Metrizable [normable] (LF)-spaces
and two classical problems in
Fréchet [Banach] spaces

by

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Abstract. It is well known that no strict (LF) -space is metrizable. Also, no (generalized)
(LB)-space is metrizable ([17], [18]). In the early 1970's, isolated examples of metrizable and/or
normable (LF)-spaces were given by Roekeke [23; p. 269 ff.], De Wilde [5; p. 84], and Saxon
[13]. This paper gives a construction for an abundance of metrizable and normable (LF)-spaces:
a Fréchet space $F$ has a dense subspace which is an (LF)-space if either

(i) $F$ splits into infinitely many parts each of which has a separable quotient, or
(ii) $F$ has a separable quotient which splits into infinitely many parts.

Note that (ii) is satisfied by every non-Banach Fréchet space, from a result of Eidelheit [4]. Thus
every non-Banach Fréchet space is the completion of some (LF)-space (Valdivia and Pérez
Carreras [22]), and the same is true for every (finite-dimensional) Banach space provided the
splitting and separable quotient problems have affirmative solutions.

No (LF)-space is both complete and metrizable, since by the Open Mapping Theorem, none is Baire, nor even a (db)-space [17]. In [11], Robertson, Tweddle and Yeomans introduced (db)-spaces, observing (with no
distinguishing examples) that

unordered Baire-like $\Rightarrow$ (db) $\Rightarrow$ Baire-like.

An (LF)-space is metrizable if and only if it is Baire-like [18]. Hence, metrizable (LF)-spaces are precisely those (LF)-spaces which distinguish between Baire-like and (db)-spaces.

Eidelheit [4] showed that every non-Banach Fréchet space $E$ has a quotient which is isomorphic to the (separable) space $\omega$ of all scalar sequences. [Choose a sequence $\{x_i\}$ in $E$ satisfying Bemerkung 3, p. 144, so

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that by Satz 2 of [4], the continuous linear operator \( x \mapsto (f_i(x)) \) from \( E \) into \( \omega \) (= \( (\omega) \) in [4]) is surjective. Whether every infinite-dimensional Banach space has a (Hausdorff, infinite-dimensional) separable quotient is a longstanding classical problem, considered e.g. in Rosenthal [12], Lacey [9] and Saxon and Wilansky [19]. Likely, it has been considered since 1932. (In [17], p. 77, we proved that every \((LF)\)-space has a separable quotient.) Here, we show that a given Banach space has a separable quotient if and only if it has a dense subspace which, with a topology finer that the relative topology, is a normable \((LF)\)-space.

Several results pertaining to the quotients of Fréchet and \((LF)\)-spaces are obtained. The paper concludes with a discussion of a number of open questions.

1. Definitions and preliminary results. In general, we assume the terminology and notations in Horváth [7]. We recall from [11], [15], [17], [19] and [21] that \( \varphi \) denotes a fixed \( \mathbb{R} \)-dimensional (real or complex) vector space endowed with the finest locally convex topology and that a locally convex space \( E \) is

1) \textit{Baire} if \( E \) is not the union of a sequence of nowhere dense sets;
2) \textit{unordered Baire-like} if \( E \) is not a union of a sequence of nowhere dense absolutely convex sets (cf. [10], [21]);
3) \((db)\)-space if \( E \) is not the union of an increasing sequence of subspaces none of which is both dense and barrelled;
4) \textit{Baire-like} if \( E \) is not the union of an increasing sequence of nowhere dense absolutely convex subsets;
5) \textit{quasi-Baire} if \( E \) is barrelled, and is not the union of an increasing sequence of nowhere dense subspaces;
6) an \( S_\sigma \)-space if it is the union of a strictly increasing sequence of closed subspaces.

Clearly,

\[
\text{Baire} \Rightarrow \text{unordered Baire-like} \Rightarrow (db) \Rightarrow \text{Baire-like} \Rightarrow \text{quasi-Baire},
\]

\[
\text{quasi-Baire} \Rightarrow \text{barrelled},
\]

\[
\text{quasi-Baire} \Rightarrow \text{not } S_\sigma.
\]

(Note: none of these implication arrows is reversible.)

