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Some properties of weakly countably determined Banach spaces

by

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Abstract. Let Y be a closed subspace of a Banach space X. If Y^{**}/Y is separable and X/Y is weakly compactly generated, then X is also weakly compactly generated. Analogous results are obtained with "weakly compactly generated" replaced by "weakly K-analytic" and also by "weakly countably determined".

The vector spaces we use here are over the field of real or complex numbers. N denotes the set of natural numbers. Our notations are standard. If (x_n) is a sequence in X, $[x_n]$ will stand for the closed linear hull of (x_n). Given a subset A of X, $\tilde{A}$ will denote its weak-star closure in X^{**}; if A is absolutely convex, i.e. convex and circled, and also closed and bounded, we shall write X_A for the Banach space on the linear hull of A with A as its closed unit ball. Given x in X and u in X^*, we shall write $\langle x, u\rangle$ instead of $u(x)$. If P is a continuous projection on X, P^* denotes the conjugate projection on X^*.

A Banach space X is said to be *weakly compactly generated* whenever there exists a weakly compact set $K \subset X$ such that the linear span of K is dense in X. In particular, every separable or reflexive Banach space is weakly compactly generated.

A Banach space X is said to be *weakly K-analytic* (respectively, *weakly countably determined*) whenever there exists a Polish topological space (respectively, a metrizable and separable topological space) F and a mapping T from F into the family of weakly compact subsets of X such that

$$X = \bigcup \{Tu:\ u \in F\}$$

with the following property: whenever (x_n) is a sequence in F converging to x_0 and U a weakly open neighbourhood of Tx_0, there exists a positive integer n_0 such that $Tx_n \in U$, $n \geq n_0$.

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Every weakly compactly generated Banach space is weakly K-analytic (a proof of this fact can be found in [8], where the concept of K-analyticity used is equivalent to the one we have just introduced; see [2], [4] and [6]). Every weakly K-analytic Banach space is, obviously, weakly countably determined.

A Banach space X is weakly K-analytic if there exists a mapping S from N^N into the family of absolutely convex and weakly compact subsets of X such that

$$X = \bigcup \{Su: u \in N^N\}$$

with the following property: whenever $a = (a_n)$ and $b = (b_n)$ are elements of N^N such that $a_n \leqslant b_n$, $n = 1, 2, \ldots$, $Sa \subset Sb$ (this result can be found in [9] in the case of a Banach space and in [1] in the case of certain locally convex spaces).

A Banach space X is weakly countably determined if there is a sequence (A_n) of absolutely convex, closed and bounded subsets of X such that whenever x belongs to X, there exists a subsequence (A_{n_j}) of (A_n) such that $\bigcap_{j=1}^{\infty} A_{n_j}$ is weakly compact and contains $\{x\}$ (see [12]).

The finite products and the separated quotients of weakly compactly generated (respectively, weakly K-analytic, weakly countably determined) Banach spaces are in the same class. In [7], an example is provided of a weakly compactly generated Banach space with a closed subspace which is not weakly compactly generated.

In the sequel we shall need the following results, which can be found in [10], [12], [3] and [11], respectively:

(a) *Let X be a Banach space such that X^{**}/X is separable. Then X is the topological direct sum of a separable Banach space and a reflexive one.*

(b) *Let X be a weakly countably determined Banach space. Let C and D be two countable subsets of X and X^*, respectively. Then there exists a continuous projection P on X such that $P(X)$ is separable, $C \subset P(X)$ and $D \subset P^*(X^*)$.*

(c) *Let A be a weakly compact subset of a Banach space X. Then there exists an absolutely convex and weakly compact subset M of X such that $A \subset M$ and X_M is reflexive.*

(d) *If Y is a closed subspace of a Banach space X, $X + \tilde{Y}$ is a Banach space (with the norm induced by X^{**}).*

THEOREM 1. *Let Y be a closed subspace of a Banach space X. If Y^{**}/Y is separable and X/Y is weakly compactly generated, X is also weakly compactly generated.*

Proof. Let B be the closed unit ball of X. Let A be an absolutely convex and weakly compact subset of X/Y whose linear hull is dense in X/Y. According to result (c), A can be chosen in such a way that $(X/Y)_A$ is reflexive. Let φ be the canonical mapping from X onto X/Y. We shall write D instead of $B \cap \varphi^{-1}(A)$. Let ψ be the canonical injection from X_D into X.

