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Complemented subspaces in tame power series spaces

by

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Abstract. The class of all power series spaces $\Lambda_\infty(\alpha)$ is characterized, for which there exists a function $S(\cdot)$ such that every continuous linear operator $A\colon \Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha)$ satisfies for large k continuity estimates $\|Ax\|_k \leqslant C_k \|x\|_{S(k)}$. It turns out that then, in terms of the canonical norms, there is a number a, depending only on α, such that every A satisfies even estimates of the form (*) $\|Ax\|_k \leqslant C_k \|x\|_{ak+b}$, b depending on A. If $\Lambda_\infty(\alpha)$ is in this class then every complemented subspace of $\Lambda_\infty(\alpha)$ has a basis. This follows from an analysis of the matrix of a projection and also from the fact, proved in § 3, that in any nuclear infinite type power series space the range of any operator satisfying continuity estimates (*) admits an absolute basis.

In this paper we introduce a new property of Fréchet spaces which turns out to be useful e.g. for investigations regarding the problem of existence of bases in complemented subspaces of nuclear Fréchet spaces.

Recall that the topology in a Fréchet space (see, e.g., [3], [7] and [11] for elementary properties of such spaces and all other unexplained ideas in this paper) can be defined by an increasing sequence of seminorms $(\|\ \|_k)_k$.

If E is a Banach space, $\|\ \|$ its norm, then the continuity of a linear operator $A\colon E \to E$ is characterized by the relation

$$\|Ax\| \leqslant C\|x\| \qquad (x \in E).$$

Here C is some constant.

If E is a Fréchet space which is not a Banach space then no such relation is possible and instead we have the characterization

$$\|Ax\|_k \leqslant C_k \|x\|_{\sigma(k)} \qquad (x \in E,\ k = 1, 2, \ldots).$$

Here $\sigma\colon N \to N$ is some function on the positive integers. In this case there are examples which show that in general $\sigma(k)$ cannot be taken equal to k and it would seem that there is no possibility of putting limits on the growth of σ.

It turns out, however, that in certain cases rather severe limits can be

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put on σ. If, for instance, $E = \Lambda_1(\alpha)$ is a power series space of finite type equipped with the norms defined by

$$\|\xi\|_k^2 = \sum_j e^{-2\alpha_j/k} |\xi_j|^2$$

then from [14], 5.1, or [4], Thm. 5, we see easily that σ can always be chosen as $\sigma(k) = ak$, $a \in N$. The linear operator $A\xi = (\xi_{ja})_j$ for $(\xi_j)_j \in \Lambda_1(\alpha)$, with $\alpha_j = j$, shows that these are in general the best possible functions σ.

Motivated by this, we put, for a fixed fundamental system of seminorms $\| \ \|_1 \leqslant \| \ \|_2 \leqslant \ldots$ and any continuous linear operator A,

$$\sigma_A(k) = \inf\{\sigma \in N: \sup_{\|x\|_\sigma \leqslant 1} \|Ax\|_k < +\infty\}$$

and call σ_A the *characteristic of continuity* (or generalized loss of derivatives function) of A. A Fréchet space E is called *tame* if there exists an increasing function $S: N \to N$ such that for any operator A we have $\sigma_A(k) \leqslant S(k)$ for large k, or equivalently, if there exist increasing functions S_α, $\alpha = 1, 2, \ldots$, such that for any operator A there is an α with $\sigma_A \leqslant S_\alpha$.

It is easily seen that this definition is independent of the choice of a fundamental system of seminorms. Power series spaces of finite type are always tame. If the norms are chosen as above, then we can choose $S_\alpha(k) = \alpha k$, $\alpha \in N$.

We will show that there are also tame power series spaces of infinite type and give a precise characterization of this class. Then we will show by means of a method of compact modification of matrices that all spaces in this class have the property that any complemented subspace has a basis. They share this property with the power series spaces of finite type (see Mityagin [8]; cf. also [10], [3]).

It turns out that in any tame power series space of infinite type with respect to the norms defined by

$$\|\xi\|_k^2 = \sum_j e^{2k\alpha_j} |\xi_j|^2$$

the functions S_α can be chosen as $S_\alpha(k) = ak + \alpha$ with suitable a. a can be made 1 by an equivalent change of $(\alpha_j)_j$. Hence every operator in such a space is tame in the sense (cf. [6], [13]) that its characteristic of continuity with respect to the above system of norms is majorized by a linear function (after some equivalent change of $(\alpha_j)_j$ even a parallel of the diagonal). This means that every operator, in particular every projection, satisfies estimates which imply for a projection in any power series space of infinite type that the range space has a basis. This is shown (at least in the nuclear case) in Section 3 by use of a generalization of the proof in Mityagin–Henkin [10] (cf. [14]). Different proofs can be found in [16] and [17].

The results of the present paper have been announced in [5].