By a \textit{Fréchet space} we always mean a complete metrizable locally convex space. We shall often use the following special form of Pták's open mapping theorem (p. 299, Proposition 2 in Horváth [7]):

A continuous linear map from a Fréchet space onto a barrelled space is open.

If \( G \) is a subspace of a locally convex space \((E, \tau)\), \( \tau_G \) denotes the relative topology on \( G \). For a subset \( A \subseteq E \), \( \text{sp}(A) \) will denote the linear span of \( A \). A continuous linear map \( P \) from \( E \) into \( E \) such that \( P(P(x)) = P(x) \) \((x \in E)\) is a projection on \( E \). A sequence \( \{P_i\} \) of projections on \( E \) is an \textit{orthogonal sequence of projections} if \( P_i(P_j(x)) = 0 \) for all \( i \neq j \). A Fréchet space \( E \) \textit{splits} if there exist infinite-dimensional closed subspaces \( M \) and \( N \) such that \( M \cap N = \{0\} \) and \( M + N = E \). We denote this situation by writing \( E = M \oplus N \). We say that \( E \) \textit{splits into infinitely many parts} \( \{M_n\} \) \((n = 1, 2, \ldots)\) if there exist sequences \( \{M_{n_1}\}, \{N_{n_1}\} \) in \( E \) such that

\[
E = M_1 \oplus N_1, \quad N_1 = M_2 \oplus N_2, \quad N_2 = M_3 \oplus N_3, \ldots
\]

**Proposition 1.** A Fréchet space \( E \) splits into infinitely many parts if and only if there exists a sequence of orthogonal projections with infinite-dimensional ranges.

**Proof:** Given \( \{M_n\} \) and \( \{N_n\} \), define the projections \( \{P_n\} \) on \( E \) by letting \( P_i \) be the identity on \( M_i \), and zero on \( N_i \), and \( P_i (1 \leq i < j \leq n) \) for each \( i \). The reverse implication is obvious. \( \blacksquare \)

A topological space \( X \) is said to be \textit{continuously included} in a topological space \( Y \) provided \( X \subseteq Y \) and the identity map on \( X \) to \( Y \) is continuous; i.e., the topology on \( X \) is finer than the topology induced by \( Y \).

The next two lemmas can be viewed as special cases of the general result on completeness of inverse limits ([7], Proposition 2.11.3, p. 153). We gratefully accept the referee's very elegant and concise proof.

**Lemma 1.** Let \( (F, \gamma) \) and \( (G, \tau) \) be Fréchet spaces and \( Q \) be a continuous linear map from \( F \) into \( G \). Further, suppose that \( (G_1, \tau_1) \) is a Fréchet space continuously included in \( (G, \tau) \). Then \( F_1 = Q^{-1}[G_1] \) is a Fréchet space under the topology \( \gamma_1 \) having as a base of neighborhoods of \( 0 \) the set \( \{U \cap Q^{-1}[V] : U \text{ and } V \text{ are neighborhoods of } 0 \text{ in } (F, \gamma) \text{ and } (G_1, \tau_1) \text{ respectively}\} \).

**Proof:** \( (F_1, \gamma_1) \cong (F, \gamma) \times (G_1, \tau_1) \) is a topological isomorphism onto a closed subspace. \( \blacksquare \)

**Lemma 2.** Let \( (F_n, \gamma_n) \) be a sequence of Fréchet spaces, each of which is continuously included in some Hausdorff space \((F, \gamma)\). Then \( F_0 = \bigcap_{n=1}^{\infty} F_n \) is a Fréchet space given the topology \( \gamma_0 \) with a subbase of neighborhoods of \( 0 \) the set

\[
\Sigma = \{F_0 \cap U : U \text{ is a } \gamma_n\text{-neighborhood of } 0 \text{ for some } n\}.
\]
Proof. Again,

\[(F_0, \tau_0) \bigcup_{n \in \mathbb{N}} \prod_{n=1}^{\infty} (F_n, \tau_n)\]