We can consider Y as a subspace of X_D. Then X_D/Y is isomorphic to the reflexive Banach space $(X/Y)_A$, hence $X_D + \tilde{Y} = (X_D)^{**}$. It is easy to prove that $\tilde{Y}/Y$ is isomorphic to Y^{**}/Y, hence $(X_D)^{**}/X_D$ is separable, being isomorphic to $\tilde{Y}/Y$. It follows from result (a) that X_D is weakly compactly generated. Let M be an absolutely convex and weakly compact subset of X_D whose linear hull is dense in this space. Then $\psi(M)$ is an absolutely convex and weakly compact subset of X whose linear hull is dense in X. ∎

THEOREM 2. *Let Y be a closed subspace of a Banach space X. Then, if X/Y is separable, there exists a separable Banach space Z such that X is isomorphic to a quotient of $Z \times Y$.*

Proof. Let φ be the canonical mapping from X onto X/Y. Let $\{u_n: n = 1, 2, \ldots\}$ be a dense subset of the unit sphere of X/Y. For every $n \in N$ we can choose an element $x_n \in X$ such that

$$\varphi(x_n) = u_n, \quad \|x_n\| < 2.$$

Let B be the closed unit ball of $[x_n]$. We denote by ψ the restriction to $[x_n]$ of φ. Given an arbitrary element v of X/Y such that $\|v\| \leqslant 1$ and a positive number ε, we can find a positive integer m and $0 \leqslant \lambda \leqslant 1$ such that

$$\|v - \lambda u_m\| = \|v - \psi(\lambda x_m)\| < \varepsilon,$$

hence the closure of $\psi(2B)$ in X/Y contains the unit ball of this space. Therefore ψ is a mapping from $[x_n]$ onto X/Y. It follows that $X = [x_n] + Y$. Writing Z for $[x_n]$ we can define

$$T(z, y) = z + y$$

for every (z, y) in $Z \times Y$. Then T is a continuous linear mapping from $Z \times Y$ onto X, hence X is isomorphic to $(Z \times Y)/T^{-1}(0)$. ∎

The following three corollaries are easy consequences of Theorem 2:

COROLLARY 1.2. *Let Y be a closed subspace of a Banach space X. Then, if Y is weakly countably determined and X/Y is separable, X is also weakly countably determined.*

COROLLARY 2.2. *Let Y be a closed subspace of a Banach space X. Then, if Y is weakly K-analytic and X/Y is separable, X is also weakly K-analytic.*

COROLLARY 3.2 [5]. *Let Y be a closed subspace of a Banach space X. Then, if Y is weakly compactly generated and X/Y is separable, X is also weakly compactly generated.*

THEOREM 3. *Let Y be a closed subspace of a Banach space X. Then, if Y^* is separable and X/Y is weakly countably determined, there exists a closed subspace Z of X with the following properties*:

(1) *Z contains Y.*

(2) *X/Z is separable.*

(3) *There exists a Banach space M which is a topological complement of $\tilde{Y}$ in $Z+\tilde{Y}$.*

Proof. Let $Y^\perp$ be the subspace of X^* orthogonal to Y. As usual, we shall identify $Y^\perp$ and $X^*/Y^\perp$ with the Banach space conjugate to X/Y and Y, respectively. $X^*/Y^\perp$ is separable, hence, arguing as in the proof of Theorem 2, there exists a separable closed subspace L of X^* such that $X^* = L+Y^\perp$. Since $L\cap Y^\perp$ is a separable Banach space we can use result (b) to get a continuous projection P in X/Y such that $P(X/Y)$ is separable and $P^*(Y^\perp)$ contains $L\cap Y^\perp$. Let Z be the subspace of X orthogonal to $P^*(Y^\perp)$. Let us prove that Z is the desired subspace:

$P^*(Y^\perp)$ is contained in $Y^\perp$. Hence property (1) is obvious.

