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# Functions in $L^\infty(G)$ and associated convolution operators

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**Abstract.** Let $G$ be a compact abelian group, let $F \in L^\infty(G)$ and let $T_F$ be the associated convolution operator $L^1(G) \to C(G)$. We relate the essential oscillation of $F$ and points of weak-norm continuity of $T_F$ or points of $w^*$-norm continuity of $T_F^*$, and get characterizations of $\mu$-Riemann integrable functions in $L^\infty(G)$. We also study geometrical properties of the image under $T_F$ of the positive face of $L^1(G)$, such as the Krein–Milman property and weak sequential precompactness.

**Introduction.** Recently many properties of operators from $L^1$ to a Banach space have been investigated (see [$T_1$], [GGMS] for example). We will be more particularly concerned here with Bochner representable operators, strongly regular operators, and operators which send the unit ball of $L^1$ into a weakly sequentially precompact set.

We restrict ourselves in this paper to convolution operators

$$T_F\colon L^1(G) \to C(G), \quad \varphi \rightsquigarrow F * \varphi,$$

where $G$ is a compact abelian group and $F$ belongs to $L^\infty(G)$. Our aim is to relate regularity properties of the function $F$ to properties of $T_F$.

Our interest in this specialization is twofold: on the one hand, the situation seems more transparent than the general setting, gives a link to classical notions on the regularity of functions and furnishes instructive examples; on the other hand, we obtain in this special setting some stronger results than in the general one.

Let us outline the organization of the paper: in part I we specify some definitions and notation and recall rather obvious facts. In part II we investigate points of weak-norm continuity for $T_F\colon P_a(G) \to C(G)$ (where $P_a(G)$ denotes the set of absolutely continuous probability measures on $G$) and points of $w^*$-norm continuity for

$$T_F^*\colon P(G) \to L^\infty(G)$$

(where $P(G)$ denotes the set of probability measures on $G$) and

$$T_F^{**}\colon \overline{P_a(G)}^{\sigma(L^{1''}, L^\infty)} \to C''(G).$$

We thus get in Corollaries II.11, II.13, II.16 characterizations of Riemann integrable and $\mu$-Riemann integrable functions which simplify and generalize a result of [GGMS]. We also give in Proposition II.4 an example (announced in [GGMS]) showing that $T_F^{**}(\varphi'')$ can belong to $C(G)$ without $\varphi''$ being a point of $w^*$-norm continuity for $T_F^{**}$.

As $\overline{F*P_a(G)}^{\|\ \|}$ and $F*P(G)$ are nice closed convex sets in $L^\infty(G)$ we study in part III the extreme points of these sets and their convex subsets. We give examples where $\overline{F*P_a(G)}^{\|\ \|}$ has extreme points without $F$ being continuous and examples where $\overline{F*P_a(G)}^{\|\ \|}$ has no extreme points. By using an operator version (Theorem III.3) of the main result of [$S_1$] we show in Corollary III.4 that if $F$ is Riemann integrable and discontinuous the set $\overline{F*P_a}^{\|\ \|}$ does not have the Krein–Milman property, i.e. contains a closed convex set without any extreme point.

As $F\in L^\infty(G)$ is continuous iff $F*P(G)$ is compact and iff $F*P(G)$ is weakly compact, it is a natural question to consider the classes of functions $F$ such that $F*P(G)$ or $F*P_a(G)$ are weakly sequentially precompact sets in $L^\infty(G)$. We give in part IV examples of functions which do or do not belong to these classes. Actually, the first is strictly included in the second (Propositions IV.7 and IV.8), which is itself strictly included in the class of Riemann integrable functions (Proposition IV.5).

A motivation for part IV is of course Rosenthal's theorem ([$R_1$], [$R_2$]).

**I. Notation, definitions and remarks.** All Banach spaces considered in this paper are vector spaces over $\mathbf{R}$. The dual of a Banach space $X$ is denoted by $X'$.

We denote by $G$ a compact abelian group. $C(G)$ is the space of continuous real-valued functions on $G$. $M(G)$ is the dual space of $C(G)$. $L^1(G)$ is the space of classes of real functions on $G$ whose modulus is integrable with respect to the Haar measure $dt$ on $G$. $L^\infty(G)$ is the dual space of $L^1(G)$. The duality between $L^1(G)$ and $L^\infty(G)$ is defined by

$$\forall\varphi\in L^1(G)\ \forall F\in L^\infty(G)\quad \langle F,\varphi\rangle=\int F(t)\varphi(-t)\,dt=F*\varphi(0).$$

If $t_0\in G$ and $\varphi\in L^1(G)$, $\varphi(t-t_0)$ is denoted by $\varphi_{t_0}(t)$ or $\varphi^{t_0}(t)$. Notice that $F*\varphi(t)=\langle F,\varphi_{-t}\rangle$.

We denote by $T_F$ the convolution operator associated to $F\in L^\infty(G)$:

$$T_F\colon L^1(G)\to C(G),\qquad \varphi\rightsquigarrow F*\varphi.$$

$T_F^*$ is the adjoint operator:

$$T_F^*\colon M(G)\to L^\infty(G),\qquad \mu\rightsquigarrow F*\mu.$$

$T_F^{**}$ is the adjoint operator of $T_F^*$:

$$T_F^{**}\colon L^{1''}(G)\to C''(G),\qquad \varphi''\rightsquigarrow T_F^{**}(\varphi''),$$

and $T_F^{**}(\varphi'')$ is denoted by $F*\varphi''$.



We have chosen the duality between $L^\infty(G)$ and $L^1(G)$ in such a way that $T_F$ and the restriction of $T_F^*$ to $L^1(G)$ coincide. The $w^*$-topology on $M(G)$ (respectively $L^\infty(G)$) is $\sigma(M(G),C(G))$ (respectively $\sigma(L^\infty(G),L^1(G))$). The Haar measure of a measurable set $A\subset G$ is denoted by $|A|$. $P(A)$ denotes the set of probability measures on $A$. If $|A|>0$, $P_a(A)$ is the subset $P(A)\cap L^1(G)$.

$\overline{P_a(A)}^{w^*}$ denotes the $w^*$-closure of $P_a(A)$ in $L^{1''}(G)$ (i.e. for the topology $\sigma(L^{1''}(G),L^\infty(G))$).

Let $F\in L^\infty(G)$. We denote by $\operatorname{Osc\,ess}(F|A)$ the *essential oscillation* of $F$ on the measurable set $A$:

$$\operatorname{Osc\,ess}(F|A)$$
$$=\sup\{\lambda\in\mathbf{R}\,|\,\exists\alpha\in\mathbf{R}\ |\{F<\alpha\}\cap A|>0\text{ and }|\{F>\alpha+\lambda\}\cap A|>0\}.$$

The essential oscillation of $F$ at $t\in G$ is

$$\operatorname{Osc\,ess}F(t)=\inf_V\operatorname{Osc\,ess}(F|V)$$

where $V$ runs through a basis of open neighborhoods of $t$ in $G$. The spectrum of $L^\infty(G)$ is denoted by $\mathscr{S}$. $\mathscr{S}_0$ is the subset of those $h\in\mathscr{S}$ whose restriction to $C(G)$ is $\delta_0$. Hence

$$\mathscr{S}=(\bigcup\mathscr{S}_0)_{t\in G}.$$

Let $\Gamma$ be the dual group of $G$. The *Fourier transform* of $\mu\in M(G)$ is defined by $\hat\mu(\gamma)=\langle\mu,\operatorname{Re}\gamma\rangle+i\langle\mu,\operatorname{Im}\gamma\rangle$ for every $\gamma\in\Gamma$. Let $\Lambda$ be a symmetric subset of $\Gamma$. We denote by $M_\Lambda(G)$ the space of real measures on $G$ whose Fourier transform is zero outside $\Lambda$. As $\Lambda$ is symmetric, $M_\Lambda(G)+iM_\Lambda(G)$ is exactly the space of complex measures on $G$ whose Fourier transform is zero outside $\Lambda$. The subspaces $C_\Lambda(G)$ and $L_\Lambda^\infty(G)$ are defined in the same way.

Definition I.1. Let $A\subset G$ with $|A|>0$. A point $t\in A$ is said to be a *point of density* 1 if $\lim_V|A\cap V|/|V|=1$ where $V$ runs through a basis of neighborhoods of $t$ in $G$. We denote by $A'$ the set of points of $A$ with density 1.

We recall that $|A\backslash A'|=0$.

Definition I.2. Let $C$ be a bounded convex subset of a real Banach space $X$. A *slice* of $C$ is a nonempty set of the form

$$S=S(x^*,\alpha)=\{x\in C\,|\,\langle x^*,x\rangle>\alpha\}$$

where $\|x^*\|_{X'}=1$. If $X$ is the dual of the space $Y$ and $x^*\in Y$, $S$ is said to be a *$w^*$-slice*.

Definition I.3. A bounded convex set $C$ in a real Banach space $X$ is *dentable* if it has slices of arbitrarily small diameter.

DEFINITION I.4. An operator $T\colon Y \to X$ is *strongly regular* if for every closed convex bounded subset $C$ of $Y$ and every $\varepsilon > 0$ there are slices $S_1, \ldots, S_n$ of $C$ and $0 < \alpha_i$, $\sum_{i=1}^n \alpha_i = 1$, such that

$$\operatorname{diam} T\Big(\sum_{i=1}^n \alpha_i S_i\Big) < \varepsilon.$$

DEFINITION I.5. A closed bounded convex set $C$ in a Banach space $X$ has the *Krein–Milman property* (denoted by KMP) if every closed bounded convex subset $D$ of $C$ has an extreme point.

DEFINITION I.6. A set $B$ in a Banach space $X$ is called *weakly sequentially precompact* (*w.s.p.*) if every sequence in $B$ has a weak Cauchy subsequence.

If the space $X$ is separable, by the Main Theorem of [$R_2$] and [OR], $B$ is w.s.p. iff its closure in $X''$ (for $\sigma(X'', X')$) is sequentially compact.

DEFINITION I.7. A sequence $(e_n)_{n \geq 1}$ in a real Banach space is an $l^1$-*sequence* if there exists $\delta > 0$ such that

$$\forall a_1, \ldots, a_n \in \boldsymbol{R} \qquad \Big\|\sum_{i=1}^n a_i e_i\Big\| \geq \delta \sum_{i=1}^n |a_i|.$$

Rosenthal's theorem [$R_1$] asserts that every bounded sequence in a Banach space has either a weak Cauchy subsequence or an $l^1$-subsequence.

Remark I.8. Without loss of generality we may assume that $G$ is metrizable and we will assume it from now on: as every $F \in L^\infty(G)$ belongs to $L^1(G)$ it has a countable spectrum which spans a countable subgroup $\Lambda$ of $\Gamma$ ($\Gamma$ is the dual group of $G$) and $F$ belongs to $L^\infty(G/\Lambda^\perp)$. This is the dual space of $L^1(G/\Lambda^\perp)$ and the quotient group $G/\Lambda^\perp$ is metrizable.

Remark I.9. Let $G$ be a compact abelian group. Every convolution operator $T\colon L^1(G) \to L^\infty(G)$ (i.e. every bounded operator which commutes with translation by every $t \in G$) is of the form $T_F$:

$$T_F\colon \varphi \rightsquigarrow F * \varphi$$

where $F \in L^\infty(G)$.

More precisely, $F = w^*\text{-}\lim_\alpha T(\varphi_\alpha)$ where $(\varphi_\alpha)$ is an approximate identity in $L^1(G)$ (i.e. $\varphi_\alpha \in P_a(G)$ and $\varphi_\alpha \xrightarrow{w^*} \delta_0$).

Remark I.10. Let $G$ be a compact abelian group. Every convolution operator $L^1(G) \to L^\infty(G)$ is a Dunford–Pettis operator. Indeed, if $F \in L^\infty(G)$, $T_F$ maps $L^2(G)$ into the space $A(G)$ of Fourier transforms of $l^1(\Gamma)$. As $l^1(\Gamma)$ has the Schur property, $T_F\colon L^2(G) \to A(G) \to C(G)$ is a compact operator.

