\|_\infty \leqslant 2^{-3m+1} \|b_{3m}\|_\infty. \tag{3.2}$$

This follows from the fact that $\tilde{b}_{3m} = b_{3m} * (z^t g)$ where $g = \sum_{k=-\infty}^{+\infty} (t+|k|)^{-1} z^k$. As the sequence $(t+k)^{-1}$, $k = 0, 1, \ldots$, is convex, by [Z], Ch. V, Th. 1.5, we have $g \in L^1$, $g \geqslant 0$ and $\|g\|_1 = g(0) = t^{-1}$.

We conclude that

$$\|E(\xi_m | \mathscr{F}_n)\|_\infty \leqslant (2\pi)^{-1} 2^{r(n)} \cdot 2 \cdot 2^{-3m+1} = 2^4 (2\pi)^{-1} \cdot 2^{-3(m-n-2(\log n)/(3\log 2))}. \tag{3.3}$$

We shall now define (analogously to [PhS]) random variables y_n, z_n which are sums of progressively longer blocks of the ξ_ν's. Define the blocks of positive integers I_j inductively by requiring that I_j contains $[j^\alpha]$ consecutive integers and there are no gaps between consecutive blocks (for some small $\alpha > 0$). Write

$$y_n = \sum_{\nu \in I_{2n}} \xi_\nu, \qquad z_n = \sum_{\nu \in I_{2n+1}} \xi_\nu.$$

We shall concentrate on the y_n's; the procedure for the z_n's is similar. Write

$$Y_n = E(y_n | \mathscr{F}_{\nu_n}) - E(y_n | \mathscr{F}_{\nu_{n-1}}), \quad \text{where}$$
$$\nu_n = \max\{\nu\colon\ \nu \in I_{2n}\}.$$

It is clear that the sequence $(\sum_{j=1}^n Y_j, \mathscr{F}_{\nu_n})_{n=1}^\infty$ forms a martingale.

In view of (3.3) and (3.1) we have the estimate

$$\begin{aligned}\|Y_n - y_n\|_\infty &\leqslant \|E(y_n | \mathscr{F}_{\nu_{n-1}})\|_\infty + \|E(y_n | \mathscr{F}_{\nu_n}) - y_n\|_\infty \\ &\leqslant \sum_{\nu \in I_{2n}} \|E(\xi_\nu | \mathscr{F}_{\nu_{n-1}})\|_\infty + \sum_{\nu \in I_{2n}} \|E(\xi_\nu | \mathscr{F}_{\nu_n}) - \xi_\nu\|_\infty \\ &\leqslant \sum_{\nu \in I_{2n}} 2^4 (2\pi)^{-1} 2^{-3(\nu - \nu_{n-1} - 2(\log \nu_{n-1})/(3\log 2))} \\ &\quad + \pi \nu_n^{-2} 2^{-(\nu_n - \nu)} \leqslant 2^{-n^\alpha} + 2\pi n^{-2}, \quad \text{for } n \text{ large enough.}\end{aligned}$$

So the series $\sum_{n=1}^\infty |Y_n - y_n|$ is convergent in L^∞.

To check the upper class part of LIL we shall make use of the following

THEOREM 2 (Stout [S]). *Let* $(\sum_{j=1}^n Y_j, \mathscr{E}_n)_{n=1}^\infty$ *be a martingale,* $E(Y_j) = 0$. *Let*

$$s_n^2 = \sum_{j=1}^n E(Y_j^2 | \mathscr{E}_{j-1}), \qquad t_n = \sqrt{2\log\log s_n^2} \quad \textit{for } n \geqslant 1.$$

Suppose

(i) $s_n^2 \to \infty$ *almost surely, and*
(ii) $Y_n \leqslant K_n s_n / t_n$ *a.s. for every* $n \geqslant 1$ *and some positive numbers* $K_n \to 0$.