Notation. Throughout this paper for any nondecreasing sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive real numbers tending to infinity we put

$$\Lambda_\infty(\alpha) = \{\xi = (\xi_1, \xi_2, \ldots): \|\xi\|_k^2 = \sum_j e^{2k\alpha_j} |\xi_j|^2 < +\infty \text{ for all } k\}.$$

Equipped with the norms $\| \ \|_k$, $k = 0, 1, 2, \ldots$, this is a Fréchet space. It is called a *power series space of infinite type*. Such a space is *nuclear* if and only if

$$\sup_n \frac{\log n}{\alpha_n} < +\infty,$$

which we do not assume unless it is explicitly stated. $\Lambda_\infty(\alpha)$ is called *shift-stable* if

$$\limsup_n \alpha_{n+1}/\alpha_n < +\infty.$$

This is equivalent to the existence of an isomorphism $\Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha) \oplus \boldsymbol{K}$, where $\boldsymbol{K}$ is the scalar field. A non-shift-stable space cannot be isomorphic to any of its subspaces or quotient spaces.

Two spaces $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\tilde{\alpha})$ are isomorphic (equal) if and only if there is a constant $C > 0$ such that

$$C^{-1}\alpha_n \leqslant \tilde{\alpha}_n \leqslant C\alpha_n.$$

In this case the sequences α and $\tilde{\alpha}$ are called *equivalent*.

A *step space* of $\Lambda_\infty(\alpha)$ is a space $\Lambda_\infty(\alpha')$, where $\alpha' = (\alpha_{n_k})_k$ is a subsequence of α. It is in a natural way a complemented subspace of $\Lambda_\infty(\alpha)$.

Two Fréchet spaces E and F are *nearly isomorphic* if there is a Fredholm operator from E to F, i.e. a continuous linear operator with (closed) finite-codimensional range and finite-dimensional kernel. In this case either E is isomorphic to a finite-codimensional subspace of F, or F is isomorphic to a finite-codimensional subspace of E.

1. Let E and F be Fréchet spaces with fixed fundamental systems of seminorms $\| \ \|_1 \leqslant \| \ \|_2 \leqslant \ldots$ and let $S_\alpha(\cdot): N \to N$ be nondecreasing functions for $\alpha = 1, 2, \ldots$ We define

$$L_\alpha(E, F) = \{A \in L(E, F): \forall k \ \exists C \ \forall x: \|Ax\|_k \leqslant C\|x\|_{S_\alpha(k)}\}.$$

If we put

$$\|A\|_{m,n} = \sup_{\|x\|_n \leqslant 1} \|Ax\|_m$$

then

$$L_\alpha(E, F) = \{A \in L(E, F): \|A\|_{k, S_\alpha(k)} < +\infty \text{ for all } k\}$$

and $L_\alpha(E, F)$ is a Fréchet space by the (not necessarily increasing) system of seminorms $\|\ \|_{k,S_\alpha(k)}$, $k = 1, 2, \ldots$.

The next proposition is a generalization of [15], Satz 1.1, where the case $S_\alpha(k) = \alpha$ was treated, which means that $\bigcup_\alpha L_\alpha(E, F) = LB(E, F)$, the class of all bounded linear operators from E to F.

1.1. Proposition. *The following are equivalent*:

(1) $\bigcup_\alpha L_\alpha(E, F) = L(E, F)$.

(2) *For every sequence $K(n)$ of positive integers tending to infinity there exists an α such that for every k we have n_0 and C with*

$$\|A\|_{k,S_\alpha(k)} \leqslant C \sup_{n \leqslant n_0} \|A\|_{n,K(n)}$$

for all $A \in L(E, F)$.

Proof. The implication (2) ⇒ (1) is clear. (1) ⇒ (2) is proved by means of Grothendieck's factorization theorem, applied as in [15], Satz 1.1.

For $E = F = \Lambda_\infty(\alpha)$ we derive from the nontrivial part of Prop. 1.1

1.2. Corollary. *If $\bigcup_\beta L_\beta(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha)) = L(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$ then for every sequence $K(n)$ of positive integers tending to infinity there exists a β such that for every k we have n_0 and C with*

$$\alpha_j k - \alpha_\nu S_\beta(k) \leqslant C + \sup_{n \leqslant n_0} (\alpha_j n - \alpha_\nu K(n))$$

for all j and ν.

Proof. We apply Prop. 1.1(2) to $A = e_j \otimes f_\nu$, where e_j and f_ν are canonical basis vectors in $\Lambda_\infty(\alpha)$ and $\Lambda'_\infty(\alpha)$ respectively, and then take logarithms.

We can now prove the main result of this section.

1.3. Theorem. *The following are equivalent*:

(1) *$\Lambda_\infty(\alpha)$ is tame.*

(2) *The set of finite limit points of $Q = (\alpha_j/\alpha_\nu)_{\nu,j \in N}$ is bounded.*

(3) *Up to equivalence α has the following form: there exist strictly increasing sequences $k(\nu) \in N$ with $k(1) = 1$ and $\beta_\nu > 0$ such that*

(a) $\lim_{\nu \to \infty} \beta_{\nu+1}/\beta_\nu = +\infty$,

(b) $\alpha_j = \beta_\nu$ *for* $k(\nu) \leqslant j < k(\nu+1)$.