Remark I.11. Let $G$ be a metric compact abelian group and $F \in L^\infty(G)$. The continuous functions lying in $F * P(G)$ belong to the norm closure of $F * P_a(G)$ in $C(G)$, denoted by $\overline{F * P_a}^{\|\,\|}$: indeed, if $\mu \in P(G)$ and if $F * \mu$ is continuous,

$$(F * \mu) * \varphi_n = F * (\mu * \varphi_n) \xrightarrow{\|\,\|} F * \mu$$

where $(\varphi_n)_{n \geq 1}$ is an approximate identity in $L^1(G)$.

Remark I.12. Let $F \in L^\infty(G)$. The set $F * P(G)$ is convex, $w^*$-compact in $L^\infty(G)$ and translation invariant. The set of its extreme points is of course translation invariant. Let us show that it is exactly $(F_t)_{t \in G}$. Let $F * \mu$ be an extreme point of $F * P(G)$, and let $\mu_0$ be an extreme point of $\{\mu' \in P(G) \mid F * \mu = F * \mu'\}$. Then $\mu_0$ is an extreme point of $P(G)$, hence $\mu_0 = \delta_t$ for a $t \in G$ and $F * \mu = F_t$.

We find in the same way that the set of extreme points of $\overline{F * P_a(G)}^{w^*}$ in $C''(G)$ is $(F * h_t)_{h \in \mathcal{S}_0, t \in G}$.

**II. Points of weak-norm continuity for $T_F$. Points of $w^*$-norm continuity for $T_F^*$ and $T_F^{**}$.** We first show that for $F \in L^\infty(G)$ the set $\overline{F * P_a(G)}^{\|\,\|}$ is never dentable unless $F$ is continuous.

PROPOSITION II.1. *Let $F \in L^\infty(G)$.*

(a) *For any $B \subset G$ with $|B| > 0$ let $S$ be a slice of $P_a(B)$. Then*

$$\operatorname{diam} F * S \geq \inf\{\operatorname{diam} F * P_a(A) \mid A \subset G,\ |A| > 0\} \geq d(F, F * P_a(G)) \geq d(F, C(G)).$$

(b) $d(F, C(G)) \geq \frac{1}{2} \sup_{t \in G} \operatorname{Osc\,ess} F(t)$.

(c) *For any $h \in \mathcal{S}_0$ and $t_0 \in G$*

$$d(F * h_{t_0}, F * P_a(G)) \geq d(F * h_{t_0}, C(G)) \geq d(F_{t_0}, C(G)) = d(F, C(G))$$

*where the first distances are taken in $C''(G)$ while the last one is taken in $L^\infty(G)$.*

Proof. (a) A slice $S \subset P_a(B)$ is of the form $S_{F',\alpha} = \{\varphi \in P_a(B) \mid \langle \varphi, F' \rangle > \alpha\}$ where $F' \in L^\infty(G)$. Hence there exists $A \subset B$ with $|A| > 0$ and $P_a(A) \subset S$. Let $A \subset G$, $|A| > 0$, and let $t_0$ be a point of density 1 for $A$. Let $(V_n)_{n \geq 1}$ be a basis of neighborhoods of $t_0$ in $G$. Then

$$\left\| \frac{1_{A \cap V_n}}{|A \cap V_n|} - \frac{1_{V_n}}{|V_n|} \right\|_{L^1(G)} \to 0 \quad \text{as } n \to +\infty,$$

hence

$$\operatorname{diam} F * P_a(A) \geqslant \lim_N \operatorname{diam}\left(F * \frac{1_{A\cap V_n}}{|A\cap V_n|}\right)_{n\geqslant N} = \lim_N \operatorname{diam}\left(F * \frac{1_{V_n}}{|V_n|}\right)_{n\geqslant N}$$
$$\geqslant \lim_n \left\| F_{t_0} - F * \frac{1_{V_n}}{|V_n|}\right\|_{L^\infty(G)} \geqslant d(F, F * P_a(G))$$
$$\geqslant d(F, C(G)).$$

(b) For any $f \in C(G)$ and $t \in G$

$$\operatorname{Osc\,ess} F(t) = \operatorname{Osc\,ess}(F-f)(t) \leqslant 2\|F-f\|_{L^\infty(G)}.$$

(c) is obvious. ■

Let us now recall some equivalent properties of continuous functions.

PROPOSITION II.2. *Let $F \in L^\infty(G)$. The following assertions are equivalent*:

(a) *$F$ is continuous.*
(b) *$(F_t)_{t\in G}$ is norm compact in $L^\infty(G)$.*
(b′) *$T_F$: $L^1(G) \to C(G)$ is compact.*
(c) *$(F_t)_{t\in G}$ is weakly compact in $L^\infty(G)$.*
(d) *$T_F$: $L^1(G) \to C(G)$ is representable.*
(e) *There exists $B \subset G$ with $|B| > 0$ such that for any $\varepsilon > 0$ there exists a slice $S$ of $P_a(B)$ with $\operatorname{diam} T_F(S) < \varepsilon$.*
(f) *For any $B \in G$ with $|B| > 0$ and for any $\varepsilon > 0$ there exists a slice $S$ of $P_a(B)$ with $\operatorname{diam} T_F(S) < \varepsilon$.*

Proof. (a)⇔(b)⇔(b′)⇔(c) are well known. (a)⇔(d) is obvious by the definition of a representable operator. (f)⇒(e) is obvious and (e)⇒(a) by Proposition II.1. (f)⇔(d) by [B, Lemma 1.1] (this is a general fact for operators $L^1 \to X$). ■

Obviously $T_F^*(\mu)$ or $T_F^{**}(\varphi'')$ belong to $C(G)$ as soon as $\mu$ or $\varphi''$ are points of $w^*$-norm continuity for $T_F^*$: $P(G) \to L^\infty(G)$ or $T_F^{**}$: $\overline{P_a(G)}^{w^*} \to L^\infty(G)$ respectively.

We will now give examples of $F \in L^\infty(G)$ and $\varphi'' \in P_a(G)$ such that $T_F^{**}(\varphi'') \in C(G)$ but $\varphi''$ is not a point of $w^*$-norm continuity for $T_F^{**}$: $\overline{P_a(G)}^{w^*} \to C''(G)$.

These examples were announced in [GGMS, Remark VIII.11]. They contrast with the key construction in [$S_1$] which provides an operator $T$: $L^1 \to X$ such that $T^{**}(\varphi'') \in X$ only if $\varphi''$ is a point of $w^*$-norm continuity of $T^{**}$: $\overline{P_a(G)}^{w^*} \to X$.

LEMMA II.3. (a) *Let $\mu \in P(G)$ be a nondiffuse measure. It cannot be a point of $w^*$-norm continuity for $T_F^*$: $P(G) \to L^\infty(G)$ unless $F$ is continuous.*



(b) *Let $\varphi'' \in \overline{P_a(G)}^{w^*}$ be such that its restriction $\mu$ to $C(G)$ is a nondiffuse measure. $\varphi''$ cannot be a point of $w^*$-norm continuity for $T_F^{**}$: $\overline{P_a(G)}^{w^*} \to C''(G)$ unless $F$ is continuous.*

Proof. The proof of (a) follows the same lines as the proof of (b) and will also be a consequence of the subsequent Theorem II.8. Hence we only prove (b). Let $0 < \lambda \leqslant 1$ and let $\varphi_1'' \in \overline{P_a(G)}^{w^*}$ be such that $0 < \lambda\varphi_1'' \leqslant \varphi''$. Then there exists $\varphi_2'' \in \overline{P_a(G)}^{w^*}$ such that

$$\varphi'' = \lambda\varphi_1'' + (1-\lambda)\varphi_2''$$

and for any $w^*$-neighborhood $V(\varphi'')$ in $\overline{P_a(G)}^{w^*}$ there exists a $w^*$-neighborhood $W(\varphi_1'')$ such that

$$V(\varphi'') \supset \lambda W(\varphi_1'') + (1-\lambda)\varphi_2''.$$

Hence

$$\operatorname{diam} F * V(\varphi'') \geqslant \lambda \operatorname{diam} F * W(\varphi_1'').$$

Let now $t_0 \in G$ be such that $\mu\{t_0\} > 0$ and let $(O_n)_{n\geqslant 1}$ be a basis of neighborhoods of $t_0$ in $G$. Let $\lambda\varphi_1''$ be a $w^*$-limit point of $(1_{O_n}\varphi'')_{n\geqslant 1}$. As $\|1_{O_n}\varphi''\|_{L^{1''}} \geqslant \mu(t_0)$ we have $0 < \lambda = \mu(t_0) \leqslant 1$ and the restriction of $\varphi_1''$ to $C(G)$ is $\delta_{t_0}$. $V$ and $W$ being as above,

$$\operatorname{diam} F * V(\varphi'') \geqslant \lambda \operatorname{diam} F * W(\varphi_1'') \geqslant \lambda d(F * \varphi_1'', C(G))$$
$$\geqslant \lambda d(F_{t_0}, C(G)) > 0. \ ■$$

PROPOSITION II.4. *Let $\Lambda_1, \Lambda_2$ be two disjoint symmetric subsets of $\Gamma$ (the dual group of $G$) such that $M_{\Lambda_2}(G)$ contains nondiffuse positive measures and $L^\infty_{\Lambda_1}(G) \setminus C_{\Lambda_1}(G)$ is nonempty. Then for $F \in L^\infty_{\Lambda_1}(G) \setminus C_{\Lambda_1}(G)$ and $\mu \in M_{\Lambda_2}(G)$, $F * \mu$ is continuous, though if moreover $\mu \in P(G)$, $\mu$ is not a point of $w^*$-norm continuity for $T_F^*$. Moreover, no $\varphi'' \in \overline{P_a(G)}^{w^*} \cap L^{1''}_{\Lambda_2}(G)$ whose restriction to $C(G)$ is such a $\mu$ can be a point of $w^*$-norm continuity for $T_F^{**}$.*

Proof. Under the above assumptions $F * \mu = 0$ and $F * \varphi'' = 0$; the remaining assertions come from Lemma II.3. We now have to prove that the assumptions of Proposition II.4 are nonempty. Indeed, let $O_1, O_2$ be two disjoint symmetric open sets in the dual group of $G_d$ (the group $G$ provided with discrete topology) and let $\Lambda_i = \Gamma \cap O_i$ $(i = 1, 2)$. By [Ru, Theorem 2.6.1], $M_{\Lambda_i}(G)$ contains positive atomic measures. In particular, $M_{\Lambda_1}(G)$ contains non-absolutely continuous measures, hence by [$LP_1$, Theorem 3.1], $L^\infty_{\Lambda_1}(G)$ contains discontinuous functions. Taking for $\Lambda_1$ an infinite subgroup of $\Gamma$ we see that $F$ may be arbitrarily irregular. ■

We now need more details on $w^*$-neighborhoods of $\varphi''$ in $\overline{P_a(G)}^{w^*}$ and of $\mu$ in $P(G)$.

As was shown in [S₁] or [GGMS, IV] a basis of $w^*$-neighborhoods of $\varphi'' \in \overline{P_a(G)}^{w^*}$ is given by

$$V_{\mathcal{P},\varepsilon}(\varphi'') = \{\psi'' \in \overline{P_a(G)}^{w^*} \mid \sum_{i=1}^{k} |\langle \varphi'', 1_{A_i}\rangle - \langle \psi'', 1_{A_i}\rangle| < \varepsilon\}$$

where $\mathcal{P} = (A_i)_{i=1}^k$ runs through the partitions of $G$ into sets of positive Haar measure and $\varepsilon \in ]0, 1]$. Moreover,

$$V_{\mathcal{P},\varepsilon}(\varphi'') = \{V_{\mathcal{P},0}(\varphi'') + \varepsilon \cdot \text{unit ball of } L^1(G)''\} \cap \overline{P_a(G)}^{w^*} \quad \text{where}$$

$$V_{\mathcal{P},0}(\varphi'') = \sum_{i=1}^{k} \langle \varphi'', 1_{A_i}\rangle \overline{P_a(A_i)}^{w^*}.$$

In particular, a finite convex combination of weakly open sets in $P_a(G)$ is still open [GGMS, Remark IV.5].

