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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Nonlinear generalizations of the Banach–Stone theorem

by

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Abstract. Let X, Y be locally compact sets. Assume there is a map T from $C_0(X)$ onto $C_0(Y)$ with $(1-\varepsilon)\|f-g\| \leqslant \|Tf-Tg\| \leqslant (1+\varepsilon)\|f-g\|$, for $f, g \in C_0(X)$, where ε is sufficiently small. Then $C_0(X)$ and $C_0(Y)$ are linearly isometric. Similar results hold for function algebras, extremely regular function spaces, and certain spaces of vector-valued functions.

§ 0. Introduction. In this paper we study the following problem:

Does there exist a universal constant $\varepsilon > 0$ such that if T is a map from a Banach space A onto a Banach space B which satisfies

$$(*)\qquad (1-\varepsilon)\|f-g\| \leqslant \|Tf-Tg\| \leqslant (1+\varepsilon)\|f-g\|, \quad \text{for } f, g \in A,$$

then

(1) A and B are linearly homeomorphic,

(2) if A and B are subspaces of $C_0(X)$ and $C_0(Y)$, respectively, then X and Y are homeomorphic?

We prove that the answer to the first problem is positive if both A and B are $C_0(X)$ spaces. The answer to the second problem is evidently negative, in general, since any Banach space can be represented in a number of ways as a subspace of a space $C(X)$, with different X. However, we prove here that the answer to this question is positive if A, B are extremely regular function spaces.

The source of our problem is the classical Banach–Stone theorem which states that the Banach spaces $C(X)$ and $C(Y)$ are linearly isometric if and only if X and Y are homeomorphic. In the sixties Amir [2] and Cambern [6, 7] proved that this result is stable: if there is a linear homeomorphism T from $C(X)$ onto $C(Y)$ with $\|T\|\|T^{-1}\| < 2$ then $C(X)$ and $C(Y)$ are actually linearly isometric. During the next years, linear isomorphisms with a small bound were studied in a number of papers; see, e.g., [3–4, 8–9, 11–16, 20].

The assumption about small bound is essential. For example, the Banach space $C[0, 1]$ is linearly homeomorphic to $C(X)$, for any compact, metric, uncountable space X. In the nonlinear case we even have the more far-reaching, well-known result of Kadec [18] that any two separable infinite-dimensional Banach spaces are homeomorphic. The situation changes significantly if we consider uniform homeomorphisms. For example, if a Banach space E is uniformly homeomorphic to a Hilbert space H, then E and H are linearly homeomorphic [19]. Hence the answer to our first problem is positive if A is a Hilbert space. On the other hand, in 1977 Aharoni and Lindenstrauss [1] gave an example of two Banach spaces $C_0(X)$ and $C_0(Y)$ which are Lipschitz equivalent but not linearly homeomorphic. This means that the assumption that ε is small is essential also in the first problem.

§ 1. **The result.** We use the standard Banach space terminology. For a closed subspace A of $C_0(X)$ we denote by $\operatorname{Ch} A$ the set of all points $x_0 \in X$ such that for any $\varepsilon > 0$ and any neighborhood U of x_0 there is an $f \in A$ with $\|f\| = f(x_0) = 1$ and $|f(x)| < \varepsilon$ for $x \in X - U$. A is called an *almost extremely regular subspace* of $C_0(X)$ if $\operatorname{Ch} A$ is dense in X and is called *extremely regular* if $\operatorname{Ch} A = X$. The main examples of almost extremely regular subspaces are function algebras. By a *function algebra* we mean any closed subalgebra of $C_0(X)$ which separates points of X. By ∂A we denote the Shilov boundary of A and $\operatorname{Ch} A$ defined above coincides with the Choquet boundary of A. A function algebra A is an almost extremely regular subspace of $C(\partial A)$ and it is extremely regular if $\operatorname{Ch} A = \partial A$.

A map T between Banach spaces A and B is called *ε-bi-Lipschitz* if the condition $(*)$ is satisfied.