(4) *There exists a constant a such that for every continuous linear operator A: $\Lambda_\infty(\alpha) \to \Lambda_\infty(\alpha)$ there is a constant b and a sequence of constants (C_k) such that for all k we have*

$$\|Ax\|_k \leqslant C_k \|x\|_{ak+b} \quad (x \in \Lambda_\infty(\alpha))$$

where the norms are as explained above.

Proof. (1) ⇒ (2). Since $\Lambda_\infty(\alpha)$ is tame there exist functions $S_\beta(\cdot)$, $\beta = 1, 2, \ldots$, such that $\bigcup_\beta L_\beta(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha)) = L(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$.

We choose a continuously differentiable strictly increasing function f on $(0, +\infty)$ with $f(1) = 1$, such that f' is strictly increasing to infinity and $f'(1) = 1$, We put $h = (f')^{-1}$. We assume that

$$\lim_k S_\beta(k+1)/f(k) = 0$$

for all β and define $K(n) = [f(n)] + 1$.

On account of Cor. 1.2 there exists a β such that for every k we have n_0 and C with

$$\frac{\alpha_j}{\alpha_\nu} k - S_\beta(k) \leqslant \frac{C}{\alpha_\nu} + \sup_{n \leqslant n_0} \left(\frac{\alpha_j}{\alpha_\nu} n - f(n)\right)$$

for all j and ν. For any finite limit point $A \geqslant 1$ of Q this gives

$$Ak - S_\beta(k) \leqslant \sup_{n \leqslant n_0} (An - f(n)).$$

Hence we obtain for every k

$$Ak - S_\beta(k) \leqslant \sup_{t > 0} (At - f(t)) = Ah(A) - f(h(A))$$

and therefore $Ak + f(h(A)) \leqslant Ah(A) + S_\beta(k)$. Putting $k = [h(A)] + 1$ we obtain

$$f(h(A)) \leqslant S_\beta([h(A)] + 1).$$

This shows that there cannot be any sequence (A_n) of finite limit points of Q, tending to infinity. This is assertion (2).

To prove the next implications we put for fixed $\beta > 0$

$$j(\nu) = \min\{j:\ \alpha_\nu \beta \leqslant \alpha_j\}.$$

Clearly $j(\nu)$ depends on β.

1.4. Lemma. (2) *is equivalent to* (2'): *There exists $\beta > 0$ such that $\lim_\nu \alpha_{j(\nu)}/\alpha_\nu = +\infty$.*

Proof. (2) ⇒ (2'). If (2') is not true, then we have for every $\beta > 0$ a subsequence of $(\alpha_{j(\nu)}/\alpha_\nu)$ converging to a finite limit. Hence there is a finite limit point of Q which is $\geqslant \beta$.

(2') ⇒ (2). From (2') it follows that there is no finite limit point of Q which is $> \beta$.

We continue now with the proof of Thm. 1.3:

(2) ⇒ (3). We choose $\beta > 1$ according to (2') in Lemma 1.4 and define $j(\nu)$ as above. We put $k(1) = 1$, $k(\nu+1) = j(k(\nu))$ for $\nu = 1, 2, \ldots$ and $\beta_\nu = \alpha_{k(\nu)}$. Then we have by 1.4

$$\lim_{\nu\to+\infty} \beta_{\nu+1}/\beta_\nu = \lim_{\nu\to+\infty} \alpha_{j(k(\nu))}/\alpha_{k(\nu)} = +\infty .$$

We put $\tilde{\alpha}_j = \beta_\nu$ for $k(\nu) \leqslant j < k(\nu+1)$. Then we have by definition $\tilde{\alpha}_j \leqslant \alpha_j$. By definition of $j(\nu)$ we have $\alpha_{k(\nu)}\beta > \alpha_j$ for $j < j(k(\nu)) = k(\nu+1)$, which implies $\alpha_j \leqslant \beta\tilde{\alpha}_j$.

(3) ⇒ (4). Let $\tilde{\alpha}$ be the equivalent sequence existing by assumption and $\| \ \|_k^\sim$ the norms in $\Lambda_\infty(\tilde{\alpha})$ $(= \Lambda_\infty(\alpha))$. For $A \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$ there exist $\sigma(\cdot)$ and $(C_k)_k$ such that $\|Ax\|_k^\sim \leqslant C_k \|x\|_{\sigma(k)}^\sim$ for all x and k.