is a topological isomorphism onto a closed subspace. □

2. Some basic properties of (LF)-spaces. Let \((E, \tau)\) be a locally convex Hausdorff space. If there exists a strictly increasing sequence \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) of Fréchet spaces such that \(E = \bigcup_{n=1}^{\infty} E_n\), each \((E_n, \tau_n)\) is continuously included in \((E_{n+1}, \tau_{n+1})\) and \(\tau\) is the finest locally convex Hausdorff topology for which \((E_n, \tau_n)\) is continuously included in \((E, \tau)\) for each \(n\), then \((E, \tau)\) is said to be an \((LF)\)-space, \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) is an inductive sequence which defines the \((LF)\)-space \((E, \tau)\) and we write

\[(E, \tau) = \lim_{n \to \infty} (E_n, \tau_n).\]

Dieudonné, Schwartz, Grothendieck and Köthe pioneered the study of \((LF)\)-spaces (cf. [3], [6], [8]). Note that an absolutely convex set \(U \subset E\) is a \(\tau\)-neighborhood of 0 if and only if \(U \cap E_n\) is a \(\tau_n\)-neighborhood of 0 for each \(n\), and a linear function \(f\) from \((E, \tau)\) into a locally convex space \(F\) is continuous if and only if \(f|E_n\) is continuous for each \(n\). If the inclusion mappings are bicontinuous onto their images (i.e., \(\tau_n \leq \tau\)), then \(E_n \equiv (E_n, \tau_n)\) for each \(n\), we say that \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) is a strict inductive sequence and \((E, \tau)\) is a \((LF)\)-space. If each \(E_n\) is a Banach space, \((E, \tau)\) is an \((LB)\)-space (a strict \((LB)\)-space if further \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) is strict). Two inductive sequences \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\), \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) in \(E\) (defining two possibly different Hausdorff topologies on \(E\)) are said to be equivalent if each member of either sequence is continuously included in some member of the other; i.e., if \(i \in [1, 2]\) and \(n\) is arbitrary, there is some \(k\) such that \(E_n^{(i)} \subset E_n^{(2)}\) and \(\tau_n^{(i)} \leq \tau_n^{(2)}\). One easily sees that equivalent inductive sequences define the same \((LF)\)-space \((E, \tau)\).

Theorem 1 (Grothendieck's Equivalence Theorem). Let \((E, \tau^{(0)}) = \lim_{i \to \infty} (E_n, \tau_n^{(0)})\) for \(i \neq 1, 2\). The following statements are equivalent:

\[(a) \quad \{(E_k, \tau_k^{(1)})\}_{k=1}^{\infty} \text{ is equivalent to } \{(E_k, \tau_k^{(2)})\}_{k=1}^{\infty};\]
\[(b) \quad \tau^{(1)} = \tau^{(2)};\]
\[(c) \quad \text{The inffimum of } \tau^{(1)} \text{ and } \tau^{(2)} \text{ is Hausdorff}.\]

Proof. Clearly (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c).

Now (c) \(\Rightarrow\) (b) by Grothendieck [6; p. 17, Thm. B 2] and (b) \(\Rightarrow\) (a) is essentially Cor. 3 to Thm. A, loc. cit. □

Note. One easily proves (c) \(\Rightarrow\) (a) directly by using Lemma 2, the fact that every Fréchet space is (db), and Pták's open mapping theorem.

Corollary 1. Let \((E, \tau)\) be a Hausdorff locally convex space. Then there is at most one topology on \(E\) finer than \(\tau\) which makes \(E\) an \((LF)\)-space.

Example 1. There exists a strict \((LB)\)-space \(E\) with a nonstrict inductive sequence of non-Banach spaces which defines \(E\).

Let \((s)\) denote the nonnormable nuclear Fréchet space of all rapidly decreasing sequences. Clearly \((s)\) is continuously included in the Banach space \(l_1\). Let

\[E_n = l_1 \times \cdots \times l_1 \times \{0\} \times \{0\} \times \cdots, \quad \text{\(n\) factors}\]
\[F_n = l_1 \times \cdots \times l_1 \times \{s\} \times \{0\} \times \{0\} \times \cdots \quad \text{\(n\) factors}\]

and let \(Y_n\) and \(\tau_n\) be the product topologies on \(E_n, F_n\) respectively. Now, \(E_n\) is continuously included in \(F_n\) and \(F_n\) is continuously included in \(E_{n+1}\) so that \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) and \(\{(F_n, \tau_n)\}_{n=1}^{\infty}\) are equivalent inductive sequences in the strict \((LB)\)-space \((E, \tau) = \lim_{n \to \infty} (E_n, \tau_n) = \lim_{n \to \infty} (F_n, \tau_n)\), with the former a strict inductive sequence of Banach spaces, the latter a nonstrict inductive sequence of non-Banach spaces.