For $\varphi \in P_a(G)$ define

$$W_{\mathcal{P},\varepsilon}(\varphi) = \{\psi \in L^1(G) \mid \sum_{i=1}^{k} |\langle \psi - \varphi, 1_{A_i}\rangle| < \varepsilon,\ \|\psi\| \leqslant 1\},$$

hence $V_{\mathcal{P},\varepsilon}(\varphi) = W_{\mathcal{P},\varepsilon}(\varphi) \cap P_a(G)$. Then it is easy to verify that

$$W_{\mathcal{P},\varepsilon}(\varphi) \subset [1-\varepsilon, 1] \times V_{\mathcal{P},2\varepsilon}(\varphi).$$

Let $\varphi \in L^1(G)$ be such that $\|\varphi\| \leqslant 1$. We have $\varphi = \lambda_1 \varphi_1 - \lambda_2 \varphi_2$ where $\lambda_1 + \lambda_2 \leqslant 1$, $\lambda_i \geqslant 0$ and $\varphi_i \in P_a(G)$ $(i = 1, 2)$. Hence $\lambda_1 W_{\mathcal{P},\varepsilon}(\varphi_1) - \lambda_2 \varphi_2$ is a weakly open neighborhood of $\varphi$ in the unit ball of $L^1(G)$ included in $[\lambda_1(1-\varepsilon), \lambda_1] V_{\mathcal{P},2\varepsilon}(\varphi_1) - \lambda_2 \varphi_2$.

We now study $w^*$-neighborhoods of $\mu$ in $P(G)$.

Definition II.5. For $0 < \varepsilon, \delta < 1$ let

$$K_{\delta,\varepsilon} = \{f \in C(G) \mid \|f\| \leqslant 1,\ \forall 0 \leqslant t \leqslant \delta\ \|f_t - f\| \leqslant \varepsilon/8\}.$$

For $\mu \in P(G)$ let

$$V_{\varepsilon,\delta}(\mu) = \{\nu \in P(G) \mid |\langle \mu - \nu, f\rangle| \leqslant \varepsilon\ \ \forall f \in K_{\delta,\varepsilon}\}.$$

As $K_{\delta,\varepsilon}$ is compact in $C(G)$, $(V_{\varepsilon,\delta}(\mu))_{\varepsilon,\delta}$ is clearly a basis of $w^*$-neighborhoods of $\mu$. Notice that $K_{\delta,\varepsilon}$ is translation invariant, hence for every $t_0 \in G$

$$V_{\varepsilon,\delta}(\mu_{t_0}) = V_{\varepsilon,\delta}^{t_0}(\mu).$$

Lemma II.6. *Let $\mu \in P(G)$ and let $(W_i)_{i=1}^n$ be a finite family of open sets in*



*$G$. Then for every $\varepsilon > 0$ there exists a family $(U_i)_{i=1}^n$ of open sets in $G$ such that:*

(i) $\forall i \leqslant n \quad U_i \subset W_i$.

(ii) $-\varepsilon + \mu(\bigcup_{i=1}^n W_i) \leqslant \mu(\bigcup_{i=1}^n U_i) \leqslant \sum_{i=1}^n \mu(U_i) \leqslant \mu(\bigcup_{i=1}^n W_i) + \varepsilon$.

Proof. We prove this lemma by induction on $n$. Assume it is true for $n-1$ open sets and consider the family $(W_i)_{i=1}^n$. Take $U_1 = W_1$. As $\mu$ is regular there exists an open set $O \subset G$ such that

(a) $$W_1^c \cap \bigcup_{i=2}^n W_i \subset O \subset \bigcup_{i=2}^n W_i,$$

(b) $$\mu(O) \leqslant \mu(W_1^c \cap \bigcap_{i=2}^n W_i) + \varepsilon/3.$$

We apply the inductive assumption to the family $(W_i \cap O)_{i=2}^n$ and $\varepsilon/3$. Hence we get $(U_i)_{i=2}^n$ and

$$\mu(\bigcup_{i=1}^n W_i) = \mu(W_1 \cup O) \leqslant \mu(W_1) + \mu(O) \leqslant \mu(W_1) + \mu(\bigcup_{i=2}^n U_i) + \varepsilon/3$$

$$\leqslant \mu(\bigcup_{i=1}^n U_i) + 2\varepsilon/3 \quad (\text{because } \mu(O \cap W_1) \leqslant \varepsilon/3 \text{ by (b)})$$

$$\leqslant \sum_{i=1}^n \mu(U_i) + 2\varepsilon/3 \leqslant \mu(W_1) + \mu(O) + 2\varepsilon/3$$

$$\leqslant \mu(W_1 \cup O) + \varepsilon = \mu(\bigcup_{i=1}^n W_i) + \varepsilon.$$

Lemma II.7. *Let $\mu \in P(G)$ and $0 < \varepsilon, \delta < 1$. There exists a finite family of open sets $(U_i)_{i=1}^n$ in $G$ such that:*

(i) $|\sum_{i=1}^n \mu(U_i) - 1| < \varepsilon/4, \quad \mu(\bigcup_{i=1}^n U_i) \geqslant 1 - \varepsilon/4$.

(ii) $(\sum_{i=1}^n \mu(U_i))^{-1} \sum_{i=1}^n \mu(U_i) P(U_i) \subset V_{\varepsilon,\delta}(\mu)$.

Proof. Let $(W_i)_{i=1}^n$ be a finite open covering of $G$ by open sets whose diameter (for the metric of $G$) is less than $\delta$, and let $(U_i)_{i=1}^n$ be chosen as in Lemma II.6 for $\mu$ and $\varepsilon/8$. Set $\alpha = \sum_{i=1}^n \mu(U_i)$. For every $\nu \in \sum_{i=1}^n \mu(U_i) P(U_i)$

and $f \in K_{\delta,\varepsilon}$

$$|\langle \nu/\alpha - \mu, f\rangle| \leqslant |1/\alpha - 1|\,\alpha + |\langle \nu - \mu, f\rangle|$$
$$\leqslant \varepsilon/4 + \mu\big((\bigcup_{i=1}^{n} U_i)^c\big) + \Big\|\sum_{i=1}^{n} 1_{U_i}\mu - 1_{\cup U_i}\mu\Big\|$$
$$+ \sum_{i=1}^{n} \mu(U_i)\,\mathrm{Osc\,ess}(f\,|\,U_i) \leqslant \varepsilon. \quad \blacksquare$$

We can now prove one of the key results of part II.

THEOREM II.8. *Let $F \in L^\infty(G)$ and $\mu \in P(G)$. Then*

$$\lim_{\varepsilon,\delta \to 0} \operatorname{diam} F * V_{\varepsilon,\delta}(\mu) = \sup_{t_0 \in G} \langle \mathrm{Osc\,ess}\, F(t), \mu_{t_0}\rangle \geqslant \int_G \mathrm{Osc\,ess}\, F(t)\,dt.$$

Proof. (a) Let us prove that for every $t_0 \in G$

$$\langle \mathrm{Osc\,ess}\, F(t), \mu_{t_0}\rangle \leqslant \lim_{\varepsilon,\delta} \operatorname{diam} F * V_{\varepsilon,\delta}(\mu).$$

By Lemma II.7 applied to $\mu$ and $V_{\varepsilon,\delta}(\mu)$ we get open sets $(U_i)_{i=1}^n$ in $G$ such that

$$\langle \mathrm{Osc\,ess}\, F_{t_0}(t), \mu\rangle \leqslant \sum_{i=1}^{n} \langle \mathrm{Osc\,ess}\, F_{t_0}(t), 1_{U_i}\mu\rangle + 2\|F\|_\infty \mu\big((\bigcup_{i=1}^{n} U_i)^c\big)$$
$$\leqslant \sum_{i=1}^{n} \mu(U_i)\,\mathrm{Osc\,ess}(F_{t_0}\,|\,U_i) + \tfrac{1}{2}\varepsilon\|F\|_\infty$$
$$= \sum_{i=1}^{n} \mu(U_i) \sup_{\varphi_i,\psi_i \in P_a(U_i)} \langle F_{t_0}, \varphi_i - \psi_i\rangle + \tfrac{1}{2}\varepsilon\|F\|_\infty$$
$$\leqslant \operatorname{diam} F * \Big(\sum_{i=1}^{n} \mu(U_i)\,P_a(U_i)\Big) + \tfrac{1}{2}\varepsilon\|F\|_\infty$$
$$\leqslant (1 + \tfrac{1}{4}\varepsilon) \operatorname{diam} F * V_{\varepsilon,\delta}(\mu) + \tfrac{1}{2}\varepsilon\|F\|_\infty.$$

(b) Let us now prove that

$$\lim_{\varepsilon,\delta} \operatorname{diam} F * V_{\varepsilon,\delta}(\mu) \leqslant \sup_{t_0 \in G} \langle \mathrm{Osc\,ess}\, F(t), \mu_{t_0}\rangle.$$

For every $n \geqslant 1$ let $\varphi_n, \psi_n \in P_a(G)$ and $t_n \in G$ be such that

$$\operatorname{diam} F * V_{1/n,1/n}(\mu) \leqslant \langle F, \varphi_n^{t_n}\rangle - \langle F, \psi_n^{t_n}\rangle + 1/n.$$

Let $(t_{n_k})_{k\geqslant 1}$ be a subsequence of $(t_n)_{n\geqslant 1}$ converging to a point $t_0 \in G$; then

$$\varphi_{n_k}^{t_{n_k}} \xrightarrow{w^*} \mu_{t_0}, \qquad \psi_{n_k}^{t_{n_k}} \xrightarrow{w^*} \mu_{t_0}.$$



As $t \rightsquigarrow \sup\mathrm{ess}\, F(t)$ and $t \rightsquigarrow -\inf\mathrm{ess}\, F(t)$ are u.s.c. we get

$$0 \leqslant \langle F, \varphi_{n_k}^{t_{n_k}}\rangle - \langle F, \psi_{n_k}^{t_{n_k}}\rangle \leqslant \langle \sup\mathrm{ess}\, F(t), \varphi_{n_k}^{t_{n_k}}\rangle - \langle \inf\mathrm{ess}\, F(t), \psi_{n_k}^{t_{n_k}}\rangle$$
$$\leqslant \langle \sup\mathrm{ess}\, F(t), \mu_{t_0}\rangle - \langle \inf\mathrm{ess}\, F(t), \mu_{t_0}\rangle = \langle \mathrm{Osc\,ess}\, F(t), \mu_{t_0}\rangle,$$

which proves the claim.

(c) Let $(f_n)_{n\geqslant 1}$ be a bounded sequence in $C(G)$ converging pointwise on $G$ to $\mathrm{Osc\,ess}\, F(t)$. Then

$$\sup_{t_0} \langle \mathrm{Osc\,ess}\, F(t), \mu_{t_0}\rangle \geqslant \int \langle \mathrm{Osc\,ess}\, F(t), \mu_{t_0}\rangle\,dt_0 = \lim_n \int (f_n * \mu)(t_0)\,dt_0$$
$$= \lim_n \int f_n(t)\,dt \int d\mu = \int \mathrm{Osc\,ess}\, F(t)\,dt. \quad \blacksquare$$

Remark II.8 bis. Let $\mu \in P(G)$ be a point of $w^*$-weak continuity for $T_F^*\colon P(G) \to L^\infty(G)$. Then $\mu$ is a point of $w^*$-norm continuity for $T_F^*$: indeed, with the notation of (b) in the above proof the assumption on $\mu$ implies $\langle F, \varphi_n^{t_n}\rangle - \langle F, \psi_n^{t_n}\rangle \to 0$ as $n \to +\infty$.