We use the decomposition $A = A^0 + A^1$ described at the beginning of Section 2. From the proof of Lemma 2.1 we obtain constants D_k such that

$$\|A^1 x\|_k^\sim \leqslant D_k \|x\|_{\sigma(0)+2}^\sim$$

for all k. By use of the estimates (*) in Section 2 we have

$$\|A^0 x\|_k^\sim \leqslant \sum_\nu e^{k\beta_\nu} \|A_{\nu,\nu}\| \|x_\nu\| \leqslant C_0 \sum_\nu e^{(k+\sigma(0))\beta_\nu} \|x_\nu\|$$
$$\leqslant C_0 \big(\sum_\nu e^{-2\beta_\nu}\big)^{1/2} \|x\|_{k+\sigma(0)+1}^\sim .$$

Hence we obtain (4) with constants $\tilde{C}_k = D_k + C_0 \big(\sum_\nu e^{-2\beta_\nu}\big)^{1/2}$ and $\tilde{b} = \sigma(0)+2$ for the norms $\| \ \|_k^\sim$. If $\alpha_j/\beta \leqslant \tilde{\alpha}_j \leqslant \beta\alpha_j$ for all j we get the assertion with $a = \beta^2$ and $b = \beta\tilde{b}$.

The implication (4) ⇒ (1) is trivial.

Remark. It should be explicitly mentioned that 1.3(4) means that if $\Lambda_\infty(\alpha)$ is tame, then with respect to the norms described under Notation the S_b can be chosen as $S_b(k) = ak+b$, or after going to an equivalent sequence $\tilde{\alpha}$, even as $S_b(k) = k+b$. This means that in $\Lambda_\infty(\tilde{\alpha})$ every continuous linear operator is tame in the sense of [6] or [13].

From Thm. 1.3(3) one derives easily:

1.5. Corollary. (1) *If α is shift-stable then $\Lambda_\infty(\alpha)$ is not tame.*
(2) *If α is strongly unstable in the sense that*

$$\lim_n \alpha_{n+1}/\alpha_n = +\infty$$

then $\Lambda_\infty(\alpha)$ is tame.

In the proof of (1) one has to notice that shift-stability is invariant under equivalence.

The converse of (1) is not true. An example of a space $\Lambda_\infty(\alpha)$ which is

not tame and not shift-stable is the following:

$$\alpha_j = (j - 2^{n-1} - 1)\, 2^{n^2} \quad \text{for } 2^{n-1} \leqslant j < 2^n \text{ and } n = 1, 2, \ldots$$

Obviously $\bar{Q} = [0, +\infty)$ and

$$\lim_n \alpha_{2^n}/\alpha_{2^n-1} = \lim_n 2^{n+2} = +\infty .$$

Moreover, $\alpha_j \geqslant 2^{n^2} \geqslant j$. Hence $\Lambda_\infty(\alpha)$ is even nuclear.

2. We will now first use the description of the tame spaces $\Lambda_\infty(\alpha)$ given in Thm. 1.3(3) to derive another characteristic property of continuous linear operators in these spaces, similar to the one given by Dragilev and Kondakov [1] for strongly unstable spaces $\Lambda_\infty(\alpha)$. This will show by a method of compact modification that any complemented subspace in a tame space $\Lambda_\infty(\alpha)$ has a basis. For another proof for that, based on Thm. 1.3(4), and hence also on Lemma 2.1 below, see Section 3.

We assume throughout this section that the exponent sequence α has the form described in Thm. 1.3(3). We put $n(\nu) = k(\nu+1) - k(\nu)$. Then

$$\Lambda_\infty(\alpha) \cong \Big\{(x_1, x_2, \ldots) \in \prod_\nu l_2(n(\nu)):$$
$$\|x\|_k^2 = \sum_\nu \|x_\nu\|^2 e^{2k\beta_\nu} < +\infty \text{ for all } k\Big\}$$

where $\|x_\nu\|$ denotes the $l_2(n(\nu))$-norm.

A map $A \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$ is described by a matrix $(A_{j,\nu})_{j,\nu\in N}$ of $n(j) \times n(\nu)$-matrices, such that for every k there is a $\sigma(k)$ and a constant C_k with

$$(*) \qquad \|A_{j,\nu}\| e^{k\beta_j} \leqslant C_k e^{\sigma(k)\beta_\nu}$$

for all j, ν. $\|A_{j,\nu}\|$ is the $L(l_2(n(\nu)), l_2(n(j)))$-operator norm.

We define $A^0 = (\delta_{j,\nu} A_{j,\nu})_{j,\nu\in N}$ and $A^1 = A - A^0$. Clearly A^0 and A^1 are in $L(\Lambda_\infty(\alpha), \Lambda_\infty(\alpha))$. The following result is a generalization of [1].