Example 2. One can easily modify Example 1 (e.g. replace \(l_1\) by \(l_2\) and \((s)\) by \(l_2\)) to obtain a strict \((LB)\)-space with a nonstrict inductive sequence of Banach spaces.

Corollary 2. If \((E, \tau) = \lim_{n \to \infty} (E_n, \tau_n)\) is a strict \((LF)\)-space and also an \((LB)\)-space, then \((E, \tau)\) is a strict \((LB)\)-space. In fact, if \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\) is a strict inductive sequence, then each \((E_n, \tau_n)\) is a Banach space \((n = 1, 2, \ldots)\).

Proof. By hypothesis, there exists a (necessarily equivalent) defining sequence \(\{(B_n, Y_n)\}_{n=1}^{\infty}\) of Banach spaces so that for any \(n\), there exist \(k\) and \(p\) with \(E_n \subset B_k \subset B_p\), where \(\tau_k \leq \tau_n \leq \tau_p\), yielding \(\tau_k \leq Y_n \leq \tau_p\). But by strictness, \(\tau_k \equiv \tau_n\) and therefore \(\tau_n\) is the norm topology \(Y_k\).

It is well known that all \((LF)\)-spaces are barreled, and we make the observation that no \((LF)\)-space is a \((db)\)-space. For, no \(E_n\) can be both dense and barreled in \((E, \tau)\) by Pták's open mapping theorem applied to the identity map from \((E_n, \tau_n)\) onto \((E_n, \tau_p)\), since \(E_n \not\subset E_{n+1}\) for each \(n\). By a similar argument, no \((LB)\)-space is Baire-like; consider an increasing sequence of multiples of the unit ball of \(\{(E_n, \tau_n)\}_{n=1}^{\infty}\). Every strict \((LF)\)-space is \(S\) and thus not quasi-Baire.

In [18], we partitioned the class of all \((LF)\)-spaces into three mutually disjoint nonempty classes:

An \((LF)\)-space \((E, \tau)\) is of type (i) or simply an \((LF)\)-space if it satisfies the condition (i) below \((i = 1, 2, 3)\):

1. \((E, \tau)\) has a defining sequence none of whose members is dense in \((E, \tau)\):
always nonmetrizable. We now give the promised Example 4, including a nonmetrizable (LF)-space which is neither an (LB)-space nor a strict (LF)-space.

**Example 4.** Let $p > 1$ and choose $N$ such that $p - 1/(N + 1) > 1$. Let $l_p^{\infty}$ denote the (LF)-space

$$\lim_{n} l_{p-1/(N+n)}.$$ 

(Note that $l_p^{\infty}$ is independent of the choice of $N$). The (LF)-space $\omega \times l_{p}^{\infty}$ has $(\omega \times l_{p-1/(N+n)}^{\infty})$, as a defining sequence. Since it contains a copy of $l_{p}^{\infty}$, it is not metrizable, and is, in fact, an (LF)$_{1}$-space. Thus, it is not an (LF)$_{2}$-space, and not a strict (LF)-space. Since there is no Hausdorff vector topology on $\omega$ strictly coarser than the product topology, there is no coarser norm topology on $\omega$. Thus by Theorem 1, $\omega \times l_{p}^{\infty}$ is not an (LF)-space.

In [18], we show that $\varphi \times l_{p}^{\infty}$ is an (LF)-space which is a nonstrict (LF)$_{2}$-space.

**Theorem 2.** Let $(E, \gamma) = \lim \{E_{n}, \gamma_{n}\}$ be an (LF)$_{1}$-space $(1 \leq i \leq 3)$ with a closed subspace $M$. If $E_{n} + M = E$ for some $n$, then the quotient space $E/M$ is a Fréchet space; otherwise, $E/M$ is an (LF)$_{i}$-space for some $j \geq i$.