Then

$$\limsup_{n\to\infty}\sum_{j=1}^{n} Y_j/(s_n t_n) \leqslant 1 \quad a.s.$$

Let us go back to our sequences Y_n, $\mathscr{F}_{v_n}$. For every positive integer M, set $N_M = \sum_{n=1}^{M} \operatorname{Card} I_{2n}$. We shall prove that

$$s_M^2 \leqslant N_M + o(N_M) \quad \text{a.s. as } M\to\infty. \tag{3.4}$$

It is clear that for every j, $E(Y_j^2 \mid \mathscr{F}_{v_{j-1}}) \leqslant E(y_j^2 \mid \mathscr{F}_{v_{j-1}})$. We write now

$$E(y_j^2 \mid \mathscr{F}_{v_{j-1}}) = A_j + B_j, \quad \text{where}$$

$$A_j = 2\sum_{\substack{v,v'\in I_{2j}\\ v<v'}} E(\xi_v \xi_{v'} \mid \mathscr{F}_{v_{j-1}}),$$

$$B_j = \sum_{v\in I_{2j}} E(\xi_v^2 \mid \mathscr{F}_{v_{j-1}}).$$

Let us consider an arbitrary A_j. On ∂D

$$\begin{aligned}\xi_v\xi_{v'} &= \operatorname{Re} b_{3v} \operatorname{Re} b_{3v'}\\ &= \tfrac{1}{4}(b_{3v}b_{3v'} + b_{3v}\bar{b}_{3v'} + \bar{b}_{3v}b_{3v'} + \bar{b}_{3v}\bar{b}_{3v'})\\ &= \tfrac{1}{4}(W_1+W_2+W_3+W_4).\end{aligned}$$

W_1 and $\bar{W}_4$ are polynomials vanishing to order $2^{3v-1}+2^{3v'-1}$, $\bar{W}_2$ and W_3 are polynomials (provided we replace $\bar{z}$ by z^{-1}) vanishing to order $2^{3v'-1} - 2^{3v+1} \geqslant 2^{3\min I_{2j}}$.

To estimate $E(\xi_v\xi_{v'} \mid \mathscr{F}_{v_{j-1}})$ we use [Z], Ch. V, Th. 1.5, for every summand W_i or its conjugate, similarly to the proof of (3.3). We obtain

$$\|E(\xi_v\xi_{v'} \mid \mathscr{F}_{v_{j-1}})\|_\infty \leqslant (2\pi)^{-1} 2^{r(j-1)} \cdot 2 \cdot 2^{-3\min I_{2j}} \leqslant 2^{-j^\alpha}.$$

So $A_j \leqslant 2j^{2\alpha} 2^{-j^\alpha}$, hence the series $\sum_{j=1}^{\infty} A_j$ is convergent.

Finally,

$$\|B_j\|_\infty \leqslant \sum_{v\in I_{2j}} \|\xi_v^2\|_\infty \leqslant (2j)^\alpha.$$

This proves the estimate (3.4).

The assertion of Theorem 2 yields

$$\limsup_{M\to\infty}\sum_{j=1}^{M} Y_j/\sqrt{2N_M \log\log N_M} \leqslant 1 \quad \text{a.s.} \tag{3.5}$$

One need not bother about assumptions (i), (ii): if necessary, just consider the random variables $Y_j + (2j)^\alpha \zeta_j$, where (ζ_j) is a Bernoulli process (independent of all Y_n) with an arbitrarily small variance. Then (i) and (ii) are satisfied.

By the convergence of $\sum |Y_n - y_n|$ in L^∞ we can replace (3.5) by

$$\limsup_{M\to\infty}\sum_{k} (\operatorname{Re} b_{3k})/\sqrt{2N_M \log\log N_M} \leqslant 1 \quad \text{a.s.}, \tag{3.6}$$

where the summation is over $k \in \bigcup_{j=1}^{M} I_{2j}$.