All our results hold both in the real and in the complex cases.

THEOREM 1. *Let X, Y be locally compact Hausdorff spaces, let A be an almost extremely regular subspace of $C_0(X)$ and B an extremely regular subspace of $C_0(Y)$. Assume there is an ε-bi-Lipschitz map T from A onto B with $T0 = 0$ and with $\varepsilon \leqslant \varepsilon_0$. Then there is a homeomorphism φ from X onto Y and*

$$(**) \qquad \big\| |(Tf)\circ\varphi| - |f| \big\| \leqslant c(\varepsilon)\|f\|, \qquad \forall f \in A,$$

where ε_0 is an absolute constant and $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We postpone the proof of this result to the next section. Now we get two corollaries.

THEOREM 2. *Let X, Y, T, A, B, φ be as in Theorem 1 and assume X is paracompact. Then there is a scalar-valued continuous function u defined on X, of modulus one, such that:*

(a) *if the spaces A and B are real, then*

$$(1) \qquad \|(Tf)\circ\varphi - u\cdot f\| \leqslant c'(\varepsilon)\|f\|, \qquad \forall f \in A;$$

(b) *if A, B are complex, then $X = X_1 \cup X_2$ and for any $f \in A$ with $\|f\| \leqslant 1$ we have*

$$(2) \qquad |Tf(\varphi(x)) - u(x)f(x)| \leqslant c'(\varepsilon), \qquad \forall x \in X_1,$$

$$(3) \qquad |Tf(\varphi(x)) - u(x)\overline{f(x)}| \leqslant c'(\varepsilon), \qquad \forall x \in X_2,$$

where $c'(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Note that (1) and (2)–(3) are formulated in different ways. In (1) and $(**)$ we have "$\ldots \leqslant c(\varepsilon)\|f\|$, $\forall f \in A$," while in (2) and (3) we have "$\ldots \leqslant c(\varepsilon)$, $\forall f \in A$ with $\|f\| \leqslant 1$". The first type of statement is stronger, in general, since T is nonlinear.

Assume we can prove Theorem 1 with the weaker statement

$$\big\| |(Tf)\circ\varphi| - |f| \big\| \leqslant c(\varepsilon), \qquad \forall f \in A,\ \|f\| \leqslant 1,$$

in place of $(**)$. If T is an ε-bi-Lipschitz map, then for any $t > 0$ the map

$$A \ni f \to (1/t)\,T(tf) \in B$$

is also ε-bi-Lipschitz. Hence for any $t > 0$ there is a homeomorphism $\varphi_t\colon X \to Y$ such that

$$\big\| |(Tf)\circ\varphi_t| - |f| \big\| \leqslant tc(\varepsilon), \qquad \forall f \in A,\ \|f\| \leqslant t.$$

Hence

$$(4) \qquad \big\| |(Tf)\circ\varphi_{t'}| - |(Tf)\circ\varphi_{t''}| \big\| \leqslant (t'+t'')c(\varepsilon), \qquad \forall f \in A,\ \|f\| \leqslant \min(t', t'').$$

Using (4) and the regularity of B it is easy to note that the map $\boldsymbol{R}^+ \ni t \to \varphi_t$ is locally constant, so it is constant.

The above means that in Theorem 1 our two types of equation are equivalent. The same situation is in the first part of Theorem 2 because two unimodular real-valued functions are identical or far from each other (with the distance equal to 2). A problem arises in the complex case. From (2) we deduce that for any $t > 0$ there is a unimodular function u_t such that

$$|Tf(\varphi(x)) - u_t(x)f(x)| \leqslant tc(\varepsilon), \qquad \forall f \in A,\ \|f\| \leqslant t,\ \forall x \in X,$$

but now u_t may depend on t. This may happen even if X and Y are just one-point sets [17]. To deduce Theorem 2 from Theorem 1 and then to prove Theorem 1, we need the following proposition, which is an immediate consequence of Proposition 2 of [10] (put, in Proposition 2, $f := T$, $x := 2f$, $y := 2g$ and also $x := 2f$, $y := 0$).