**Proof.** Case 1. Suppose for some $n$, $E_{n} + M = E$. Then, if $Q$ is the continuous quotient map from $E$ onto $E/M$, we have $Q(E_{n}) = E/M$. Since $\gamma_{n} \leq \gamma$, $Q(\gamma_{n})$ is a continuous surjection from the Fréchet space $(E_{n}, \gamma_{n})$ onto the barreled space $E/M$; therefore by Pták’s open mapping theorem, $E/M$ is a Fréchet space.

Case 2. Suppose $E_{n} + M \not\subseteq E$ for each $n$. Then $[E_{n}] = E/M$ for each $n$.

Thus for some subsequence $\{(F_{n}, \eta_{n})\}_{n=1}^{\infty}$ of $\{E_{n}\}_{n=1}^{\infty}$, if $G_{p} = Q(F_{p})$ for $p = 1, 2, \ldots$, then $\{G_{p}\}$ is a strictly increasing sequence. If $T$ denotes the quotient topology on $E/M$, and $\tau_{p}$ the quotient topology of $\eta_{p}$ on $G_{p}$, then $Q|_{F_{p}}$ is continuous from $(F_{p}, \eta_{p})$ onto $(G_{p}, \tau_{p})$ and therefore $\tau_{p} \leq \tau_{p}$.

Let $(E/M, \xi) = \lim (G_{p}, \tau_{p})$. Since $(E, \gamma) = \lim (F_{p}, \eta_{p})$ and since $Q|_{F_{p}}$ is continuous onto $(G_{p}, \tau_{p})$, it is continuous onto $(G_{p}, \xi_{p})$, $p = 1, 2, \ldots$ Thus we see that $Q$ is continuous from $(E, \gamma)$ onto $(E/M, \xi)$. Also the mapping is clearly open for $\xi$. Therefore, $\xi = \tau$, since there is only one topology on $E/M$ that makes $Q$ continuous and open; i.e., the quotient space $E/M$ is an (LF)$_{1}$-space for some $j$, $1 \leq j \leq 3$. If $F_{p}$ is dense in $E$, then so is $G_{p}$ in $E/M$, by continuity and surjectivity of $Q$, and $E/M$ is metrizable whenever $E$ is. Thus $j \geq i$.

By relaxing the requirement that inductive sequences must be strictly increasing, one could regard Fréchet spaces as the remaining class of (LF)-spaces of type (4), in respect of the above theorem. Every (LF)-space of types
(1), (2) or (3) has a (Hausdorff, infinite-dimensional) separable quotient [17]; in [18] we prove that every (LF)-space of type (3) has a quotient which is a separable infinite-dimensional Fréchet space. Also in [18], we prove that the cartesian product of an (LF)-space with an (LF)-space is an (LF)_k-space where k = \min(i, j); an infinite product of (LF)-spaces is never an (LF)-space; the Hausdorff inductive limit of an increasing sequence of (LF)-spaces is an (LF)-space; a countable-codimensional subspace of an (LF)-space is an (LF)-space if and only if it is closed, and not contained in any member of the defining sequence.

**Corollary 3** (Köthe’s Open Mapping Theorem [7; vol. II, p. 43]). If f: F \to G is a continuous linear surjection from an (LF)-space F onto an (LF)-space G, then f is open.

**Proof.** Let \( \bar{f} \) denote the (continuous) associated injection from F/M onto G, where M = f^{-1}(0). Note that f is open if and only if \( \bar{f} \) is an isomorphism. Since G is a non-Fréchet barrelled space, F/M cannot be a Fréchet space by Pták’s open mapping theorem. Thus, F/M is an (LF)-space by Theorem 2. Since \( \bar{f} \) is continuous, it carries a finer (LF)-topology onto G so that by Corollary 1, the two topologies on G coincide. I.e., \( \bar{f} \) is an isomorphism, and f is open.

Note. If G is not an (LF)-space, the conclusion can fail, even under the hypothesis that F is an (LF)_k-space, and G is metrizable and barrelled, by Example 2 of [17].