For $\mu \in P(G)$ define

$$D(\mu) = \{\varphi'' \in \overline{P_a(G)}^{w^*} \,|\, \mu \text{ is the restriction of } \varphi'' \text{ to } C(G)\}.$$

We have the following estimation:

PROPOSITION II.9. *Let $\mu \in P(G)$ and $F \in L^\infty(G)$. Then*

$$\lim_{\varepsilon,\delta \to 0} \operatorname{diam} F * V_{\varepsilon,\delta}(\mu) = \operatorname{diam} F * D(\mu).$$

Proof. (a) Let $\varphi_{n_k}^{t_{n_k}}, \psi_{n_k}^{t_{n_k}}$ $(k \geqslant 1)$ and $t_0 \in G$ be defined as in the proof of Theorem II.8(b). Let $\varphi''$ and $\psi''$ be limit points of $(\varphi_{n_k}^{t_{n_k}})_{k\geqslant 1}$ and $(\psi_{n_k}^{t_{n_k}})_{k\geqslant 1}$ respectively in $\overline{P_a(G)}^{w^*}$. Then $\varphi''$ and $\psi''$ belong to $D(\mu_{t_0}) = D(\mu)_{t_0}$, hence

$$\lim_{\varepsilon,\delta} \operatorname{diam} F * V_{\varepsilon,\delta}(\mu) \leqslant \langle F, \varphi''\rangle - \langle F, \psi''\rangle \leqslant \operatorname{diam} F * D(\mu).$$

(b) For every $\varepsilon > 0$ let $\varphi'', \psi'' \in D(\mu)$ and $\nu \in P(G)$ be such that

$$\operatorname{diam} F * D(\mu) \leqslant \langle F * \nu, \varphi'' - \psi''\rangle + \varepsilon$$

and let $\varphi_\alpha \xrightarrow{w} \varphi''$, $\psi_\alpha \xrightarrow{w} \psi''$, $(\varphi_\alpha), (\psi_\alpha) \in P_a(G)$. In particular, $\varphi_\alpha \xrightarrow{w^*} \mu$ and $\psi_\alpha \xrightarrow{w^*} \mu$. Hence

$$\langle F * \nu, \varphi'' - \psi''\rangle = \lim_\alpha \langle F * \nu, \varphi_\alpha - \psi_\alpha\rangle \leqslant \lim_\alpha \|F * \varphi_\alpha - F * \psi_\alpha\|$$
$$\leqslant \lim_{\varepsilon,\delta} \operatorname{diam} F * V_{\varepsilon,\delta}(\mu). \quad \blacksquare$$

We now give consequences of Theorem II.8. We first complete the result of Proposition II.1:

PROPOSITION II.10. *Let* $F\in L^\infty(G)$ *and* $\varphi''\in D(\delta_0)$. *Then*

$$\tfrac{1}{2}\sup_t \operatorname{Osc\,ess} F(t) \leqslant d(F, F*P_a) \leqslant d(F*\varphi'', F*P_a) \leqslant \lim_{\mathscr{P},\varepsilon} \operatorname{diam} F*V_{\mathscr{P},\varepsilon}(\varphi'') \leqslant \lim_{\varepsilon,\delta} \operatorname{diam} F*V_{\varepsilon,\delta}(\delta_0) \leqslant \sup_t \operatorname{Osc\,ess} F(t).$$

Thus in particular

$$\tfrac{1}{2}\lim_{\varepsilon,\delta} \operatorname{diam} F*V_{\varepsilon,\delta}(\delta_0) \leqslant \lim_{\mathscr{P},\varepsilon} \operatorname{diam} F*V_{\mathscr{P},\varepsilon}(\varphi'') \leqslant \lim_{\varepsilon,\delta} \operatorname{diam} F*V_{\varepsilon,\delta}(\delta_0).$$

We will see in Theorem II.15 below that this is still true for $\varphi\in P_a(G)$ in place of $\delta_0$ and $\varphi''$. We do not know whether this equivalence is true in other cases or not.

Theorem II.8 also implies:

COROLLARY II.11. *Let* $F\in L^\infty(G)$ *and* $\mu\in P(G)$. $\mu$ *is a point of* $w^*$*-norm continuity for* $T_F^*$ *iff for every* $t_0\in G$, $\langle \operatorname{Osc\,ess} F(t), \mu_{t_0}\rangle = 0$, *i.e. iff for every* $t_0\in G$, $F$ *is* $\mu_{t_0}$*-Riemann integrable. If* $F$ *is not continuous such a* $\mu$ *is necessarily a diffuse measure.*

COROLLARY II.12. *Let* $F\in L^\infty(G)$. *The set* $\Delta_F$ *of points of essential discontinuity for* $F$ *is countable iff every diffuse measure* $\mu\in P(G)$ *is a point of* $w^*$*-norm continuity for* $T_F^*$: $P(G)\to L^\infty(G)$.

COROLLARY II.13. *The following properties for* $F\in L^\infty(G)$ *are equivalent:*

(a) $F$ *is Riemann integrable.*

(b) *Every* $\varphi\in P_a(G)$ *is a point of* $w^*$*-norm continuity for* $T_F^*$: $P(G)\to L^\infty(G)$.

(c) *There exists* $\mu\in P(G)$ *which is a point of* $w^*$*-norm continuity for* $T_F^*$: $P(G)\to L^\infty(G)$.

(d) *For every* $\varepsilon>0$ *there exists a* $w^*$*-open set* $U$ *in* $P(G)$ *such that* $\operatorname{diam} F*U<\varepsilon$.

The following lemma will permit us to compare points of $w^*$-norm continuity and points of weak-norm continuity in $P_a(G)$.

LEMMA II.14. *Let* $A\subset G$ *be such that* $|A|>0$ *and let* $\mathscr{V}$ *be a symmetric neighborhood of* 0 *in* $G$.

(a) *For every* $\mathscr{V}$, $1_A*1_{\mathscr{V}}$ *is continuous and strictly positive on the open set* $A'+\mathscr{V}$.

(b) *There exists* $\mathscr{V}_0$ *such that* $|A\cap(A+t)|>0$ *for every* $t\in\mathscr{V}_0$.

(c) *Let* $C\subset G$ *and* $\mathscr{V}$ *be such that* $|C\cap(A'+\mathscr{V})|>0$. *There exists* $t\in\mathscr{V}$ *such that* $|C\cap(A+t)|>0$.



(d) *Let* $F\in L^\infty(G)$. *Define* $h$: $G\to\mathbf{R}$ *by* $h(t)=\operatorname{Osc\,ess}(F\mid A+t)$. *If* $\mathscr{V}$ *is small enough (depending only on* $A$)

$$\operatorname{Osc\,ess}(F\mid A'+\mathscr{V})\leqslant 2\sup_{t\in\mathscr{V}} h(t).$$

(e) $h$ *is a function of first Baire class on* $G$; *for any* $\varepsilon>0$, $t_0$ *and* $\mathscr{V}_0$ *there exists* $\mathscr{V}_1+t_1\subset\mathscr{V}_0+t_0$ *such that*

$$\operatorname{Osc}(h\mid\mathscr{V}_1+t_1)<\varepsilon.$$

Proof. (a) Let $a'\in A'$, $t\in\mathscr{V}$. Then

$$1_A*1_{\mathscr{V}}(a'+t)=|A'\cap(\mathscr{V}+a'+t)|\geqslant|A'\cap(a'+\mathscr{V}_0)|>0$$

if $\mathscr{V}_0\subset\mathscr{V}+t$ is a small enough symmetric neighborhood of 0, because $a'$ is a point of density 1 for $A$.

(b) $|A\cap(A+t)|=1_A*1_{-A}(t)$ and $1_A*1_{-A}$ is a continuous function which is strictly positive at $t=0$.

(c) By assumption and (a)

$$0<|C\cap(A'+\mathscr{V})|=|C'\cap(A'+\mathscr{V})|\leqslant|C'\cap\{1_A*1_{\mathscr{V}}>0\}|.$$

Take $x\in C'$ such that $\alpha=1_A*1_{\mathscr{V}}(x)>0$ and take $\mathscr{V}_0$ such that $1_A*1_{\mathscr{V}}>\frac{1}{2}\alpha$ on $x+\mathscr{V}_0$ and $|C'\cap(x+\mathscr{V}_0)|>0$. Then

$$\int_{\mathscr{V}} 1_C*1_{-A}(t)\,dt=(1_C*1_{-A})*1_{\mathscr{V}}(0)=1_C*(1_{-A}*1_{\mathscr{V}})(0)$$
$$=\int_{C'}(1_A*1_{\mathscr{V}})(t)\,dt\geqslant\tfrac{1}{2}\alpha|C'\cap(\mathscr{V}_0+x)|>0.$$

As $1_C*1_{-A}$ is continuous there exists $t\in\mathscr{V}$ such that

$$0<1_C*1_{-A}(t)=|C\cap(A+t)|.$$

(d) By (b) choose $\mathscr{V}_0$ such that $|A\cap(A+t)|>0$ for any $t\in\mathscr{V}_0$ and let $\mathscr{V}$ be such that $\mathscr{V}+\mathscr{V}\subset\mathscr{V}_0$. By (c)

$$\sup\operatorname{ess}(F\mid A'+\mathscr{V})\leqslant\sup_{t\in\mathscr{V}}\sup\operatorname{ess}(F\mid A+t)=\alpha.$$

For every $\varepsilon>0$ there exists $t_1\in\mathscr{V}$ such that $\alpha\leqslant\varepsilon+\sup\operatorname{ess}(F\mid A+t_1)$. In the same way there exists $t_2\in\mathscr{V}$ such that

$$\inf\operatorname{ess}(F\mid A'+\mathscr{V})\geqslant\inf\operatorname{ess}(F\mid A+t_2)-\varepsilon.$$

Put $C=(A+t_1)\cap(A+t_2)$. Hence $|C|=|A\cap(A+t_2-t_1)|>0$ by our choice for $\mathscr{V}$. Hence

$$\sup\operatorname{ess}(F\mid A+t_1)\leqslant h(t_1)+\inf\operatorname{ess}(F\mid C),$$
$$\inf\operatorname{ess}(F\mid A+t_2)\geqslant -h(t_2)+\sup\operatorname{ess}(F\mid C),$$

$$\operatorname{Osc\,ess}(F \mid A' + \mathscr{V}) \leqslant 2\varepsilon + 2 \sup_{t \in \mathscr{V}} h(t).$$

$\varepsilon$ being arbitrary, the last inequality implies (d).