PROPOSITION. *Let T be an ε-bi-Lipschitz map from a Banach space E onto a Banach space F, with $\varepsilon < 1/3$ and $T0 = 0$. Then for any $f, g \in E$ with $\|f\|, \|g\| \leqslant 2$ we have*

$$\|(Tf + Tg) - T(f+g)\| \leqslant \varepsilon'$$

and

(G) $$\|f+g\|-\varepsilon' \leqslant \|Tf+Tg\| \leqslant \|f+g\|+\varepsilon',$$

where $\varepsilon' = 100\varepsilon^{1/10}$.

Proof of Theorem 2. We define a carrier $F: X \to 2^B$ by

$$F(x) = \{g \in B: \|g\| = 1 = g(\varphi(x))\}$$

Since B is extremely regular $F(x)$ is nonempty for any $x \in X$. It is also easy to check that F is a norm lower semicontinuous, convex, complete carrier, so by the Michael Selection Theorem [5] there is a norm continuous function $F: X \to B$ such that $\|F(x)\| = 1 = F(x)(\varphi(x))$ for all $x \in X$. We define v on X by

$$v(x) = T^{-1}(F(x))(x), \quad \text{for } x \in X.$$

The map v is continuous and by (**)

(5) $$\big|\,|v(x)|-1\big| \leqslant c(\varepsilon)(1+2\varepsilon), \quad \forall x \in X.$$

Put $\tilde{A} = A/v := \{f/v: f \in A\}$ and define $\tilde{T}: \tilde{A} \to B$ by $\tilde{T}(f/v) = T(f)$. The space $\tilde{A}$ is an almost extremely regular subspace of $C_0(X)$ and $\tilde{T}$ is an $(\varepsilon + 2c(\varepsilon))$-bi-Lipschitz map. By the definition of $\tilde{T}$ and (5) it is sufficient to prove Theorem 2 for $\tilde{T}$ and $\tilde{A}$ in place of T and A. We have also $\tilde{T}^{-1}(F(x))(x) \equiv 1$. Hence to simplify notation we can just assume that $v \equiv 1$, $X = Y$, $\varphi = \mathrm{id}$ and that

$$\|\,|Tf|-|f|\,\| \leqslant \varepsilon, \quad \forall f \in A,\ \|f\| \leqslant 3.$$

Fix $x \in X$. For any $f \in A$ with $\|f\| \leqslant 1$ we have

(6) $$\big\|\,|T(f+T^{-1}(F(x)))|-|f+T^{-1}(F(x))|\,\big\| \leqslant \varepsilon.$$

By the Proposition and since $\varphi(x) = x$ we have also

$$\big|\,|1+Tf(x)|-|T(f+T^{-1}(F(x)))(x)|\,\big|$$
$$\leqslant \|(F(x)+Tf)-T(f+T^{-1}(F(x)))\| \leqslant \varepsilon'.$$

Hence evaluating (6) at the point x we get

$$\big|\,|1+Tf(x)|-|1+f(x)|\,\big| \leqslant \varepsilon+\varepsilon'.$$

Hence, since $|f(x)| \leqslant 1$ and $|Tf(x)| \leqslant 1+\varepsilon$, by a direct computation we get, in the real case,

$$|Tf(x)-f(x)| \leqslant 2\varepsilon+\varepsilon',$$

and in the complex case:

(a) $|Tf(x)-f(x)| \leqslant 2\varepsilon+\varepsilon'$, or

(b) $|Tf(x)-\overline{f(x)}| \leqslant 2\varepsilon+\varepsilon'$.