The following is an alternative version of Theorem 2.

**Theorem 2A.** Let \( \{(F_n, Y_n)\}_{n=1}^\infty \) be a defining sequence for an (LF)-space \( (F_0, Y_0) \) and let M be a closed subspace of \( F_0 \) with \( M \subseteq F_1 \). Then \( (G_0, \tau_0) \) is an (LF)-space defined by the sequence \( \{(G_n, \tau_n)\}_{n=1}^\infty \), where \( G_n = (F_n, Y_n)/M \) and \( \tau_n \) is the quotient topology, \( n = 0, 1, 2, \ldots \).

**Corollary 4.** If \( F_0 \) is a metrizable (LF)-space and \( M \) is a complete subspace, then the quotient \( F_0/M \) is a metrizable (LF)-space.

**Proof.** Let \( (F_0, Y_0) \) be as in Theorem 2A. The Fréchet space \( M \) is a (db)-space. That is to say, one of the covering spaces \( M \cap F_N = E_N \) is both dense and barrelled in \( M \). Clearly, \( Y_N \mid E_N \) is finer than \( Y_0 \mid E_N \), so that \( E_N \subseteq Y_0 \mid E_N \) is a Fréchet space (by Lemma 2) mapped continuously onto \( E_N \), \( Y_0 \mid E_N \), a dense barrelled subspace of \( M \). Hence by Pták’s open mapping theorem, the spaces are isomorphic so that \( E_N \) is a dense complete subspace of \( M \); i.e., \( M \subseteq F_0 \). Since \( \{(F_n, Y_n)\}_{n=1}^\infty \) is a defining sequence for \( (F_0, Y_0) \), the theorem applies.

**Corollary 5.** No metrizable (LF)-space is complete.

**Proof.** If \( F_0 \) is Fréchet, then by Corollary 4, \( F_0/M \) is an (LF)-space contradicting the fact that the defining sequences are strictly increasing.

**Remark.** Some nonmetrizable (e.g., all strict) (LF)-spaces are complete (Köthe [8; vol. I, p. 225]).

**3. Constructing metrizable (LF)-spaces.** We use the following theorem, of independent interest, in the main result. A former Banach space version appeared in [19]. For a non-Banach Fréchet space \( E \) the stated equivalence merely says that conditions \( (i)-(v) \) hold, since by the result of Eidelheit [4], \( E \) has a quotient isomorphic to \( \tilde{E} \).

**Theorem 3 (see Corollary 8).** Let \( (E, \mathcal{T}) \) be a Fréchet space. The following statements are equivalent:

(i) \( E \) has a separable (infinite-dimensional) quotient (by a closed subspace).

(ii) \( E \) has a dense \( S_0 \)-subspace.

(iii) \( E \) has a dense nonbarrelled subspace.

(iv) \( E \) has a dense non-(db)-subspace.

(v) \( E \) densely, properly and continuously includes a Fréchet space \( M \) (\( M \) can be chosen so as to contain any specified countable subset of \( E \)).

Note. The equivalence of (iii) and (v) is essentially due to Bennett and Kalton [2; p. 512, Prop. 1].

**Proof.** (i) \( \Rightarrow \) (ii). If \( E \) is a closed subspace of \( E/M \) is infinite-dimensional and separable, then there exists a linearly independent sequence \( \{x_n\} \subseteq E \) such that \( M \cap \text{sp}(\{x_n\}) = \{0\} \) and \( M + \text{sp}(\{x_n\}) \) is dense in \( E \). Then \( \bigcup_{n=1}^\infty \{M + \text{sp}(\{x_1, \ldots, x_n\})\} \) is a dense \( S_0 \)-subspace of \( E \).

(ii) \( \Rightarrow \) (iii). This is clear since by [1] or [14], metrizable barrelled spaces are Baire-like (cf. Section 1).

Trivially, (iii) \( \Rightarrow \) (iv).