(e) Let us verify that $g(t) = \sup\operatorname{ess}(F \mid A+t)$ is l.s.c. on $G$: for $\lambda \in \mathbf{R}$

$$\{g > \lambda\} = \{t \in G \mid |\{F > \lambda\} \cap (A+t)| > 0\} = \{1_{\{F>\lambda\}} * 1_{-A} > 0\}$$

is open since $1_{\{F>\lambda\}} * 1_{-A}$ is continuous. Hence $h(t) = \sup\operatorname{ess}(F \mid A+t) - \inf\operatorname{ess}(F \mid A+t)$ is l.s.c. and in particular belongs to the first Baire class. ■

We can now state the second key result of part II:

THEOREM II.15. *Let $F \in L^\infty(G)$ and $\varphi \in P_a(G)$. Then:*

(a) $\lim_\alpha \operatorname{diam} F * W_\alpha(\varphi) \leqslant \lim_n \operatorname{diam} F * V_n(\varphi) \leqslant 2 \lim_\alpha \operatorname{diam} F * W_\alpha(\varphi)$ *where $(W_\alpha(\varphi))_\alpha$ is a basis of weak neighborhoods of $\varphi$ in $P_a(G)$ and $(V_n(\varphi))_{n \geqslant 1}$ is a basis of $w^*$-neighborhoods of $\varphi$ in $P(G)$.*

(b) *For every weakly open set $U$ in $P_a(G)$*

$$\operatorname{diam} F * U \geqslant \tfrac{1}{2} \int_G \operatorname{Osc\,ess} F(t)\, dt.$$

Proof. (a) The left inequality is obvious. By Theorem II.8 we only have to show that

$$\sup_{t_0 \in G} \langle \operatorname{Osc\,ess} F(t), \varphi_{t_0} \rangle \leqslant 2 \lim_\alpha \operatorname{diam} F * W_\alpha(\varphi).$$

As $t_0 \rightsquigarrow \langle \operatorname{Osc\,ess} F(t), \varphi_{t_0} \rangle$ is a continuous function on $G$, it is sufficient to majorize its supremum on a dense subset of $G$. By the result of [$S_1$] we have recalled we can take

$$W_\alpha(\varphi) = W_{A_1,\ldots,A_n,\varepsilon}(\varphi) \supset \sum_{i=1}^{n} \langle \varphi, 1_{A_i} \rangle P_a(A_i).$$

Put $\langle \varphi, 1_{A_i} \rangle = \alpha_i$ and $h_i(t) = \operatorname{Osc\,ess}(F \mid A_i + t)$ $(t \in G,\ i \leqslant n)$. Hence

$$\operatorname{diam} F * \sum_{i=1}^{n} \alpha_i P_a(A_i) = \sup_{t \in G} \sum_{i=1}^{n} \alpha_i \operatorname{Osc\,ess}(F \mid A_i + t) = \sup_{t \in G} \sum_{i=1}^{n} \alpha_i h_i(t).$$

For every open set $\mathscr{O} \subset G$ and every $\varepsilon > 0$ there exist, by Lemma II.14(e), $t_0 \in \mathscr{O}$ and a neighborhood $\mathscr{V}$ of 0 in $G$ such that

$$\forall i \leqslant n \quad \operatorname{Osc}(h_i \mid \mathscr{V} + t_0) < \varepsilon \alpha_i^{-1}.$$

Hence by Lemma II.14(d)

$$\sum_{i=1}^{n} \alpha_i \operatorname{Osc\,ess}(F_{-t_0} \mid A_i' + \mathscr{V}) \leqslant 2 \sum_{i=1}^{n} \alpha_i \sup_{t \in \mathscr{V}} h_i(t + t_0) \leqslant 2 \sum_{i=1}^{n} \alpha_i h_i(t_0) + \varepsilon.$$



As

$$\langle \operatorname{Osc\,ess} F_{-t_0}(t), \varphi \rangle \leqslant \sum_{i=1}^{n} \alpha_i \operatorname{Osc\,ess}(F_{-t_0} \mid A_i' + \mathscr{V})$$

the claim is proved.

(b) Let $U$ be a weakly open set in $P_a(G)$ and $\varphi \in U$. By (a)

$$\operatorname{diam} F * U \geqslant \lim_\alpha \operatorname{diam} F * W_\alpha(\varphi) \geqslant \tfrac{1}{2} \sup_{t_0 \in G} \langle \operatorname{Osc\,ess} F(t), \varphi_{t_0} \rangle.$$

By Theorem II.8, $\sup_{t_0} \langle \operatorname{Osc\,ess} F(t), \varphi_{t_0} \rangle \geqslant \int \operatorname{Osc\,ess} F(t)\, dt$. ■

Theorems II.8 and II.15 imply the following Corollary II.16 which must be compared to the above Corollary II.13. Corollary II.16 is a slight generalization of [GGMS, Example VIII.10; see also the general Theorem IV.10]. Our proof here is much simpler. In particular, it does not use the deep characterization of R.I. functions proved in [$T_1$, Theorem 15.4].

COROLLARY II.16. *Let $F \in L^\infty(G)$. The following assertions are equivalent:*

(a) *$F$ is Riemann integrable.*

(b) *Every $\varphi \in P_a(G)$ is a point of w-norm continuity for $T_F$: $P_a(G) \to C(G)$.*

(c) *There exists $\varphi'' \in \overline{P_a(G)}^{w^*}$ which is a point of $w^*$-norm continuity for $T_F^{**}$: $\overline{P_a(G)}^{w^*} \to C''(G)$.*

(d) *For every $\varepsilon > 0$ there exists a weakly open set $U$ in $P_a(G)$ such that* $\operatorname{diam} F * U < \varepsilon$.

(e) *For every $A \subset G$, $|A| > 0$, and every $\varepsilon > 0$ there exists a weakly open set $U$ in $P_a(A)$ such that* $\operatorname{diam} F * U < \varepsilon$.

(f) *$T_F$: $L^1(G) \to C(G)$ is a strongly regular operator.*

Proof. (a)$\Rightarrow$(b) by Theorem II.8; (b)$\Rightarrow$(c)$\Rightarrow$(d) and (b)$\Rightarrow$(e)$\Rightarrow$(d) are obvious; (d)$\Rightarrow$(a) by Theorem II.15.

(b)$\Rightarrow$(f) by a lemma of Bourgain [GGMS, Lemma II.1] showing that every relatively weakly open subset of a bounded convex set $C$ contains a convex combination of slices of $C$, and the structure of weakly open sets in the unit ball of $L^1(G)$.

(f)$\Rightarrow$(d) because a convex combination of slices of $P_a(G)$ is a weakly open set in $P_a(G)$.

The equivalence (a)$\Leftrightarrow$(f) in the above corollary has an analogue for the $w^*$-topology on $P(G)$:

PROPOSITION II.17. *Let $F \in L^\infty(G)$. The following assertions are equivalent:*

(a) *$F$ is Riemann integrable.*

(b) *For every convex subset $C$ of $P_a(G)$ and every $\varepsilon > 0$ there are $w^*$-*

*slices* $S_1, \ldots, S_n$ *of* $C$ *and* $0 < \alpha_i$, $\sum_{i=1}^n \alpha_i = 1$, *such that*

$$T_F\Big(\sum_{i=1}^n \alpha_i S_i\Big) < \varepsilon.$$

(c) *For every* $\varepsilon > 0$ *there are* $w^*$*-slices* $S_1, \ldots, S_n$ *of* $P(G)$ *and* $0 < \alpha_i$, $\sum_{i=1}^n \alpha_i = 1$, *such that*

$$\operatorname{diam} T_F^*\Big(\sum_{i=1}^n \alpha_i S_i\Big) < \varepsilon.$$

Proof. (a)$\Rightarrow$(b) by Corollary II.13 and the same lemma of Bourgain [GGMS, Lemma II.1] showing that every relatively $w^*$-open subset of $C$ contains a convex combination of $w^*$-slices of $C$.

(b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(a). By Corollary II.13 it is sufficient to verify that a convex combination of $w^*$-slices of $P(G)$ contains a $w^*$-open set. This is easy to check directly. It is also a consequence of the subsequent Lemma IV.6.

Remark II.18. We have considered so far convolution operators $L^1(G) \to C(G)$. Let us now consider convolution operators $L^1(G) \to L^1(G)$. They are exactly convolution operators $T_\mu$ by a measure $\mu \in M(G)$. It is known that every strongly regular operator $L^1 \to X$ has a Pettis density with values in $X''$ [GGMS, Proposition IV.19], hence by [GGMS, Proposition VII.4] when $X = L^1$ such an operator is representable. Thus if $\mu \notin L^1(G)$, $T_\mu$ is not strongly regular. It is interesting to notice that for these particular operators the result is also an obvious consequence of the much easier [GM, Lemma V.2]: let $A$ be a bounded subset of $L^1(G)$ and define

$$\delta(A) = \limsup_{\varepsilon \to 0} \{|\langle \varphi, 1_B \rangle| \mid \varphi \in A,\ B \subset G,\ |B| < \varepsilon\}.$$

Let $S_1, \ldots, S_n$ be slices of $P_a(G)$ and let $\alpha_i > 0$ $(i \leqslant n)$ be such that $\sum_{i=1}^n \alpha_i = 1$. There exists $A_i \subset G$ $(1 \leqslant i \leqslant n)$ such that $|A_i| > 0$ and $P_a(A_i) \subset S_i$. Then

$$\operatorname{diam} \mu * \sum_{i=1}^n \alpha_i S_i \geqslant \operatorname{diam} \mu * \sum_{i=1}^n \alpha_i P_a(A_i) \geqslant \delta\Big(\mu * \sum_{i=1}^n \alpha_i P_a(A_i)\Big)$$

$$= \sum_{i=1}^n \alpha_i \delta\big(\mu * P_a(A_i)\big) \geqslant \sum_{i=1}^n \alpha_i d(\mu_{t_i}, L^1) = d(\mu, L^1)$$

where $t_i \in A_i$, hence $\mu_{t_i}$ belongs to the $w^*$-closure of $\mu * P_a(A_i)$ in $M(G)$.

**III. Extreme points of $\overline{F * P_a(G)}^{\|\ \|}$ and its closed convex subsets.** We begin with examples showing that for $F \in L^\infty(G) \backslash C(G)$, $\overline{F * P_a(G)}^{\|\ \|}$ may have no extreme points or may have plenty of extreme points.

Example III.1. Let us identify the group $T$ of complex numbers with



modulus 1 with $[-\pi, \pi]$ and let $0 < a < \pi$ be such that $a/\pi$ is irrational. Let $I = [-a, a]$ and let $F = 1_I$. Then $\overline{1_I * P_a(T)}^{\|\ \|}$ has no extreme points in $C(T)$:

Actually, for $0 \neq n \in Z$, $\hat F(n) = (\sin na)/(n\pi)$, and $\hat F(0) = a/\pi$. Hence $\hat F$ is never zero on $Z$ and $T_F^*$: $M(T) \to L^\infty(T)$ is one-to-one. By Corollary II.12, $F * \mu \in \overline{F * P_a(T)}^{\|\ \|}$ for every diffuse measure $\mu \in P(T)$, but cannot be an extreme point of this set. Let us show that if $\mu \in P(T)$ is atomic $F * \mu$ is not continuous. This will prove the claim.

Let $\varphi_n = n 1_{[0,1/n]}$ and $\psi_n = n 1_{[-1/n,0]}$ $(n \geqslant 1)$ in $L^1(T)$, and assume $F * \mu$ is continuous. Then for every $t \in T$

$$0 = \lim_n \langle 1_I * (\varphi_n - \psi_n), \mu_t \rangle = \mu_t(a) - \mu_t(-a).$$

Hence $\mu_a = \mu_{-a}$, $(e^{2ina} - 1)\hat\mu(n) = 0$ for every $n \in Z$, and $\hat\mu(n) = 0$ for every $n \in Z$, $n \neq 0$, which is impossible.

The example of the next proposition goes along the same lines as [$S_2$, Proposition 3.1].

Proposition III.2. *Let* $F \in L^\infty(T) \backslash C(T)$ *be such that:*

(a) $\hat F(n) \neq 0$ *for all* $n \in Z$.

(b) $F + F_\pi \in C(T)$.

*Then* $F_t * \frac{1}{2}(\delta_0 + \delta_\pi)$ *is an extreme point of* $\overline{F * P_a(T)}^{\|\ \|}$ *for every* $t \in T$.

Proof. By Remark I.11, (b) implies that $F_t * \frac{1}{2}(\delta_0 + \delta_\pi) \in \overline{F * P_a(T)}^{\|\ \|}$. Let $\nu = \frac{1}{2}(\delta_0 + \delta_\pi)$. By (a), $T_F^*$: $P(T) \to L^\infty(T)$ is a one-to-one operator. Hence if $F_t * \nu$ is not an extreme point of $\overline{F * P_a(T)}^{\|\ \|}$ there exist $\nu_1, \nu_2 \in P(T)$ such that $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ and $F_t * \nu_1, F_t * \nu_2 \in C(T)$. Clearly $\nu_1, \nu_2$ belong to $L^1(\nu) \cap P(T) = \{\lambda\delta_0 + (1-\lambda)\delta_\pi \colon 0 \leqslant \lambda \leqslant 1\}$. Let $\nu_1 = \lambda\delta_0 + (1-\lambda)\delta_\pi$ with $\lambda \neq \frac{1}{2}$. Then $\nu_1 - 2\lambda\nu = (1-2\lambda)\delta_\pi$, hence $F_\pi = F * \nu_1 - 2\lambda F * \nu$ is continuous, which is a contradiction. ∎

Indeed, there are plenty of examples of functions $F$ satisfying the assumptions: let $F_1 \in L^\infty(T) \backslash C(T)$ be such that $F_1 + F_1 * \delta_\pi = 0$ and define a perturbation $F_2$ by

$$\hat F_2(n) = 0 \quad \text{if } \hat F_1(n) \neq 0, \qquad \hat F_2(n) = 2^{-|n|} \quad \text{if } \hat F_1(n) = 0.$$

Then $F_2$ is continuous with real values and we put $F = F_1 + F_2$. The assumption on $F_1$ is satisfied if $\hat F_1$ is supported by $2Z + 1$. Note that we may choose the set $\Delta_F$ of discontinuity points of $F$ arbitrary except for the condition $\Delta_F = \Delta_F + \pi$.