Now to end the proof we have to show that in the complex case, for any $x \in X$, we have always $Tf(x) \approx f(x)$ or always $Tf(x) \approx \overline{f(x)}$ independently of f. Assume that

(7) $$T(iF(x))(x) \approx iF(x)(x) = i,$$

and let $f \in A$ with $\|f\| \leqslant 1$. Assume $Tf(x) \approx \overline{f(x)}$. Hence, by the previous result and the Proposition, we get

$$f(x)+i = f(x)+iF(x)(x) \approx T(f+iF(x))(x) \approx Tf(x)+i$$

or

$$f(x)+i = f(x)+iF(x)(x) \approx \overline{T(f+iF(x))(x)}$$
$$\approx \overline{Tf(x)}-i \approx f(x)-i.$$

The latter is impossible, so (7) implies $Tf(x) \approx f(x)$ for all $f \in A$. If $T(iF(x))(x) = -i$, then by exactly the same argument we get $Tf(x) \approx \overline{f(x)}$ for all $f \in A$.

Corollary. *Let A, B be uniform algebras such that $\partial B = \mathrm{Ch}\, B$. Assume that there is an ε-bi-Lipschitz map from A onto B, with $\varepsilon \leqslant \varepsilon_0$ (absolute constant). Then ∂A and ∂B are homeomorphic. Furthermore, if $B = C(\partial B)$ then $A = C(\partial A)$.*

§ 2. **Proof of Theorem 1.** To prove the theorem we first need some notation. For x_0 in X a net $(f_\alpha)_{\alpha \in \Lambda} \subset C_0(X)$ is called *peaking* at x_0 if:

(i) for any $\alpha \in \Lambda$, $\|f_\alpha\| = 1 = f_\alpha(x_0)$, and

(ii) $f_\alpha \to 0$ uniformly off any neighborhood of x_0.

We denote by $P_A^1(x_0)$ the set of all nets $(f_\alpha)_{\alpha \in \Lambda}$ in A such that $(|f_\alpha|)_{\alpha \in \Lambda}$ peaks at x_0 and by $P_A^2(x_0)$ the subset of $P_A^1(x_0)$ consisting of nets which peak at x_0.

Fix $M \geqslant 0$. For any $x_0 \in X$ and $i = 1, 2$ we define

$$S_{x_0}^i = \{y \in Y: \exists (f_\alpha)_{\alpha \in \Lambda} \in P_A^i(x_0)\ \exists (y_\alpha)_{\alpha \in \Lambda} \subset Y,$$
$$y_\alpha \to y \text{ and } \forall \alpha \in \Lambda\ |Tf_\alpha(y_\alpha)| \geqslant M\}.$$

Evidently $S_{x_0}^2 \subset S_{x_0}^1$.

We divide the proof into a number of simple steps and at various points of the proof we use inequalities involving ε which are valid only if ε is sufficiently small; in these circumstances we will merely assume that ε is near zero. These assumptions are the source of ε_0.

The strategy of the proof is the following: Steps 1 and 2 show that, for a suitable M and any $x \in \mathrm{Ch}\, A$, the sets S_x^1, S_x^2 coincide and contain exactly one point. This gives a function φ: $\mathrm{Ch}\, A \to Y$ defined by $\{\varphi(x)\} = S_x^1 = S_x^2$. Steps 3 and 4 show that

$$|Tf(\varphi(x))| \leqslant |f(x)| + o(\varepsilon_0),$$

and Steps 5,6 prove that

$$|Tf(\varphi(x))| \geqslant |f(x)| - o(\varepsilon),$$

which together give (**). In the last four steps we show that φ is continuous and injective and can be extended to a continuous and still injective map from X onto Y. The main tools of the proof are the Proposition, which says that T is "almost" additive, and the peaking functions, which, as we show, are mapped to "almost" peaking functions.