In fact, we can consider lim sup here over all N. This is so because the blocks I_{2j} are short, so breaking into them does not change the estimate.

Applying the whole procedure to $z_n = \sum_{v\in I_{2n+1}} \xi_v$ and to the sequences $\operatorname{Re} b_{3k+1}$, $\operatorname{Re} b_{3k+2}$ we obtain the LIL estimate from above for the sequence $\operatorname{Re} b_n$.

§ 4. **Estimate for C_M, other remarks and questions.** (a) We can now estimate the Makarov universal constant C_M (see § 1).

First observe that in the division of the sequence $\operatorname{Re} b_{3k}$ (similarly $\operatorname{Re} b_{3k+\tau}$, $\tau = 1, 2$) into blocks y_j, z_j we could assume that each z_j is short, say $\operatorname{Card} I_{2j} = j^\varepsilon$, $\operatorname{Card} I_{2j+1} = j^{\varepsilon/K}$, K arbitrarily large. So N_M's, in the analog of (3.6) for (z_n), are small compared with $\max I_{2M+1}$, hence z_n is negligible in the estimates. We could also divide b_n into only two sequences b_{2k} and b_{2k+1} if instead of $b_n = b * W_n$ we considered the convolutions with the modified kernels:

$$\hat{W}_n' \equiv 1 \quad \text{on } [2^n - 2^{(n-1)(1-\varepsilon)},\ 2^n + 2^{n(1-\varepsilon)}],$$

$$\hat{W}_n' \equiv 0 \quad \text{outside } (2^{n-1} + 2^{(n-1)(1-\varepsilon)},\ 2^{n+1} - 2^{n(1-\varepsilon)}),$$

$\hat{W}_n'$ is linear on the complementary intervals (including their ends).

So

$$\limsup_{n\to\infty}\sum_{j=0}^{n} (\operatorname{Re} b_j)/\sqrt{2n\log\log n} \leqslant \sqrt{2} \quad \text{a.s.}$$

We could prove the same estimate for the sequence $\sum_{j=0}^{n} \operatorname{Re}(\alpha b_j)$ for any α with $|\alpha| = 1$. We choose a countable set of α's, dense in ∂D, and then the estimate holds almost surely for all α's. We conclude that

$$\limsup_{n\to\infty}\Big|\sum_{j=0}^{n} b_j\Big|/\sqrt{n\log\log n} \leqslant 2 \quad \text{a.s. on } \partial D$$

provided $\|b\|_{\mathscr{B}}' \leqslant 1$.

Hence, because of the estimate from the Appendix and because of Proposition 1

$$C_M \leqslant 16/\sqrt{\log 2}.$$

(b) The question arises whether the assertion of Theorem 1 holds for a holomorphic function b on D with the sequence $\|b * W_n\|_p$ bounded (for $p \neq \infty$, p sufficiently large). To get (3.3) and the estimate of A_j in L^p one can use the Strong Marcinkiewicz Multiplier Theorem (see e.g. [EG]). To get almost sure estimates one makes use of the Borel–Cantelli Lemma (see the Warwick preprint version of the paper). So the unique place where our proof does not go through for an arbitrary (large) p is the estimate for B_j. The question is: does a strong law of large numbers for the sequence (B_j) hold?

(c) An easy case of Theorem 1 is the case of the lacunary series $l(z) = \sum_{n \geq 0} z^{2^n}$. Then $\operatorname{Re} l_j(z) = \varphi(g^j(z))$, where $g^j = g \circ \ldots \circ g$ (j times), $g(z) = z^2$ and $\varphi(z) = \operatorname{Re} z$. So the sequence of random variables $\operatorname{Re} l_j$ on D is stationary; one can refer to a more classical version of LIL.