In order to establish our next result on the Krein–Milman property for $\overline{F * P_a(G)}^{\|\ \|}$ we need an operator version of the main result of [$S_1$]. This will be proved in the appendix to this paper.

THEOREM III.3. *Let $X$, $Y$ be Banach spaces, $U$ a bounded operator $Y \to X$, $\alpha > 0$ and $D \subset Y$ a bounded convex set such that:*

(a) *For every slice $S$ of $D$*

$$\operatorname{diam} U(S) > 2\alpha.$$

(b) *For every slice $S$ of $D$ and every $\varepsilon > 0$ there exist slices $S_1, \ldots, S_n$ in $S$ and $0 < \alpha_i$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and*

$$\operatorname{diam} U\Big(\sum_{i=1}^{n} \alpha_i S_i\Big) < \varepsilon.$$

*Then $\overline{U(D)}^{\|\ \|}$ contains a closed convex set without extreme points.*

Recall that, by a lemma of Bourgain [GGMS, Lemma II.1], assumption (b) is satisfied as soon as every slice of $D$ contains a weakly open set $V$ such that $\operatorname{diam} U(V) < \varepsilon$.

As every slice of $D = P_a(G)$ contains $P_a(A)$ for a set $A \subset G$, $|A| > 0$, Corollary II.16 and Theorem III.3 imply the following corollary:

COROLLARY III.4. *Let $F \in L^\infty(G) \backslash C(G)$ be Riemann integrable. Then $\overline{F * P_a(G)}^{\|\ \|}$ does not have the Krein–Milman property.*

This result was motivated by the following question of G. Godefroy: Let $\Lambda$ be a subset of $\Gamma$ and assume that $C_\Lambda(G)$ has KMP. Does this imply that $C_\Lambda(G)$ has the Radon–Nikodym property (i.e. $C_\Lambda(G) = L_\Lambda^\infty(G)$ [LP1])?

Let us also recall the following result:

PROPOSITION III.5 [$LP_2$, Proposition 5]. *Let $F \in L^\infty(G)$. The following assertions are equivalent:*

(a) *$F * P_a(G)$ lies in the norm closure of the convex hull of the extreme points $(F_t)_{t \in G}$ of $F * P(G)$.*

(b) *$F$ is totally ergodic (i.e. for every continuous character $\gamma$ on $G$ and every mean $m$ on $L^\infty(G)$, $\langle \gamma F, m \rangle = \int \gamma F\, dt$).*

By [$T_3$] there are totally ergodic non-Riemann integrable functions.

**IV. When is $F * P_a(G)$ or $\{F_t\}_{t \in G}$ weakly sequentially precompact in $L^\infty(G)$?**

PROPOSITION IV.1. *The set of functions $F \in L^\infty(G)$ such that $F * P_a(G)$ (respectively $(F_t)_{t \in G}$, $F * P(G)$) is w.s.p. in $L^\infty(G)$ is a norm closed subspace of $L^\infty(G)$.*



Proof. The set of functions $F \in L^\infty(G)$ such that $F * P_a(G)$ (for example) contains an $l^1$-sequence in $L^\infty(G)$ is a norm open set in $L^\infty(G)$. Rosenthal's theorem finishes the proof. ∎

Another easier proof would go as follows: identify $L^\infty(G)$ with $C(\mathscr{S})$ and apply the following well-known lemma which we will need again later on:

LEMMA IV.2. *Let $K$ be a compact topological space and $(f_n)_{n \geq 1}$ a bounded sequence in $C(K)$. Let $(\alpha_k)_{k \geq 1}$ be a decreasing sequence of real numbers, $\alpha_k \to 0$. Assume that for every $k \geq 1$ there exists a sequence $(f_n^k)_{n \geq 1}$ in $C(K)$ such that:*

(i) $\limsup \|f_n^k - f_n\| \leq \alpha_k$ $(n \to +\infty)$.
(ii) *$(f_n^k)_{n \geq 1}$ is w.s.p.*

*Then $(f_n)_{n \geq 1}$ is w.s.p.*

Proof. Use a diagonal process. ∎

By applying [GGMS, Theorems VI.6 and VI.16] to our context we get:

PROPOSITION IV.3. *Let $F \in L^\infty(G)$. The following assertions are equivalent:*

(a) *$F * P_a(G)$ is w.s.p. in $C(G)$.*
(b) *$F * P(G)$ is $w^*$-strongly regular in $L^\infty(G)$, i.e. for every $\varepsilon > 0$ and every $w^*$-compact subset $C$ of $F * P(G)$ there exist $w^*$-slices $S_1, \ldots, S_n$ of $C$ and $0 < \alpha_i$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $\operatorname{diam} \sum_{i=1}^{n} \alpha_i S_i < \varepsilon$.*

COROLLARY IV.4. *Let $F \in L^\infty(G)$ be a non-Riemann integrable function. Then $F * P_a(G)$ contains an $l^1$-sequence.*

Proof. If $F$ is not R.I., $F * P(G)$ is not $w^*$-strongly regular by Proposition II.17. Hence Proposition IV.3(a) $\Rightarrow$ (b) (or [B, Lemma 3.7]) proves the claim. ∎

However, an example of M. Talagrand shows the following:

PROPOSITION IV.5. *There exists a Riemann integrable function $F \in L^\infty(\boldsymbol{T})$ such that $F * P_a(\boldsymbol{T})$ is not w.s.p.*

Proof. In [$T_2$] a R.I. function $F \in L^\infty(\boldsymbol{T})$ is constructed in such a way that $F = 1_A$, the boundary of $A$ is the union of the Cantor set and a countable set, and for every $h \in \mathscr{S}_0$ the function $f_h$: $t \mapsto F_t(h)$ is not $\mu$-measurable, where $\mu$ is the canonical measure on the Cantor set. Hence $f_h$ does not belong to the first Baire class on $\boldsymbol{T}$, though $f_h$ lies in the pointwise closure of $F * P_a(\boldsymbol{T})$ (more precisely, $f_h(t) = \lim_\alpha \langle F_t, \varphi_\alpha \rangle = \lim_\alpha F * \varphi_\alpha(t)$ where $t \in \boldsymbol{T}$ and $\varphi_\alpha \in P_a(\boldsymbol{T})$, $\varphi_\alpha \xrightarrow{w^*} h$). By the main theorem of [$R_2$], $F * P_a(\boldsymbol{T})$ is not w.s.p. ∎

We now give an example where $F*P_a(G)$ is w.s.p. We first need

LEMMA IV.6. *Let $\mu_j \in P(G)$ $(j=1, 2)$, $0<\lambda<1$, and $\mu=\lambda\mu_1+(1-\lambda)\mu_2$.*

(a) *For $0<\varepsilon, \delta<1$, $\lambda V_{\varepsilon,\delta}(\mu_1)+(1-\lambda)V_{\varepsilon,\delta}(\mu_2)+4\varepsilon\cdot$ unit ball of $M(G)$ contains a w*-neighborhood of $\mu$ in $P(G)$.*

(b) *Let $(\varphi_k)_{k\geq 1}$ be a sequence in $P_a(G)$ such that $\varphi_k \xrightarrow{w^*} \lambda\mu_1+(1-\lambda)\mu_2$. Then there exist sequences $(\varphi_k^j)_{k\geq 1}$ $(j=1, 2)$ in $P_a(G)$ such that:*

(i) $\forall k\geq 1 \quad \varphi_k=\lambda\varphi_k^1+(1-\lambda)\varphi_k^2$.

(ii) $\varphi_k^j \xrightarrow{w^*} \mu_j \quad (j=1, 2)$.

Proof. (b) is an obvious consequence of (a). Let us prove (a). By revisiting the proof of Lemmas II.6 and II.7 we get the assertion of Lemma II.7 both for $\mu_1$ and $\mu_2$ and the same family of open sets $(U_i)_{i=1}^n$. Hence, writing $\alpha_j=(\sum_{i=1}^n \mu_j(U_i))^{-1}$,

$$\sum_{i=1}^n \mu(U_i)P(U_i)=\lambda\sum_{i=1}^n \mu_1(U_i)P(U_i)+(1-\lambda)\sum_{i=1}^n \mu_2(U_i)P(U_i)$$

$$\subset \frac{\lambda}{\alpha_1}\sum_{i=1}^n \mu_1(U_i)P(U_i)+\frac{(1-\lambda)}{\alpha_2}\sum_{i=1}^n \mu_2(U_i)P(U_i)+\frac{\varepsilon}{3}\cdot\text{unit ball of } M(G)$$

$$\subset \lambda V_{\varepsilon,\delta}(\mu_1)+(1-\lambda)V_{\varepsilon,\delta}(\mu_2)+\frac{\varepsilon}{3}\cdot\text{unit ball of } M(G).$$

We will now define a $w^*$-neighborhood $V(\mu)$ in $P(G)$ such that

$$V(\mu)\subset \sum_{i=1}^n \mu(U_i)P(U_i)+(3\varepsilon+2\varepsilon/3)\cdot\text{unit ball of } M(G).$$

Let $K_i\subset U_i'\subset \overline{U_i'}\subset U_i$ $(1\leq i\leq n)$ where $K_i$ is compact, $U_i'$ is open, $\mu(U_i\setminus K_i)\leq \varepsilon/(3n)$ and let $f_0, f_i, f_i'$ $(1\leq i\leq n)$ be functions in $C(G)$ such that:

(i) $\forall i$ $(1\leq i\leq n)$ $\quad 0\leq f_i\leq 1, \quad 0\leq f_i'\leq 1, \quad 0\leq f_0\leq 1$.

(ii) $f_0=1$ on $\bigcup_{i=1}^n K_i, \quad f_0=0$ outside $\bigcup_{i=1}^n U_i'$.

(iii) $\forall i$ $(1\leq i\leq n)$ $\quad f_i=1$ on $K_i, \quad f_i=0$ outside $U_i'$.

(iv) $\forall i$ $(1\leq i\leq n)$ $\quad f_i'=1$ on $\overline{U_i'}, \quad f_i'=0$ outside $U_i$.

Let us define

$$V(\mu)=\left\{\nu\in P(G)\,\middle|\, |\langle\nu-\mu, f_i\rangle|\leq\frac{\varepsilon}{3n}\ (0\leq i\leq n),\ \langle\nu-\mu, f_i'\rangle|\leq\frac{\varepsilon}{3n}\ (1\leq i\leq n)\right\};$$



hence for $1\leq i\leq n$ and $\nu\in V(\mu)$

$$\mu(U_i)-\frac{2\varepsilon}{3n}\leq\langle\mu, f_i\rangle-\frac{\varepsilon}{3n}\leq\langle\nu, f_i\rangle\leq\langle\nu, U_i'\rangle$$

$$\leq\langle\nu, f_i'\rangle\leq\langle\mu, f_i'\rangle+\frac{\varepsilon}{3n}\leq\mu(U_i)+\frac{\varepsilon}{3n}$$

and it is easily checked that

$$\Big\|\nu-\sum_{i=1}^n \mu(U_i)1_{U_i'}\nu\,\|1_{U_i'}\nu\|^{-1}\Big\|\leq 3\varepsilon+2\varepsilon/3. \ ■$$

PROPOSITION IV.7. *Let $F\in L^\infty(G)$ be such that its set $\Delta_F$ of points of essential discontinuity is countable. Then $F*P_a(G)$ is w.s.p.*

Proof. (a) Let $(\varphi_k)_{k\geq 1}$ be a sequence in $P_a(G)$. By extracting a subsequence we may assume that $\varphi_k\xrightarrow{w^*}\mu=\lambda\mu_1+(1-\lambda)\mu_2$ where $0\leq\lambda\leq 1$, $\mu_1\in P(G)$ is diffuse and $\mu_2\in P(G)$ is discrete. By the above lemma $F*\varphi_k=\lambda F*\varphi_k^1+(1-\lambda)F*\varphi_k^2$ and by Corollary II.12, $(F*\varphi_k^1)_{k\geq 1}$ is norm convergent in $C(G)$. Hence we are left with the case where $\mu$ is discrete.