Step 1. *If $M \leqslant 1-\varepsilon-\varepsilon'$, then for any $x_0 \in \mathrm{Ch}\, A$ we have $S_{x_0}^2 \neq \emptyset$.*

Proof. If Y is compact, this is an immediate consequence of (G) with $g = 0 = Tg$. To consider the general case, let $(f_\alpha)_{\alpha\in\Lambda} \in \boldsymbol{P}_A^2(x_0)$. Fix $\alpha_0 \in \Lambda$. For any $\alpha \in \Lambda$ we have $\|f_\alpha + f_{\alpha_0}\| = 2$, so by (G), $\|Tf_\alpha + Tf_{\alpha_0}\| \geqslant 2-\varepsilon'$. On the other hand, $\|Tf_\alpha\| \leqslant 1+\varepsilon$, so we get

$$\sup\{|Tf_\alpha(y)|:\ y \in Y,\ |Tf_{\alpha_0}(y)| \geqslant 1-\varepsilon\} \geqslant 1-\varepsilon-\varepsilon'.$$

The set $\{y \in Y:\ |Tf_{\alpha_0}(y)| \geqslant 1-\varepsilon\}$ is compact, so its intersection with $S_{x_0}^2$ is not empty.

Step 2. *If $M \geqslant \sqrt{2}/2+2\varepsilon'$, then for any $x_0 \in \mathrm{Ch}\, A$ the set $S_{x_0}^1$ has at most one point.*

Proof. Assume y^1, y^2 are two distinct points of $S_{x_0}^1$ and let $(f_\alpha^i)_{\alpha\in\Lambda_i} \in \boldsymbol{P}_A^1(x_0)$, $(y_\alpha^i)_{\alpha\in\Lambda_i} \subset Y$, $i = 1, 2$, be the corresponding nets given by the definition of $S_{x_0}^1$. Without loss of generality we can assume that $Tf_\alpha^i(y_\alpha^i) \xrightarrow[\alpha\in\Lambda_i]{} \lambda^i$ with $|\lambda^i| \geqslant M$ for $i = 1, 2$. Since B is extremely regular there are $g_1, g_2 \in B$ such that

$$g_i(y^i) = \lambda^i/|\lambda^i| \quad \text{for } i = 1, 2, \qquad \|\,|g_1|+|g_2|\,\| \leqslant 1+\varepsilon'/3.$$

We have

$$\liminf_\alpha \|Tf_\alpha^i + g_i\| \geqslant 1+M, \qquad i = 1, 2;$$

hence, by the Proposition,

$$\liminf_\alpha \|f_\alpha^i + T^{-1}g_i\| \geqslant 1+M-\varepsilon', \qquad i = 1, 2.$$

Hence, by the definition of the peaking sequence, $|T^{-1}g_i(x_0)| \geqslant M-\varepsilon'$ for $i = 1, 2$, and we get

$$\max(\|T^{-1}g_1 + T^{-1}g_2\|,\ \|T^{-1}g_1 - T^{-1}g_2\|) \geqslant \sqrt{2}(M-\varepsilon')$$

(in the real case we even have $\max(\ldots) \geqslant 2(M-\varepsilon')$). Now, by the Proposition or just by (*) and since $M > \sqrt{2}/2+2\varepsilon'$, we get

$$\max(\|g_1+g_2\|,\ \|g_1-g_2\|) \geqslant \sqrt{2}(M-\varepsilon')-\varepsilon' > 1+\varepsilon'/3,$$

which contradicts the assumption $\|\,|g_1|+|g_2|\,\| \leqslant 1+\varepsilon'/3$.

In the remaining part of the proof we assume that $\sqrt{2}/2+2\varepsilon' < M = 1-\varepsilon-\varepsilon'$, and we define φ: $\mathrm{Ch}\, A \to Y$ by $\{\varphi(x)\} = S_x^1$.

We denote by Γ the set of all scalars of modulus one; this means Γ is a two-point set or a circle.

Step 3. *Fix $x_0 \in \mathrm{Ch}\, A$ and $f_0 \in A$ with $\|f_0\| = 1 = f_0(x_0)$. For any $\lambda \in \Gamma$ we define $\chi(\lambda) = T(\lambda f_0)(\varphi(x_0))$. We have:*

(i) $\forall \lambda \in \Gamma \quad |\chi(\lambda)| \geqslant M$.