2.1. Lemma. *A^1 is compact.*

Proof. Let k be given. For sufficiently large $j_0 = j_0(k)$ we have the following estimates:

1. For $\nu > j \geqslant j_0$ we conclude from $\|A_{j,\nu}\| \leqslant C_0 e^{\sigma(0)\beta_\nu}$ that

$$\|A_{j,\nu}\| e^{k\beta_j} \leqslant C_0 e^{\sigma(0)\beta_\nu + k\beta_j} \leqslant C_0 e^{(\sigma(0) + k\beta_j/\beta_{j+1})\beta_\nu}$$
$$\leqslant C_0 e^{(\sigma(0)+1)\beta_\nu}.$$

2. For $\nu < j$, $j \geqslant j_0$ we conclude from $\|A_{j,\nu}\| e^{2k\beta_j} \leqslant C_{2k} e^{\sigma(2k)\beta_\nu}$ that

$$\|A_{j,\nu}\| e^{k\beta_j} \leqslant C_{2k} e^{\sigma(2k)\beta_\nu - k\beta_j} \leqslant C_{2k} e^{(\sigma(2k) - k\beta_j/\beta_{j-1})\beta_\nu} \leqslant C_0 .$$

3. For $j < j_0$

$$\|A_{j,\nu}\| e^{k\beta_j} \leqslant \|A_{j,\nu}\| e^{k\beta_{j_0}} \leqslant (C_0 e^{k\beta_{j_0}}) e^{\sigma(0)\beta_\nu}.$$

We apply these estimates to $k+1$ instead of k and obtain for all k

$$\|A^1 x\|_k \leqslant \sum_{j \neq \nu} \|A_{j,\nu}\| e^{k\beta_j} \|x_\nu\|$$

$$\leqslant (C_0 e^{(k+1)\beta_{j_0(k+1)}}) \sum_{j,\nu} e^{-\beta_j - \beta_\nu} e^{(\sigma(0)+2)\beta_\nu} \|x_\nu\|$$

$$\leqslant D_k \|x\|_{\sigma(0)+2}.$$

This proves the assertion.

We apply the previous results now to a projection P in the tame space $\Lambda_\infty(\alpha)$. It follows that $P = P^0 + P^1$ and P^1 is compact. Therefore $(P^0)^2 - P^0$ is also compact. From the results of Prada [12], Thm. 6, we see that there is a projection Q in $\Lambda_\infty(\alpha)$ with $Q_{j,\nu} = 0$ for $j \neq \nu$ such that $P^0 - Q$ is compact. That follows because the construction of the projection in [12] is by means of a resolvent integral. Hence we have:

2.2. Proposition. *Under the assumption of this section for every projection P in $\Lambda_\infty(\alpha)$ there is a projection Q with $Q_{j,\nu} = 0$ for $j \neq \nu$ such that $P - Q$ is compact.*

According to [12], Thm. 2, $\operatorname{im} P$ and $\operatorname{im} Q$ are nearly isomorphic spaces, i.e. there is a Fredholm map from $\operatorname{im} P$ to $\operatorname{im} Q$. But clearly

$$\operatorname{im} Q = \{(x_1, x_2, \ldots) \in \prod_\nu E_\nu \colon \sum_\nu \|x_\nu\|^2 e^{t\beta_\nu} < +\infty \text{ for all } t\}$$

where $E_\nu = \operatorname{im} Q_{\nu\nu} \cong l_2(m(\nu))$ isometrically, $m(\nu) = \dim E_\nu$. Hence $\operatorname{im} Q$ is isomorphic to a step space in $\Lambda_\infty(\alpha)$. We proved:

2.3. Theorem. *In a tame space $\Lambda_\infty(\alpha)$ every complemented subspace has a basis.*

By means of the proof of Bessaga's conjecture for infinite type power series spaces of Mityagin [9] (see also [2] for the strongly unstable case), Thm. 2.3 is equivalent to the following stronger formulation. However, in our case it follows quite easily from the information on Q and the fact that the tame spaces $\Lambda_\infty(\alpha)$ are not shift-stable (Cor. 1.5).

2.3′. Theorem. *In a tame space $\Lambda_\infty(\alpha)$ every complemented subspace is isomorphic to a step space.*

Proof. We have $\operatorname{im} Q \cong \Lambda_\infty(\alpha')$, where α' is a subsequence of α. Either $\operatorname{im} P$ is isomorphic to a finite (m-) codimensional subspace E of $\Lambda_\infty(\alpha')$. Then we obtain, by eliminating m appropriate terms of α', a subsequence α'' such that the "restriction map" $E \to \Lambda_\infty(\alpha'')$ is an isomorphism. Or $\Lambda_\infty(\alpha')$ is isomorphic to a finite-codimensional subspace F of $\operatorname{im} P$. If the codimension of $\Lambda_\infty(\alpha')$ in $\Lambda_\infty(\alpha)$ is larger than or equal to the codimension of F in $\operatorname{im} P$ again we are through. But otherwise we would obtain an isomorphism imbedding $\Lambda_\infty(\alpha)$ as a proper subspace of $\operatorname{im} P \subset \Lambda_\infty(\alpha)$. Since $\Lambda_\infty(\alpha)$ is not shift-stable, this is impossible.