Let now u be an arbitrary element of $P(X/Y)$. Let v be an element in X such that $\varphi(v)=u$, φ the canonical mapping from X onto X/Y. If ψ denotes the canonical mapping from X onto X/Z, let $Tu=\psi(v)$. We then have

$$\langle u, z\rangle = 0, \quad z\in P^{*-1}(0),$$

hence, for $u\neq 0$, there exists w in $P^*(Y^\perp)$ such that $\langle u, w\rangle \neq 0$. Then

$$\langle v, w\rangle = \langle \varphi(v), w\rangle = \langle u, w\rangle \neq 0,$$

which implies that v is not in Z, hence Tu is different from zero. It is plain that Tu does not change when v varies in $\varphi^{-1}(u)$. From these remarks it follows that T is an injective linear mapping from $P(X/Y)$ into X/Z, its continuity being easy to prove. Let us now choose an arbitrary element x of X/Z. Write x_1 for an element in X such that $\psi(x_1)=x$, and

$$t = (P\circ\varphi)x_1.$$

Then $Tt = x$, hence T is an isomorphism from $P(X/Y)$ onto X/Z. Thus (2) has been established.

In order to prove (3), recall that $Z+\tilde{Y}$ is a Banach space, according to result (d). Let M be $L^\perp\cap(Z+\tilde{Y})$, $L^\perp$ being the subspace of X^{**} orthogonal to L. It is enough to prove that M and $\tilde{Y}$ form an algebraic decomposition of $Z+\tilde{Y}$. Obviously, $M\cap\tilde{Y}=\{0\}$. Let now s be an arbitrary element of $Z+\tilde{Y}$. Let m be the linear form on X^* defined by $m(u)=s(w)$ where $u=v+w$, $v\in L$, $w\in Y^\perp$, whenever $u\in X^*$. Obviously the definition is consistent. Recalling that $L+Y^\perp = X^*$, m is easily seen to be continuous on $(X^*, \|\cdot\|)$. Hence $m\in M$ and $\tilde{y}=s-m\in\tilde{Y}$. Then $s=m+\tilde{y}\in M+\tilde{Y}$. ∎

COROLLARY 1.3. *Let Y be a closed subspace of a Banach space X. Then, if Y^{**} is separable and X/Y is weakly countably determined, X is also weakly countably determined.*

Proof. Theorem 3 allows us to choose a closed subspace Z of X with properties (1)–(3) stated there. Z/Y is isomorphic to a closed subspace of X/Y, hence weakly countably determined. Using the notation of the proof of Theorem 3, we find that $(Z+\tilde{Y})/\tilde{Y}$ is isomorphic to M and also to Z/Y, hence M is weakly countably determined. Since Y^{**} is separable and $\tilde{Y}$ is isomorphic to Y^{**}, $M+\tilde{Y}=Z+\tilde{Y}$ is weakly countably determined. The Banach space Z is a subspace of $Z+\tilde{Y}$, hence Z is also weakly countably determined. Finally, X/Z is separable. Corollary 1.2 gives the desired conclusion. ∎

COROLLARY 2.3. *Let Y be a closed subspace of a Banach space X. Then, if Y^{**} is separable and X/Y is weakly K-analytic, X is weakly K-analytic.*

Proof. The proof goes along the lines of the proof of Corollary 1.3 with "weakly countably determined" replaced by "weakly K-analytic" and with Corollary 2.2 used instead of Corollary 1.2. ∎

LEMMA. *Let Y be a reflexive subspace of a Banach space X. Let φ be the canonical mapping from X onto X/Y. Then, if A is a bounded subset of X such that $\varphi(A)$ is weakly relatively compact, A is also weakly relatively compact.*