(ii) $\{\chi(\lambda)/|\chi(\lambda)|:\ \lambda \in \Gamma\} = \Gamma$.

Proof. Let $(f_\alpha)_{\alpha\in\Lambda} \in \boldsymbol{P}_A^1(x_0)$. As in Step 1 we have $\|\lambda f_\alpha + \lambda f_0\| = 2$, $\forall \alpha \in \Lambda$, hence

$$\|T(\lambda f_\alpha) + T(\lambda f_0)\| \geqslant 2-\varepsilon', \qquad \forall \alpha \in \Lambda,$$

and therefore

$$\sup\{|T(\lambda f_0)(y)|:\ y \in Y,\ |T(\lambda f_\alpha)(y)| \geqslant 1-\varepsilon\} \geqslant 1-\varepsilon'-\varepsilon.$$

It follows that there is a net (y_α) such that $|T(\lambda f_\alpha)(y_\alpha)| \geqslant 1-\varepsilon$ and $|T(\lambda f_0)(y_\alpha)| \geqslant 1-\varepsilon'-\varepsilon$. Hence, by the definition of $S_{x_0}^1$, we have

$$\sup\{|T(\lambda f_0)(y)|:\ y \in S_{x_0}^1\} \geqslant 1-\varepsilon'-\varepsilon,$$

so, since $S_{x_0}^1 = \{\varphi(x_0)\}$, we get $|\chi(\lambda)| \geqslant M$.

To prove (ii), note that by our assumption (*), χ is continuous, and $|\chi(\lambda) + \chi(-\lambda)| \leqslant \varepsilon'$; hence

$$\tilde{\chi}:\ \Gamma \ni \lambda \to \chi(\lambda)/|\chi(\lambda)| \in \Gamma$$

is a continuous function such that $|\tilde{\chi}(\lambda)+\tilde{\chi}(-\lambda)| \leqslant 2(1-1/M)+\varepsilon' < 1$, so $\tilde{\chi}$ is surjective.

Step 4. *For any $f_0 \in A$ with $\|f_0\| \leqslant 2$ and $x \in \mathrm{Ch}\, A$ we have*

$$|Tf_0(\varphi(x))| \leqslant |f_0(x)| + 2\varepsilon' + \varepsilon.$$

Proof. We have

$$1+|f_0(x)| = \inf\{\sup\{\|f_0+\lambda f\|:\ \lambda \in \Gamma\}:\ f \in A,\ \|f\| = f(x) \leqslant 1\},$$

and by Step 3.

$$M+|Tf_0(\varphi(x))| \leqslant \inf\{\sup\{\|Tf_0+T(\lambda f)\|:\ \lambda\in\Gamma\}:\ f\in A,\ \|f\|=f(x)\leqslant 1\}.$$

Hence, by the assumption that $M=1-\varepsilon-\varepsilon'$ and the Proposition,

$$|Tf_0(\varphi(x))| \leqslant |f_0(x)|+(1-M)+\varepsilon' = |f_0(x)|+2\varepsilon'+\varepsilon.$$

Step 5. *Fix $x_0\in \operatorname{Ch} A$ and $g\in B$ with $g(\varphi(x_0))=\|g\|=1$. For any $\lambda\in\Gamma$ we define $K(\lambda)=T^{-1}(\lambda g)(x_0)$. We have:*

(i) $\forall\lambda\in\Gamma\quad |K(\lambda)|\geqslant 1-2\varepsilon'-\varepsilon.$

(ii) $\{K(\lambda)/|K(\lambda)|:\ \lambda\in\Gamma\}=\Gamma.$

Proof. (i) is an immediate consequence of Step 4, and we get (ii) exactly as in the proof of (ii) in Step 3.