3. We will denote by $(|\ |_k)_{k=0,1,2,\ldots}$ the sequence of Hilbert norms in (s) given by

$$|\xi|_k^2 = \sum_j j^{2k} |\xi_j|^2 \qquad (\xi = (\xi_j) \in (s)).$$

A continuous linear operator A in (s) will be called *linear-tame* if there are $a, b \in N$ such that

$$|A\xi|_k \leqslant C_k |\xi|_{ak+b} \qquad (\xi \in (s))$$

with constants C_k and $k = 0, 1, 2, \ldots$

If $E \subset (s)$ is a subspace, then by E_k we denote the Hilbert space $(E, |\ |_k)^\wedge$.

3.1. Theorem. *If A is a linear-tame operator in (s) and E the range of A, then there exists a complete orthonormal system $(e_n)_n$ in E_0 consisting of elements of E and an equicontinuous sequence $(y_n)_n$ in $(s)'$ such that for every k there is an l with*

$$\sum_n |y_n|_l^* |e_n|_k < +\infty,$$

$$Ax = \sum_n y_n(x) e_n \qquad (x \in (s)).$$

Proof. The proof will be in two parts. First we establish the conclusion under the assumption that there exist a function $f\colon [1, \infty) \to [1, \infty)$ and $\beta, \sigma \in N$ such that:

(1) f is strictly increasing.

(2) For every k there is an $n(k)$ such that

$$\frac{n^k}{f(n)} \geqslant \frac{(n+1)^k}{f(n+1)}$$

for $n \geqslant n(k)$.

(3) $\|A\xi\|_1 \leqslant C\|\xi\|_2$ $(\xi \in (s))$ where C is some positive number and

$$\|\xi\|_1^2 = \sum_j (f(j))^2 |\xi_j|^2, \qquad \|\xi\|_2^2 = \sum_j (f(j^\beta))^2 |\xi_j|^2.$$

(4) $|A\xi|_0 \leqslant C|\xi|_\sigma$ $(\xi \in (s))$ where C is some positive number.

For the first part of the proof we will use nothing about A except (1)–(4). The second part of the proof consists of a construction of f and choice of β, σ to satisfy (1)–(4).

So let us begin by assuming that we have f and β, σ. From (3) it follows that $B_i = \{\xi: \|\xi\|_i \leqslant 1\}$ $(i = 1, 2)$ is a bounded subset of (s). The norms $|\ |_k, \|\ \|_i$ have their duals $|\ |_k^*, \|\ \|_i^*$ $(k = 0, 1, \ldots; i = 1, 2)$ on $(s)'$ and on E' respectively. We use the same notation for the dual norms and distinguish them only by the vectors to which they are applied.

For the first step of the argument we derive two inequalities which we need later. First we note that for $\xi \in (s)$ and $m \geqslant n(k)$

$$\sum_{j=1}^{\infty} j^{2k}|\xi_j|^2 \leqslant \sum_{j=1}^{m} j^{2k}|\xi_j|^2 + \sum_{j=m+1}^{\infty} j^{2k}|\xi_j|^2$$
$$\leqslant m^{2k}|\xi|_0^2 + \frac{(m+1)^{2k}}{\big(f(m+1)\big)^2}\|\xi\|_1^2$$

so for any $\xi \in (s)$ and $m \geqslant n(k)$ we have

(5) $$|\xi|_k \leqslant m^k|\xi|_0 + \frac{(m+1)^k}{f(m+1)}\|\xi\|_1.$$

Turning to the second inequality we obtain for $\eta \in (s)'$, $k > \sigma$ and $m \geqslant n(k)$

$$\sum_{j=1}^{\infty} j^{-2k}|\eta_j|^2$$
$$\leqslant \sum_{j=1}^{n(k)} |\eta_j|^2 + \sum_{j=n(k)}^{m} \frac{\big(f(j^\beta)\big)^2}{j^{2k}}\big(f(j^\beta)\big)^{-2}|\eta_j|^2 + \sum_{j=m+1}^{\infty} \frac{1}{j^{2(k-\sigma)}} j^{-2\sigma}|\eta_j|^2$$
$$\leqslant \left[f\big((n(k))^\beta\big) + \frac{f(m^\beta)}{m^k}\right]^2 \|\eta\|_2^{*2} + \frac{1}{(m+1)^{2(k-\sigma)}}|\eta|_\sigma^{*2}.$$

Hence we have for $\eta \in (s)'$ and any m with appropriate C_k

(6) $$|\eta|_k^* \leqslant C_k \frac{f(m^\beta)}{m^k}\|\eta\|_2^* + \frac{1}{(m+1)^{k-\sigma}}|\eta|_\sigma^*.$$

Now let H_∞ be the continuously imbedded Hilbert subspace of (s) generated by the bounded Hilbert ball $B_2 = \{\xi \in (s): \|\xi\|_2 \leqslant 1\}$, i.e. $H_\infty = l_2\big((f(n^\beta))_n\big)$. Let $A_\infty: H_\infty \to E_0$ be the map generated by A. It is compact because the space (s) is nuclear and its range is dense because H_∞ is dense in (s).