Proof. Let M be a weakly compact absolutely convex subset of X/Y containing $\varphi(A)$ and such that $(X/Y)_M$ is reflexive. Let B be the closed unit ball of X. We can find a positive integer m such that $A\subset mB$. Put

$$D = \varphi^{-1}(M)\cap mB.$$

X_D is a Banach space and Y is a reflexive subspace of X_D. Since X_D/Y is isomorphic to $(X/Y)_M$, X_D is reflexive, hence D is weakly relatively compact in X_D, thus weakly relatively compact in X. Finally, D contains A, so we get the conclusion. ∎

Remark. A method analogous to the one used in the proof of the last lemma gives the following more general result: *Let Y be a closed subspace of a Banach space X. Let φ be the canonical mapping from X onto X/Y. Then, if A is a bounded subset of X such that $\varphi(A)$ is weakly relatively compact, $\tilde{A}$ is contained in $X+\tilde{Y}$.*

PROPOSITION 1. *Let Y be a reflexive subspace of a Banach space X. Then, if X/Y is weakly K-analytic, X is also weakly K-analytic.*

Proof. Let φ be the canonical mapping from X onto X/Y. There exists a mapping T from N^N into the family of weakly compact absolutely convex subsets of X/Y such that

$$X/Y = \bigcup \{Tu:\ u \in N^N\}$$

and whenever $a = (a_n)$ and $b = (b_n)$ are elements of N^N such that $a_n \leqslant b_n$, $n = 1, 2, \ldots$, we have $Ta \subset Tb$.

Let B be the closed unit ball of X. If $a = (a_n)$ belongs to N^N, write $a' = (a_n)_{n=2}^{\infty}$ and let

$$Sa = \varphi^{-1}(Ta') \cap a_1 B.$$

Using the Lemma we see that Sa is a weakly compact absolutely convex subset of X. Given $b = (b_n)$ in N^N such that $a_n \leqslant b_n$, $n = 1, 2, \ldots$, we have

$$X = \bigcup \{Su:\ u \in N^N\}, \quad Sa \subset Sb.$$

Thus X is weakly K-analytic. ∎

PROPOSITION 2. *Let Y be a reflexive subspace of a Banach space X. Then, if X/Y is weakly countably determined, X is also weakly countably determined.*

Proof. Let φ be the canonical mapping from X onto X/Y. Let (M_n) be a sequence of bounded, closed and absolutely convex subsets of X/Y such that whenever z is an element of X/Y, there is a subsequence (M_{n_j}) of (M_n) such that $\bigcap_{j=1}^{\infty} M_{n_j}$ is weakly compact and contains $\{z\}$. Let B be the closed unit ball of X. The double sequence

$$(\varphi^{-1}(M_p) \cap qB)_{p,q=1}^{\infty}$$

can be arranged in a sequence (P_n).

Let now x be an arbitrary element of X. We can choose a positive integer p such that x is in pB as well as a subsequence (M_{m_j}) of (M_n) such that $\bigcap_{j=1}^{\infty} M_{m_j}$ is a weakly compact set containing $\{\varphi(x)\}$. Let (P_{n_j}) be a subsequence of (P_n) consisting exactly of the elements $\{\varphi^{-1}(M_{m_j}) \cap pB:\ j = 1, 2, \ldots\}$. Then

$$x \in \bigcap_{j=1}^{\infty} P_{n_j}.$$

Moreover.

$$\varphi\Big(\bigcap_{j=1}^{\infty} P_{n_j}\Big) \subset \bigcap_{j=1}^{\infty} M_{m_j}$$

and, in view of the Lemma, $\bigcap_{j=1}^{\infty} P_{n_j}$ is weakly compact. Hence X is weakly countably determined. ∎

THEOREM 4. *Let Y be a closed subspace of a Banach space X. Then, if Y^{**}/Y is separable and X/Y is weakly K-analytic, X is also weakly K-analytic.*