(b) Assume first $\mu=\delta_{t_0}$ $(t_0\in G)$ and let $(V_k)_{k\geq 1}$ be a basis of neighborhoods of $t_0$ in $G$. For every $k\geq 1$ there exists $N_k$ such that

$$\forall n\geq N_k \quad \langle\varphi_n, 1_{V_k^c}\rangle<1/k.$$

Put $\varphi_{N_k}'=\varphi_{N_k}1_{V_k}/\langle\varphi_{N_k}, 1_{V_k}\rangle$. Hence

$$\lim_k\|F*\varphi_{N_k}-F*\varphi_{N_k}'\|=0.$$

Moreover, $(F*\varphi_{N_k}')_{k\geq 1}$ converges at every point $t\in G$ such that $\operatorname{Osc\,ess} F(t)=0$, hence outside a countable set. Thus a subsequence of $(F*\varphi_{N_k}')_{k\geq 1}$ (and of $(F*\varphi_{N_k})_{k\geq 1}$) converges pointwise on $G$.

(c) If $\mu$ is finitely supported, (b) and the above lemma imply that $(F*\varphi_n)_{n\geq 1}$ has a pointwise convergent subsequence.

(d) If $\mu$ is discrete and not finitely supported, then for every $\varepsilon>0$, $\mu=\lambda\mu_1+(1-\lambda)\mu_2$ where $\mu_i\in P(G)$ $(i=1, 2)$, $0<\lambda\leq\varepsilon$ and $\mu_2$ is finitely supported. Then by the above lemma, (c) and Lemma IV.2 we see again that $(F*\varphi_n)_{n\geq 1}$ is w.s.p. ■

On the other hand, we have:

PROPOSITION IV.8. *There exists $F\in L^\infty(T)$ such that $\Delta_F$ is countable but $(F_t)_{t\in T}$ contains an $l^1$-sequence.*

Proof. We identify $L^\infty(T)$ with $L^\infty[0, 1]$. Let $(F_k)_{k\geq 1}$ be a sequence of functions which are continuous on $\boldsymbol{R}\setminus(0)$, supported on $[0, 3^{-k}]$ and $\|F_k\|_\infty \leqslant 1$ $(k \geqslant 1)$. Define $t_k = 2^{-1} - \sum_{j=1}^{k} 3^{-j}$. Hence $(t_k)_{k\geqslant 1}$ decreases to 0 and $F_k(u-t_k)$ is supported by $]t_k, t_{k-1}]$. Define

$$F(u) = \sum_{k\geqslant 1} F_k(u-t_k)$$

(this series is $w^*$-convergent in $L^\infty[0, 1]$). Clearly $\Delta_F \subset \{0\} \cup \bigcup_{k\geqslant 1} \{t_k\}$. We will choose $(F_k)_{k\geqslant 1}$ in such a way that $(F_{-t_k})_{k\geqslant 1}$ is an $l^1$-sequence. We have

$$F_{-t_k}(u) = \sum_{n\geqslant 1} F_n(u-t_n+t_k).$$

As $F_n$ is zero in a neighborhood of $-t_n+t_k$ except if $k = n$ or $k+1 = n$ and $F_{k+1}$ is zero and continuous at $t_k - t_{k+1} = 3^{-k-1}$ we get

$$\forall h \in \mathscr{S}_0 \quad F_{-t_k}(h) = F_k(h).$$

In order that $(F_{-t_k})_{k\geqslant 1}$ be an $l^1$-sequence in $L^\infty(\boldsymbol{T})$ it is sufficient that there exists $\delta > 0$ such that

$$\forall a_1, \ldots, a_K \in \boldsymbol{R}\ \exists h \in \mathscr{S}_0 \quad \Big|\sum_{k=1}^{K} a_k F_k(h)\Big| \geqslant \delta \sum_{k=1}^{K} |a_k|.$$

Hence it is sufficient that there exists a real sequence $(u_m)_{m\geqslant 1}$ such that $u_m \to 0^+$, $u_m \leqslant 3^{-K}$ and

$$\forall m \geqslant 1 \quad \Big|\sum_{k=1}^{K} a_k F_k(u_m)\Big| \geqslant \delta \sum_{k=1}^{K} |a_k|.$$

Take $F_k(u) = \sin(2\pi 3^k u^{-1})$ on $]0, 3^{-k}]$ and $F_k(u) = 0$ outside. It is well known that $(\sin(2\pi 3^k x))_{k\geqslant 1}$ is an $l^1$-sequence in $C([0, 1])$ [Ru, 5.7.6]. Hence there exists $\delta > 0$ such that

$$\forall a_1, \ldots, a_K \in \boldsymbol{R}\ \exists x \in [0, 1] \quad \Big|\sum_{k=1}^{K} a_k \sin(2\pi 3^k x)\Big| \geqslant \delta \sum_{k=1}^{K} |a_k|$$

and this inequality remains true with $x+m$ $(m \in \boldsymbol{N})$ instead of $x$. Putting $u_m = (x+m)^{-1}$ proves the desired inequality. ∎

Remark IV.9. Let $F \in L^\infty(G)$. By [GGMS, IV.C] the set $(F_t)_{t\in G}$ is said to be a *set of small oscillation* if for every $\varepsilon > 0$ there exists a finite partition $A_1, \ldots, A_n$ of $G$ into sets of positive Haar measure such that

$$\forall t \in G \quad \sum_{i=1}^{n} |A_i| \operatorname{Osc\,ess}(F_t \mid A_i) < \varepsilon.$$

This means exactly that $\lim_\alpha \operatorname{diam} F * W_\alpha(1) = 0$, hence by Corollary II.16 that $F$ is Riemann integrable.



The function $F$ in Proposition IV.8 thus gives an example of a set $(F_t)_{t\in\boldsymbol{T}}$ which is of small oscillation but not w.s.p. in $L^\infty(\boldsymbol{T})$. Another example of such a set was given in [GGMS, Example VIII.12].

We will now consider examples where $(F_t)_{t\in G}$ is w.s.p. We recall some facts which will be useful to establish the next two results.

Let $F \in L^\infty(G)$. We identify $L^\infty(G)$ with $C(\mathscr{S})$. A sequence $(F_{-t_n})_{n\geqslant 1}$ is weak Cauchy in $L^\infty(G)$ iff

$$\forall h \in \mathscr{S}_0\ \ \forall t \in G \quad F_{-t_n}(h_t) = F(h_{t+t_n}) \text{ converges}.$$

If $t_n \to 0$ every limit point of $h_{t+t_n}$ belongs to $\mathscr{S}_0$. If $t$ is a point of continuity for $F$, i.e. if $t \notin \Delta_F$,

$$\forall h, h' \in \mathscr{S}_0 \quad F(h_t) = F(h'_t) = F(t).$$

Hence if $t_n \to 0$ and $t \notin \Delta_F$,

$$\forall h \in \mathscr{S}_0 \quad F(h_{t+t_n}) \to F(t).$$

In order to prove that $(F_t)_{t\in G}$ is w.s.p. we can restrict ourselves to sequences $(F_{t_n})_{n\geqslant 1}$ such that $t_n \to 0$: indeed, if $(t_n)_{n\geqslant 1}$ is a sequence in $G$, a subsequence $(t_{n_k})_{k\geqslant 1}$ tends to $t_0$ and if $(F_{t_{n_k}-t_0})_{k\geqslant 1}$ has a weak Cauchy subsequence, so does $(F_{t_{n_k}})_{k\geqslant 1}$.

Proposition IV.10. *Let $F \in L^\infty(G)$ be such that $\Delta_F$ is finite. Then $(F_t)_{t\in G}$ is w.s.p.*

Proof. Assume $t_n \neq 0$ $(n \geqslant 1)$ and $t_n \to 0$. We have already proved above that $(F_{-t_n})_{n\geqslant 1}$ converges pointwise on $\{h_t \mid h \in \mathscr{S}_0,\ t \in \Delta_F^c\}$. Let $t \in \Delta_F$. Hence $t+t_n$ belongs to $\Delta_F^c$ if $n$ is large enough and $F(t+t_n) = F(h_{t+t_n})$ for any $h \in \mathscr{S}_0$. There exists a subsequence $(t_{n_k})_{k\geqslant 1}$ such that $F(t+t_{n_k})$ converges for every $t \in \Delta_F$, hence $(F_{-t_{n_k}})_{k\geqslant 1}$ is weak Cauchy. ∎

We now give another kind of example:

Proposition IV.11. *Let $I^2 = [-\frac{1}{2}, \frac{1}{2}]^2$ and $D \subset I^2$ the closed disk centered at the origin with radius $4^{-1}$. Let $F = 1_D$ in $L^\infty(I^2)$ and let $\tilde{F}$ be the canonical image of $F$ in $L^\infty(\boldsymbol{T}^2)$. Then $(\tilde{F}_t)_{t\in\boldsymbol{T}^2}$ is w.s.p.*

Proof. *First step.* By the observation above we only have to show that $(\tilde{F}_{t_n})_{n\geqslant 1}$ has a weak Cauchy subsequence in $L^\infty(\boldsymbol{T}^2)$ when $t_n \to 0$ in $\boldsymbol{T}^2$. Let $J$ be the canonical isomorphism $L^\infty(I^2) \to L^\infty(\boldsymbol{T}^2)$. If $t_n \to 0$ there exists $x_n \to 0$ in $I^2$ such that $J(F_{x_n}) = \tilde{F}_{t_n}$. Hence it is sufficient to show that $(F_{x_n})_{n\geqslant 1}$ has a weak Cauchy subsequence in $L^\infty(I^2)$ when $x_n \to 0$. Moreover, if $\boldsymbol{R}^2$ is provided with an orthonormal basis $(0, \vec{i}, \vec{j})$ we can assume that $\theta_n = (\overrightarrow{0x_n}, \vec{i}) \to \theta$.

*Second step.* Denote by $\mathscr{B}(I^2)$ the algebra of first Baire class functions on $I^2$, provided with the norm $\|f\| = \sup_{x \in I^2} |f(x)|$. Consider $f = 1_D \in \mathscr{B}(I^2)$; $f$ is continuous outside the boundary $\Delta$ of $D$. Let $\Delta_1$, $\Delta_2$ be the two open arcs of $\Delta$ with endpoints $a$, $b$ where the tangent to the circle has direction $\theta$. Then:

($\alpha$) $\forall x \in \Delta^c$ $f$ is continuous at $x+x_n$ and $f$ is constant on the sequence $(x+x_n)$ if $n$ is large enough.