In [PUZ] we considered the case of a univalent function R on $\boldsymbol{D}$ such that $f = R \circ g \circ R^{-1}$ extends holomorphically beyond $\partial R(\boldsymbol{D})$. Then for $z \in \boldsymbol{D}$

$$\log R'(z) = \log R'(0) + \lim_{n \to \infty} \log\left((g^n)'(z)/(f^n)'(R(z))\right)$$

$$= \log R'(0) + \sum_{n=0}^{\infty} \log\left(g'(g^n(z))/f'(R \circ g^n(z))\right).$$

This is a Bloch function of the form of a series

$$\sum_{n=0}^{\infty} \varphi(z^{2^n}) \tag{4.1}$$

for $\varphi = \log(g'/f' \circ R)$. (In [PUZ] we consider LIL for the partial sums $\sum_{j=0}^{n} \operatorname{Re} \tilde{\varphi}(z^{2^j})$ on $\partial \boldsymbol{D}$ rather than $\sum_{j=0}^{n} \operatorname{Re} b_j$ where $\tilde{\varphi}$ denotes the radial limit of φ a.e.)

Is it possible to characterize the Bloch functions of the form (4.1) (i.e. the univalent mappings R with f extending holomorphically beyond $\partial R(\boldsymbol{D})$)?

(d) One would like to be able to decide for any individual $b \in \mathscr{B}$ whether

$$\limsup_{t \to 1-} |b(tz)| / \sqrt{\log\left(\frac{1}{1-t}\right) \log\log\log\left(\frac{1}{1-t}\right)} = 0 \quad \text{a.s.} \tag{4.2}$$

or not.

It is not hard to see that if $b \in \mathscr{B}_0$, i.e. if

$$\lim_{|z| \to 1-} (1-|z|^2)|b'(z)| \to 0,$$

then $\|b * W_n\|_\infty \to 0$, so (4.2) holds.

On the other hand, if we set $b = \sum_{n \geq 0} a_n z^n$ I conjecture that if

$$\limsup_{n \to \infty} \Big(\sum_{j=1}^{n} a_j^2\Big) / \log n > 0$$

then (4.2) does not hold.

In the case where $f = R \circ g \circ R^{-1}$ extends holomorphically beyond $R(\boldsymbol{D})$ a dichotomy happens: Either (4.2) is not true or $\partial R(\boldsymbol{D})$ is a real-analytic curve (see [PUZ] and [Zd]).

Appendix. We shall estimate $\sup_{b \in \mathscr{B}} \|b\|'_{\mathscr{B}} / \|b\|_{\mathscr{B}}$. To this end it is enough to estimate from above, for every $n \geq 0$,

$$\|W_n\|_{\mathscr{I}} \equiv \int_0^1 \Big(\int_{\partial \boldsymbol{D}} |W_n'(tz)|\, d\mu(z) \Big)\, dt$$

by a constant independent of n and use the inequality

$$|(b * W_n)(z)| \leq 2\|b\|_{\mathscr{B}} \|W_n\|_{\mathscr{I}} \quad \text{for every } z \text{ with } |z| \leq 1.$$

(This inequality is easily computable, see for e.g. [ACPo]). We have

$$W_n = z^{2^n} F_{2^{n-1}-1} + \tfrac{1}{2} z^{2^n + 2^{n-1}} F_{2^{n-1}-1},$$

where $F_m(z)$ denotes the mth Fejér kernel:

$$F_m(z) = \frac{1}{m+1} \sum_{j=0}^{m} \sum_{s=-j}^{j} z^s \quad \text{for } m \geq 0.$$

Set $A_n = z^{2^{n-1}-1} F_{2^{n-1}-1}$. By S. Bernstein's inequality

$$\int_{\partial \boldsymbol{D}} |A_n'|\, d\mu \leq (2^n - 2) \int_{\partial \boldsymbol{D}} |A_n|\, d\mu \leq 2^{n-2}$$