Step 6. *For any $f_0\in A$ with $\|f_0\|\leqslant 1+2\varepsilon$ and $x_0\in\operatorname{Ch} A$ we have*

$$|Tf_0(\varphi(x_0))| \geqslant |f_0(x_0)|-4\varepsilon'. \tag{8}$$

Proof. Since B is extremely regular, there is a $g\in B$ such that $\|g\|=1=g(\varphi(x_0))$ and

$$\|\,|Tf_0|+|g|\,\| \leqslant 1+|Tf_0(\varphi(x_0))|+4\varepsilon. \tag{9}$$

By Step 5 there is a $\lambda\in\Gamma$ such that

$$\|f_0+T^{-1}(\lambda g)\| \geqslant |f_0(x_0)|+(1-2\varepsilon'-\varepsilon).$$

Hence, by the Proposition,

$$\|Tf_0+\lambda g\| \geqslant |f_0(x_0)|+(1-2\varepsilon'-\varepsilon)-\varepsilon'. \tag{10}$$

From (9) and (10) we get (8).

Step 7. *φ can be extended to a continuous function from X into Y. We denote the extended function by the same symbol.*

Proof. Assuming the contrary, there is an $x_0\in X$ and two nets $(x_\gamma^i)_{\gamma\in\Gamma_i}$, $i=1,2$, in $\operatorname{Ch} A$, both converging to x_0, such that

$$y_\gamma^i=\varphi(x_\gamma^i)\xrightarrow[\gamma\in\Gamma_i]{} y^i,$$

where $y^1\neq y^2\in Y$ (by Step 6, if Y is not compact, $\varphi(x_\gamma)$ still cannot be divergent to ∞). In particular, to prove the continuity of the original function φ we assume that $(x_\gamma^2)_{\gamma\in\Gamma_2}$ is a sequence constantly equal to x_0. By Step 4, for any $g\in B$ with $\|g\|\leqslant 1$ we have

$$|g(y_\gamma^i)| \leqslant |T^{-1}g(x_\gamma^i)|+2\varepsilon'+\varepsilon,\quad \text{for } i=1,2,\ \gamma\in\Gamma_i,$$

hence

$$|g(y^i)| \leqslant |T^{-1}g(x_0)|+2\varepsilon'+\varepsilon,\quad \text{for } i=1,2. \tag{11}$$

Now, as in the proof of Step 2, we let $g_1, g_2\in B$ be such that $g_i(y^i)=1$ for $i=1,2$ and $\|\,|g_1|+|g_2|\,\|\leqslant 1+\varepsilon$. From (11) we get

$$|T^{-1}g_i(x_0)|\geqslant 1-2\varepsilon'-\varepsilon,\quad i=1,2.$$

Hence $\max\|T^{-1}g_1\pm T^{-1}g_2\|\geqslant\sqrt{2}(1-2\varepsilon'-\varepsilon)$, so

$$\max\|g_1\pm g_2\|\geqslant\sqrt{2}(1-2\varepsilon'-\varepsilon)-\varepsilon',$$

which contradicts the assumption $\|g_1\pm g_2\|\leqslant 1+\varepsilon$.

Step 8. *If X is noncompact then neither is Y, and φ can be extended to a continuous map from $X^*=X\cup\{\infty\}$ (the one-point compactification of X) into $Y^*=Y\cup\{\infty\}$ (the one-point compactification of Y).*

Proof. Assuming the contrary, there is a net $(x_\gamma)_{\gamma\in\Gamma}$ in X tending to ∞ such that $\varphi(x_\gamma)=y_\gamma$ tends to some point y_0 of Y. By Step 4 and since $\operatorname{Ch} A$ is dense in X, for any $g\in B$, $\|g\|\leqslant 1$, we get

$$|g(y_0)|=\lim_\gamma|g(y_\gamma)|\leqslant\lim_\gamma|T^{-1}g(x_\gamma)|+2\varepsilon'+\varepsilon=2\varepsilon'+\varepsilon<1,$$

which contradicts the assumption that B is extremely regular.