Therefore we may apply the spectral theorem to A_∞ to obtain an orthonormal system (h_n) in H_∞, a complete orthonormal system (e_n) in E_0

and a decreasing null sequence (a_n) of positive scalars such that

(7) $$Ax = \sum_{n=1}^{\infty} a_n \langle x, h_n\rangle_\infty e_n \quad (x \in H_\infty)$$

with the understanding that this series converges in E_0. Here $\langle\ ,\ \rangle_\infty$ denotes the scalar product in H_∞.

Since $Ah_n = a_n e_n$ the vector e_n is in E. We put for $x \in (s)$

$$y_n(x) = \langle Ax, e_n\rangle_0$$

where $\langle\ ,\ \rangle_0$ is the scalar product in E_0 and obtain, because of (7), an equicontinuous sequence in $(s)'$ with

$$Ax = \sum_{n=1}^{\infty} y_n(x) e_n \quad (x \in (s))$$

again with the understanding that the series converges in E_0. We have to show the convergence required in the theorem.

We have because of (3) and (7)

(8) $$|e_n|_0 = 1, \quad \|e_n\|_1 = (1/a_n)\|Ah_n\|_1 \leqslant C/a_n.$$

Moreover, (4) gives $|y_n(x)| \leqslant |Ax|_0 \leqslant C|x|_\sigma$ and (7) yields $y_n(x) = a_n\langle x, h_n\rangle_\infty$ for $x \in H_\infty$. Hence

(9) $$|y_n|_\sigma^* \leqslant C, \quad \|y_n\|_2^* = a_n.$$

Since A_∞ can be written as $A_\infty = B \circ J$ where J: $H_\infty = l_2\big((f(n^\beta))_n\big) \to l_2\big((n^\sigma)_n\big)$ is the identity map, and, because of (4), $\|B\| \leqslant C$, it follows from standard facts about diagonal operators on Hilbert spaces (see [11] for details) that

(10) $$a_n \leqslant Cn^\sigma/f(n^\beta)$$

for large n.

Because of (2) which implies convergence to zero we can for fixed n large enough find an m_n such that

(11) $$a_n f(m_n^\beta)/m_n^\sigma \leqslant C < a_n f\big((m_n+1)^\beta\big)/(m_n+1)^\sigma.$$

Then (10) implies $m_n \geqslant n$.

For $k > \sigma$ we apply (5) to $\xi = e_n$ and $m = (m_n+1)^\beta - 1$. We obtain by use of (8) and (11)

$$|e_n|_k \leqslant m^k + C\frac{(m+1)^k}{f(m+1)}\cdot\frac{1}{a_n} \leqslant m^k + (m_n+1)^{\beta k-\sigma}$$
$$\leqslant (1+2^{\beta k})\, m_n^{\beta k}.$$

For $l > \sigma$ (instead of k) we apply (6) to $\eta = y_n$ and $m = m_n$. We obtain by use of (9) and (11)

$$|y_n|_l^* \leqslant C_l \frac{f(m^\beta)}{m^l} a_n + \frac{C}{(m+1)^{l-\sigma}} \leqslant C(C_l+1)\frac{1}{m_n^{l-\sigma}}.$$

We put $l = \beta k + \sigma + 2$ and obtain for large n_0 and $M_k = (1+2^{\beta k})\, C(C_l+1)$

$$\sum_{n=n_0}^{\infty} |y_n|_l^* |e_n|_k \leqslant M_k \sum_{n=n_0}^{\infty} m_n^{-2} \leqslant M_k \sum_{n=n_0}^{\infty} n^{-2} < +\infty,$$

which proves the assertion under assumptions (1)–(4).

Now we turn to the second half of the proof in which (since the existence of σ satisfying (4) is a simple consequence of continuity) we have to construct f and β to satisfy (1)–(3).

For $\xi \in (s)$ we write $A\xi = \left(\sum_n t_{j,n} \xi_n\right)_j \in (s)$. From the continuity and the assumption of tameness we have $a \in N$, $b \in N$, $\beta \geqslant 2$ (take any $\beta > 2a$) and constants C'_k, C_k such that for $k \in N$, $n \in N$

$$|t_{j,n}|\, j^k \leqslant |A\vartheta_n|_k \leqslant C'_k |\vartheta_n|_{ak+b} = C'_k n^{ak+b} \leqslant C_k n^{\beta k/2}.$$

Here, $\vartheta_n = (\delta_{n,j})_j$ is the usual basis vector in (s).

We choose a sequence $(m_k)_{k=0}^{\infty}$ strictly increasing to infinity such that

$$m_0 = 1, \qquad m_k \geqslant C_{k+1} \qquad (k = 1, 2, \ldots).$$

We put

$$M_k = \prod_{l=0}^{k} m_l, \qquad k = 0, 1, \ldots,$$

and define for $t \geqslant 1$

$$f(t) = \sup_k t^k / M_k.$$

Moreover, we define for $t \geqslant 1$ a number $q = q(t)$ by $m_q \leqslant t < m_{q+1}$. $q(t)$ is well defined, nondecreasing and $\lim_{t\to\infty} q(t) = \infty$.