(since $\int_{\partial \boldsymbol{D}} F_m\, d\mu = 1$ for every $m \geq 0$ and $F_m \geq 0$ on $\partial \boldsymbol{D}$). By Hardy's Convexity Theorem (see [D]), for $0 \leq t \leq 1$

$$\int_{\partial \boldsymbol{D}} |A_n'(tz)|\, d\mu(z) \leq \int_{\partial \boldsymbol{D}} |A_n'(z)|\, d\mu(z),$$

$$\int_{\partial \boldsymbol{D}} |A_n(tz)|\, d\mu(z) \leq \int_{\partial \boldsymbol{D}} |A_n(z)|\, d\mu(z).$$

Since $W_n = (z^{2^{n-1}+1} + \frac{1}{2} z^{2^n+1}) A_n$, we obtain

$$\|W_n\|_{\mathscr{I}} \leq \tfrac{3}{2} \int_{\partial \boldsymbol{D}} |A_n|\, d\mu + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2(2^n+1)} \right) \int |A_n'|\, d\mu \leq \tfrac{3}{2} + \tfrac{5}{2} = 4.$$

The conclusion is that for every $b \in \mathscr{B}$

$$\|b\|'_{\mathscr{B}} / \|b\|_{\mathscr{B}} \leq 8.$$

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Added in proof (January 1989). N. Makarov has informed me that the conjecture stated in § 4(d) is false but the question remains open if $\limsup_{n\to\infty}$ is replaced by $\liminf_{n\to\infty}$.

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On the Hausdorff dimension of some fractal sets

by

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Dedicated to the memory of M. Irwin

Abstract. We describe a method of estimating from below the Hausdorff dimension of some fractal sets. These include compact connected subsets of tori with nondense orbit under a hyperbolic toral automorphism, graphs of Weierstrass nowhere differentiable functions, e.g. $\sum_{n=0}^{\infty}\lambda^n \sin 2^n x$, $1/2<\lambda<1$, and also graphs of $\sum_{n=0}^{\infty}\lambda^n r_n$, r_n the nth Rademacher function. On the other hand, we prove that for λ^{-1} a Pisot–Vijayaraghavan number, the latter graph has Hausdorff dimension less than $2-\log\lambda^{-1}/\log 2$.

1. Introduction. This paper concerns the Hausdorff dimension and limit capacity of three types of related fractal sets. Our estimates of Hausdorff dimension from below rely on a fact formulated in § 2 as Lemma 1. Here it is as applied to the plane $\boldsymbol{R}^2$.

LEMMA 0. *Let K be a Borel subset of the x, y plane $\boldsymbol{R}^2$ whose projection to the x axis has positive 1-dimensional Lebesgue measure. Assume that there exist constants $C_1, C_2>0$, $0<\alpha<1$ such that for every horizontal interval $[x_1, x_2]\times\{y\}$ there exist a_1, a_2 with $x_1\leqslant a_1<a_2\leqslant x_2$ such that $a_2-a_1=C_1(x_2-x_1)$ and the rectangle*

$$[a_1, a_2]\times[y-\tfrac{1}{2}C_2(x_2-x_1)^\alpha,\ y+\tfrac{1}{2}C_2(x_2-x_1)^\alpha]$$

is disjoint from K. Then the Hausdorff dimension $\mathrm{HD}(K)$ satisfies

$$\mathrm{HD}(K)\geqslant C(\alpha, C_1)>1 \tag{0}$$

where $C(\alpha, C_1)$ is a constant depending only on α and C_1.

We recall some definitions: For a metric space (X,ϱ), $A\subset X$, $r>0$ we denote by $N(A,r)$ the minimum number of balls in X with radii $\leqslant r$, needed to cover A. The *lower* and *upper capacities* of A are defined as

$$\underline{\mathrm{Cap}}\,A=\liminf_{r\to 0}\frac{\log N(A,r)}{-\log r},\qquad \overline{\mathrm{Cap}}\,A=\limsup_{r\to 0}\frac{\log N(A,r)}{-\log r}.$$