Step 9. *φ maps X onto Y.*

Proof. By the previous step φ is a closed map, so it is sufficient to show that $\varphi(X)$ is dense in Y. Assuming the contrary, there is a $g\in B$ with $\|g\|=1$ such that

$$\sup\{|g(\varphi(x))|:\ x\in X\}<\varepsilon.$$

But on the other hand, by Step 6 and the Proposition,

$$\sup\{|g(\varphi(x))|:\ x\in X\}\geqslant\sup\{|T^{-1}g(x)|:\ x\in X\}-4\varepsilon'$$
$$=\|T^{-1}g\|-4\varepsilon'\geqslant\|g\|-5\varepsilon'=1-5\varepsilon'.$$

Step 10. *φ is injective.*

Proof. Let $x_1\in X_0:=\operatorname{Ch} A$, $x_2\in X$, $x_1\neq x_2$, and let $f\in A$ be such that $\|f\|=f(x_1)=1$ and $|f(x_2)|<\varepsilon$. By Steps 3 and 4,

$$|Tf(\varphi(x_1))|\geqslant 1-\varepsilon'-\varepsilon,\qquad |Tf(\varphi(x_2))|\leqslant 2(\varepsilon'+\varepsilon).$$

Hence $\varphi(x_1)\neq\varphi(x_2)$, so in particular $\varphi|_{X_0}$ is injective.

Let us now pick one point from each set $\varphi^{-1}(y)$, $y\in Y$, i.e. let us consider a function $\psi:\ Y\to X$ such that $\varphi\circ\psi$ is the identity map. By Step 8, φ is a closed map, so ψ is continuous. To prove that φ is injective we have to prove that ψ is surjective. Since $\varphi|_{X_0}$ is injective $\psi(Y)$ contains X_0, so is a dense subset of X. Hence if Y is compact then we are done. If Y is not compact, then by Step 8, ψ can be extended to a continuous function from Y^* into X^* with $\psi(\infty) = \infty$; now the domain is again compact and so is the image.

Now by Steps 7–10, φ is a homeomorphism from X onto Y and (**) follows from Steps 4 and 6 with $c(\varepsilon) = 4\varepsilon' = 400\,\varepsilon^{1/10}$.

§ 3. **Remark.** Note that the complex Banach space $C_0(X)$ is isometric to $C_0^R(X) \otimes H_2$ (the injective tensor product of the real $C_0(X)$ space and the real two-dimensional Hilbert space) as well as to $C_0^R(X, H_2)$ (the real Banach space of H_2-valued continuous functions on X vanishing at infinity). The inspection of the proof of Theorem 1 immediately shows that we have only used the following properties of $A \subset C_0^R(X) \otimes H_2$, $B \subset C_0^R(Y) \otimes H_2$:

(i) A is almost extremely regular and B is extremely regular.

(ii) $\exists c > 1 \ \ \forall h_1, h_2 \in H_2, \ \|h_1\| = 1 = \|h_2\|$,
$$\max(\|h_1 + h_2\|, \|h_1 - h_2\|) \geqslant c.$$

(iii) The unit ball of H_2 is compact.

Hence by exactly the same arguments we can get the following more general result.

THEOREM 4. *Let X, Y be locally compact Hausdorff spaces, let A be an almost extremely regular subspace of $C_0(X)$ and B an extremely regular subspace of $C_0(Y)$, and let E be a strictly convex finite-dimensional Banach space. Assume T is an ε-bi-Lipschitz map from $A \otimes E \subseteq C_0(X, E)$ onto $B \otimes E \subseteq C_0(Y, E)$ with $T0 = 0$ and $\varepsilon \leqslant \varepsilon_0$. Then there is a homeomorphism φ: $X \to Y$ such that*
$$\big|\|Tf(\varphi(x))\| - \|f(x)\|\big| \leqslant c(\varepsilon)\|f\|, \qquad \forall f \in A \otimes E,$$
where $\varepsilon_0 > 0$ as well as the function $c(\cdot)$, with $c(\varepsilon) \to 0$ as $\varepsilon \to 0$, depend on the modulus of convexity of E only.

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