Proof. Result (a) establishes the existence of two closed subspaces of Y, one reflexive, U, the other one separable, V, such that $U+V = Y$ and $U \cap V = \{0\}$. Let φ be the canonical mapping from X onto X/V. The space X/Y is isomorphic to $(X/V)/\varphi(U)$ and, since $\varphi(U)$ is a reflexive subspace of X/V, we can use Proposition 1 to deduce that X/V is weakly K-analytic. Finally, Y^{**}/Y is isomorphic to V^{**}/V, hence V^{**} is separable. Thus X is weakly K-analytic, in view of Corollary 2.3. ∎

THEOREM 5. *Let Y be a closed subspace of a Banach space X. Then, if Y^{**}/Y is separable and X/Y is weakly countably determined, X is also weakly countably determined.*

Proof. The proof goes along the lines of that of Theorem 4, with Proposition 2 and Corollary 1.3 used instead of Proposition 1 and Corollary 2.3, respectively. ∎

PROPOSITION 3. *Let Y be a closed subspace of a Banach space X. Then, if X/Y is separable and X is weakly countably determined, there exist two closed subspaces U and V of X such that*

$$U \cap V = \{0\}, \quad U+V = X, \quad V \subset Y,$$

U separable.

Proof. As in the proof of Theorem 2, we can find a separable closed subspace Z of X such that $Z+Y = X$. But Y is weakly countably determined and $Z \cap Y$ is separable, thus we can use result (b) to get a separable subspace Z_1 of Y which contains $Z \cap Y$ and with a topological complement V relative to Y. Denoting $Z+Z_1$ by U, U is closed. Obviously U and V satisfy the other required conditions. ∎

PROPOSITION 4 [5]. *Let Y be a closed subspace of a Banach space X. Then, if X/Y is separable and X is weakly compactly generated, Y is also weakly compactly generated.*

Proof. Since X is weakly countably determined, we can use Proposition 3 to get two closed subspaces U and V of X with the aforesaid properties. Then V is isomorphic to X/U, hence weakly compactly generated. Finally, $U \cap Y$ is separable and V is its topological complement in Y. Thus Y is weakly compactly generated. ∎

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On the law of iterated logarithm for Bloch functions*

by

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Abstract. We present a proof of the law of iterated logarithm for Bloch holomorphic functions on the unit disc D by approximating the sequence of sums of trigonometric polynomials which are convolutions of a Bloch function with Fejér type kernels by a martingale on ∂D.

§ 1. Introduction. A holomorphic function b on the unit disc $D \subset C$ is called a *Bloch function* if

$$\|b\|_{\mathcal{B}} \equiv |b(0)| + \sup_{z \in D} (1-|z|^2)|b'(z)| < \infty. \tag{1.1}$$

Denote the class of all Bloch functions by $\mathcal{B}$.

The following theorem was recently proved by N.G. Makarov in [M].

THEOREM 1 (Makarov). *There exists a universal constant $C_M > 0$ such that if $b \in \mathcal{B}$ then*

$$\limsup_{t \to 1-} |b(tz)| / \sqrt{\log\left(\frac{1}{1-t}\right) \log\log\log\left(\frac{1}{1-t}\right)} \leqslant C_M \|b\|_{\mathcal{B}} \tag{1.2}$$

for almost all $z \in \partial D$.

For every holomorphic univalent function f on D with $f'(0) = 1$, the function $\log f'$ is a Bloch function with $\|\log f'\|_{\mathcal{B}} \leqslant 6$ (see [H], L. 17.4.1). So (1.2) yields for almost every $z \in \partial D$

$$|f'(tz)| \leqslant \exp\left((6C_M + o(1)) \sqrt{\log\left(\frac{1}{1-t}\right) \log\log\log\left(\frac{1}{1-t}\right)}\right) \quad \text{as } t \to 1-.$$

This provides information about the harmonic measures on the boundary of $f(D)$ (see [M]).

* This is a considerably revised version of the paper with the same title published as a preprint of the University of Warwick, January 1986.