($\beta$) $\forall x \in \Delta_1 \cup \Delta_2$ $f$ is continuous at $x+x_n$ if $n$ is large enough,

$$\forall x \in \Delta_1 \quad f(x+x_n) = 1 \text{ if } n \text{ is large enough},$$

$$\forall x \in \Delta_2 \quad f(x+x_n) = 0 \text{ if } n \text{ is large enough}.$$

($\gamma$) Let $x = a$. Either there exists $(x_{n_k})_{k \geq 1}$ such that $(a+x_{n_k})_{k \geq 1} \in \mathring{D}$ or $(a+x_{n_k})_{k \geq 1} \in \bar{D}^c$. Then $f$ is continuous at $a+x_{n_k}$ and $f$ is constant on this sequence, hence $f(a+x_{n_k})$ converges. Or there exists $(x_{n_k})_{k \geq 1}$ such that $(a+x_{n_k})_{k \geq 1} \in \Delta_1$ or $(a+x_{n_k})_{k \geq 1} \in \Delta_2$. Assume $(a+x_{n_k})_{k \geq 1} \in \Delta_1$. Let $(u_l)_{l \geq 1}$ be such that $u_l \to 0$ in $I^2$ and $(\overrightarrow{0u^l}, \vec{i}) \to \theta'$. If $k$ is large enough all $a+x_{n_k}$ lie on the same arc of $\Delta$ with endpoints $c$, $d$ where the tangent has direction $\theta'$. By ($\beta$) applied to $x = a+x_{n_k}$, $\lim_l f(a+x_{n_k}+u_l)$ exists and does not depend on $k$.

*Third step.* Let $R$ be the canonical map $\mathscr{B}(I^2) \to L^\infty(I^2)$. As $F_{-x_n} = R(f_{-x_n})$ it is sufficient to show that $(f_{-x_n})_{n \geq 1}$ has a weak Cauchy subsequence in $\mathscr{B}(I^2)$. Let $\mathscr{T}$ be the spectrum of $\mathscr{B}(I^2)$ and $\mathscr{T}_0 = \{h \in \mathscr{T} \mid$ the restriction of $h$ to $C(I^2)$ is $\delta_0\}$. It is clear that $I^2$ is dense in $\mathscr{T}$, and that if $f$ is continuous at $x$,

$$\forall h \in \mathscr{T}_0 \quad f(h_x) = f(x).$$

Hence by ($\alpha$), ($\beta$), for every $h \in \mathscr{T}_0$, $f_{-x_n}(h_x)$ converges if $x \neq a, b$. By ($\gamma$) (first case), for every $h \in \mathscr{T}_0$, $f_{-x_{n_k}}(h_a)$ converges.

In the second case of ($\gamma$) consider the separable subalgebra of $\mathscr{B}(I^2)$ spanned by $(f_{-a-x_{n_k}})_{k \geq 1}$. For every $h \in \mathscr{T}_0$ there exists a sequence $(u_l)_{l \geq 1}$ such that $u_l \to 0$ in $I^2$ and

$$\forall k \geq 1 \quad f_{-x_{n_k}}(a+u_l) = f(a+x_{n_k}+u_l) \to f(h_{a+x_{n_k}}) = f_{-x_{n_k}}(h_a) \quad \text{as } l \to +\infty.$$

By ($\gamma$), $f_{-x_{n_k}}(h_a)$ does not depend on $k$ if $k$ is large enough, hence it converges. The case $x = b$ is similar to the case $x = a$. We have proved the claim. ∎

Remark IV.12. The same kind of proof would work for $F = 1_D$ where $D$ is a (not necessarily convex) set in $I^2$ whose boundary is regular enough (e.g. $D$ a square).



Proposition IV.13. *Let $F \in L^\infty(G)$. Then:*

(a) *$F * P_a(G)$ contains an $l^1$-sequence iff $(F_t)_{t \in G}$ contains an $l^1$-sequence which is w*-dense in itself in $L^\infty(G)$.*

(b) *$F * P(G)$ contains an $l^1$-sequence iff $(F_t)_{t \in G}$ does.*

Proof. Proposition IV.13 is a particular case of the following two results applied to $C = F * P(G)$ and $\operatorname{Extr} C = (F_t)_{t \in G}$: either $F * P_a(G)$ is w.s.p. and Proposition IV.14 below applies, or $F * P_a(G)$ is not w.s.p. and Proposition IV.15 below applies. We denote by $\operatorname{Extr} C$ the set of extreme points of a convex set $C$.

Proposition IV.14. *Let $X$ be a separable Banach space and $C$ a w*-compact convex subset of $X'$ such that every bounded sequence in $X$ has a subsequence which converges pointwise on $C$. Then $C$ is the norm closed convex hull of $\operatorname{Extr} C$ and $C$ contains an $l^1$-sequence iff $\operatorname{Extr} C$ does.*

The first assertion is proved in [H, Proposition 3.1], the second in [$R_2$, Addendum]. See also [GGMS, Theorem VI.6 and Proposition VI.7].

Proposition IV.15. *Let $X$ be a separable Banach space and $C$ a w*-compact set in $X'$. The following assertions are equivalent:*

(a) *$C$ contains an $l^1$-sequence $(x'_n)_{n \geq 1}$ which is w*-dense in itself.*

(a') *$\operatorname{Extr} C$ contains an $l^1$-sequence $(x'_n)_{n \geq 1}$ which is w*-dense in itself.*

(b) *There exists a bounded sequence in $X$ such that no subsequence converges pointwise on $C$.*

(b') *There exist a bounded sequence in $X$ and a w*-compact set $K$ in $\operatorname{Extr} C$ such that no subsequence converges pointwise on $K$.*

(c) *There exist a bounded sequence $(f_k)_{k \geq 1}$ in $X$ and $\delta > 0$ such that*

$$\forall a_1, \ldots, a_K \in \mathbf{R} \quad \sup_n \Big|\sum_{k=1}^{K} a_k \langle f_k, x'_n \rangle\Big| \geq \delta \sum_{k=1}^{K} |a_k|.$$

Proof. (a')$\Rightarrow$(a) is obvious; (a)$\Rightarrow$(c) is proved in [$St_1$, Theorem II]; (c)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(b') is proved in [$St_2$, Lemma 6.54 and Theorem 6.46]. (In the particular case of Proposition IV.13 this is just the dominated convergence theorem.)

(b')$\Rightarrow$(a'). The proof is a modification of the argument in [$R_2$, Theorem 2 and Proposition 1]. By assumption and [$R_2$, Theorem 2] there exists a bounded sequence $(f_k)_{k \geq 1}$ in $X$ which converges pointwise to a limit $f$ on a countable subset $L \subset K$ dense in itself and there exist $r$, $\delta > 0$ such that for every relatively open subset $U$ in $L$ there exist $y, z \in U$ such that $f(y) > r+\delta$ and $f(z) < r$. For $k \geq 1$ let $(\theta_{k,j})_{j \leq 2^k}$ be an enumeration of $\{+1, -1\}^k$ and let

$\theta_{k,j}(i)$ be the $i$th coordinate. For every $k \geqslant 1$ we can construct pairwise disjoint open sets $(V_{k,i})_{i \leqslant 2^k}$ in $K$ such that

$$V_{k,2i-1} \cup V_{k,2i} \subset V_{k-1,i} \quad (i \leqslant 2^{k-1}),$$

the diameter of each $V_{k,i}$ is less than $2^{-k}$, the trace of each $V_{k,i}$ on $L$ is not empty, and for every $\theta_{k,j}$ $(j \leqslant 2^k)$ there exists $f_{k,j} \in (f_k)_{k \geqslant 1}$ such that for every $i \leqslant 2^k$

$$f_{k,j} > r+\delta \quad \text{on } V_{k,i} \text{ if } \theta_{k,j}(i) = +1,$$

$$f_{k,j} < r \quad \text{on } V_{k,i} \text{ if } \theta_{k,j}(i) = -1.$$

Now define

$$H = \bigcap_{k \geqslant 1} \bigcup_{i=1}^{2^k} \bar{V}_{k,i}.$$

By a standard argument one verifies, using the $(f_{k,j})$, that

$$\forall x'_1, \ldots, x'_N \in H \;\; \forall a_1, \ldots, a_N \in R \quad \sup_{k \geqslant 1} \Big| \sum_{n=1}^{N} a_n \langle f_k, x'_n \rangle \Big| \geqslant \tfrac{1}{2}\delta \sum_{n=1}^{N} |a_n|. \;\blacksquare$$

COROLLARY IV.16. *Let $X$ be a separable Banach space and $C$ a $w^*$-compact convex set in $X'$. Then $C$ contains an $l^1$-sequence iff* $\operatorname{Extr} C$ *does.*

We had first proved the particular case of Proposition IV.13 and we are indebted to C. Stegall for having pointed out to us the above general result.

**Appendix**

THEOREM. *Let $X$, $Y$ be Banach spaces, $D \subset Y$ a closed bounded convex set, $\alpha > 0$, and $U$: $Y \to X$ a bounded operator such that:*

(a) *For every slice $S$ of $D$*

$$\operatorname{diam} U(S) > 2\alpha.$$

(b) *For every slice $S$ of $D$ and every $\varepsilon > 0$ there exist slices $S_1, \ldots, S_n$ in $S$ and $0 < \alpha_i$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and*

$$\operatorname{diam} U\Big(\sum_{i=1}^{n} \alpha_i S_i\Big) < \varepsilon.$$

*Then $\overline{U(D)}^{\|\,\|}$ contains a closed convex set without extreme points.*

This operator version of the result of $[S_1]$ requires a weaker hypothesis than in $[S_1]$: it was observed in $[R_3]$ that it is sufficient to require (b) as above for every slice $S$ instead of every closed convex subset of $D$. The proof follows exactly the same lines as in $[S_1]$, hence we will only sketch the necessary modifications.



By [GGMS, Proposition III.9] we may as well assume that $D$ is separable. The theorem follows from the following proposition just as Theorem 2.1 follows from Proposition 2.6 in $[S_1]$:

PROPOSITION. *Under the assumptions of the theorem above there is an operator $T$: $L^1[0,1] \to X$ such that $T(P_a[0,1]) \subset \overline{U(D)}$ and assertions* (ii), (iii) *of* $[S_1$, *Proposition* 2.6] *are satisfied.*

The proof of the proposition is parallel to that of $[S_1, 2.6]$, and we use the same notation. $(x_i)_{i=1}^{\infty}$ is now a dense sequence in the linear span of $U(D)$. All slices are slices of $D \subset Y$. In all formulas involving diameter or distance the slices will be replaced by their images under $U$. With this modification the proof carries over verbatim. Finally, choose the $x_\omega$ at the end of the proof to be arbitrary elements of $U(T_\omega)$.

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# Some properties of weakly countably determined Banach spaces

by

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**Abstract.** Let $Y$ be a closed subspace of a Banach space $X$. If $Y^{**}/Y$ is separable and $X/Y$ is weakly compactly generated, then $X$ is also weakly compactly generated. Analogous results are obtained with "weakly compactly generated" replaced by "weakly $K$-analytic" and also by "weakly countably determined".

The vector spaces we use here are over the field of real or complex numbers. $N$ denotes the set of natural numbers. Our notations are standard. If $(x_n)$ is a sequence in $X$, $[x_n]$ will stand for the closed linear hull of $(x_n)$. Given a subset $A$ of $X$, $\tilde{A}$ will denote its weak-star closure in $X^{**}$; if $A$ is absolutely convex, i.e. convex and circled, and also closed and bounded, we shall write $X_A$ for the Banach space on the linear hull of $A$ with $A$ as its closed unit ball. Given $x$ in $X$ and $u$ in $X^*$, we shall write $\langle x, u\rangle$ instead of $u(x)$. If $P$ is a continuous projection on $X$, $P^*$ denotes the conjugate projection on $X^*$.

A Banach space $X$ is said to be *weakly compactly generated* whenever there exists a weakly compact set $K \subset X$ such that the linear span of $K$ is dense in $X$. In particular, every separable or reflexive Banach space is weakly compactly generated.

A Banach space $X$ is said to be *weakly $K$-analytic* (respectively, *weakly countably determined*) whenever there exists a Polish topological space (respectively, a metrizable and separable topological space) $F$ and a mapping $T$ from $F$ into the family of weakly compact subsets of $X$ such that

$$X = \bigcup \{Tu:\ u \in F\}$$

with the following property: whenever $(x_n)$ is a sequence in $F$ converging to $x_0$ and $U$ a weakly open neighbourhood of $Tx_0$, there exists a positive integer $n_0$ such that $Tx_n \in U$, $n \geq n_0$.

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