Since for any $t \geqslant 1$ and $k \geqslant 1$ and $k \in N$

$$\frac{t^{k-1}}{M_{k-1}} (\lesseqgtr) \frac{t^k}{M_k} \Leftrightarrow m_k (\lesseqgtr) t$$

the sup in the definition of f always occurs at $q(t)$.

Now f is obviously strictly increasing. To obtain (2) we write $q = q(n)$, $p = q(n+1)$, so $p \geqslant q$ and we have $m_p \leqslant n+1$. So

$$\frac{M_p}{M_q} = m_p m_{p-1} \ldots m_{q+1} \leqslant m_p^{p-q} \leqslant (n+1)^{p-q}.$$

Hence, if $q > k$ then

$$\frac{(n+1)^k}{f(n+1)} = \frac{M_p}{(n+1)^{p-k}} \leqslant \frac{M_q}{(n+1)^{q-k}} \leqslant \frac{M_q}{n^{q-k}} = \frac{n^k}{f(n)}$$

so it suffices for (2) to take $n(k)$ sufficiently large that $q(n) > k$ for $n \geqslant n(k)$.

Finally, (3) follows from the following estimate:

$$\begin{aligned}\sum_{j \geqslant m_4} (f(j))^2 \Big|\sum_n t_{j,n}\xi_n\Big|^2 &\leqslant \sum_{j \geqslant m_4} j^{-2} \Big(\sum_n \frac{|t_{j,n}|\, j^{q(j)+1}}{M_{q(j)}} |\xi_n|\Big)^2 \\ &\leqslant \sum_{j \geqslant m_4} j^{-2} \Big(\sum_n \frac{n^{\beta(q(j)+1)/2}}{M_{q(j)-1}} |\xi_n|\Big)^2 \\ &\leqslant \Big(\sum_{\nu=1}^{\infty} \nu^{-2}\Big)^2 \sup_{j \geqslant m_4} \sum_n \Big(\frac{n^{\beta(q(j)+2)/2}}{M_{q(j)-1}}\Big)^2 |\xi_n|^2 \\ &\leqslant \frac{\pi^4}{36} \sum_n (f(n^\beta))^2 |\xi_n|^2.\end{aligned}$$

In the last inequality we used the fact that $\beta(q(j)+2)/2 \leqslant \beta(q(j)-1)$ for $q(j) \geqslant 4$, which is the case for $j \geqslant m_4$.

Remark. In the second part of the previous proof we derived assumptions (1)–(4) from the continuity and tameness of A. On the other hand, in the first part of this proof from (1)–(4) the following is derived, with $l = \beta k + (\sigma+2)$ and $C = \sum_n |y_n|_l^* |e_n|_k < +\infty$:

$$|Ax|_k \leqslant \sum_n |y_n(x)|\,|e_n|_k \leqslant \Big(\sum_n |y_n|_l^* |e_n|_k\Big)|x|_l = C|x|_{\beta k + (\sigma+2)}.$$

Hence the existence of f, β and σ satisfying (1)–(4) is equivalent to the tameness of A.

An easy consequence of Thm. 3.1 is the following result:

3.2. Theorem. *The range of a tame operator in* (s) *has an absolute basis.*

Proof. Any $x = A\xi$ in the range of A has an expansion

$$x = \sum_n \lambda_n e_n$$

with $\lambda_n = y_n(\xi)$. Since λ_n can be calculated as $\lambda_n = \langle x, e_n\rangle_0$ where $\langle\ ,\ \rangle_0$ is the scalar product in E_0, the expansion is unique.

The basis is absolute since on account of the estimate in the previous remark we have

$$\sum_n |\lambda_n|\,|e_n|_k < +\infty.$$

Remark. The basis need not be equicontinuous since the estimate says only

$$\sum_n |\lambda_n| \, |e_n|_k \leqslant C \, |\xi|_{\beta k+(\sigma+2)}$$

and there is not necessarily a ξ such that the norm of ξ can be estimated by a norm of x. If, however, A has closed range, then this follows from the open mapping theorem. So we have the following consequence of Thm. 3.2.

3.3. Theorem. *The range of a tame operator in* (s) *with closed range is isomorphic to a power series space of infinite type.*

Proof. With the help of the previous remark and e.g. [11], 10.1.4, we see that the range E is isomorphic to a nuclear Köthe space, which is isomorphic to a quotient and to a subspace of (s). Hence it is a power series space of infinite type.

Since (s) contains all nuclear power series spaces of infinite type as step spaces, any tame operator in such a space can be considered as a tame operator in (s). A consequence is:

3.4. Theorem. *In a tame nuclear power series space of infinite type the range of any continuous linear operator has an absolute basis.*

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