(iv) \( \Rightarrow \) (iii). Suppose \( E \) is a dense subspace and \( M \) is the union of an increasing sequence \( \{M_n\} \) of subspaces none of which is both dense in \( E \) and barrelled. Then if \( M \) is nonbarrelled, we are through. If \( M \) is barrelled, then \( M \) is quasi-Baire by [14], so that some \( M_n \) is dense in \( M \), hence dense and nonbarrelled in \( E \).

Thus (iii) \( \Rightarrow \) (iv) are equivalent.

(iii) \( \Rightarrow \) (v). Suppose \( N \) is any dense nonbarrelled subspace of \( E \), and let \( C \) be any countable subset of \( E \). Then \( \text{sp}(N \cup C) \) is nonbarrelled (see [16]) so there exists a closed absolutely convex set \( V \) such that \( M = \text{sp}(V) \subseteq N \cup C \), and \( V \) is not a \( \mathcal{Y}_M \)-neighborhood of 0. The collection \( \{k^{-1}V \cap U; k \text{ is a positive integer and } U \text{ is a closed neighborhood of 0 in } (E, \mathcal{T})\} \) forms a base of \( \mathcal{Y} \)-complete neighborhoods of 0 for a metrizable topology \( \mathcal{Y}_0 \) on \( E \). By [7; Prop. 5, p. 207], \( (M, Y_0) \) is complete, and thus is a Fréchet space.
continuously included in \( E \). \( M \) is a dense proper subspace, since it contains \( N \) and since \( V \) is not a neighborhood of 0 on the barrelled space \( E \).

Clearly, (vi) \( \Rightarrow \) (iii) by the Open Mapping Theorem.

(iii) \( \Rightarrow \) (i). If \( E \) is Banach, the implication is given by Saxon and Wilansky [19]; if \( E \) is non-Banach, (i) holds by Eidelheit [4], since \( \omega \) is separable. \( \blacksquare \)

We note without proof the following elementary lemma.

**Lemma 3.** Let \( G \) be a dense subset of a topological vector space \( E \) and let \( U \) be a closed absolutely convex neighborhood of 0 in \( E \). Then \( U \cap G = U \).

**Theorem 4 (Main Result).** Let \( (F, \gamma) \) be a Fréchet space with a sequence \( \{p_n \} \) of orthogonal projections such that each of the (necessarily closed) subspaces \( P_n[F] \) has a separable (Hausdorff, infinite-dimensional) quotient. Then \( F \) contains a dense subspace \( F_0 \) which, with the relative topology, is a metrizable (LF)-space.

(In short, a Fréchet space has a dense (LF)-subspace if it splits into infinitely many parts, each of which has a separable quotient.)

**Proof.** By (i) \( \Rightarrow \) (vi) of Theorem 3, for each \( n \), there exists a dense proper subspace \( Q_n \) of \( P_n[F] \) which, with a topology \( \tau_n \), finer than the relativization of \( \gamma \), is a Fréchet space. Set \( F_n = P_n^{-1}[Q_n] \) and let \( E_n = \cap_k P_n \) for all \( n, k \). Then by Lemmas 1 and 2, each \( E_n \) is a Fréchet space with the topology \( \gamma_n \) having a base of neighborhoods of 0 \( \{E_n \cap U \cap \cap P_n^{-1} [K_n] : p \leq k, U \) is a neighborhood of 0 in \( F, \gamma \) and \( V_n \) is a \( \tau_n \)-neighborhood of 0 in \( Q_n \) for \( k \leq n \leq p \). Clearly, \( E_n \subseteq E_{n+1} \) and \( \gamma_{n+1} \) induces on \( E_n \) a topology coarser than \( \gamma_n \).

Now, there exists \( x \in P_n[F] \setminus Q_n \) and since the projections \( P_n \) are orthogonal, \( x \in E_{n+k} \setminus E_n \) is properly contained in \( E_{n+k} \). To see that \( E_n \) is dense in \( F \), let \( U_{n+1} \) be a neighborhood base of 0 for \( \gamma \) such that \( U_n \) is absolutely convex and closed, and \( U_{n+1} \subset U_{n+1} \) for \( n \geq 1 \) and let \( x \in F \). For an arbitrary positive integer \( k \), choose \( x_r \in G_{r+1} \) such that \( [x_r-P_n(x)] \in U_{n+k} \) for \( n \geq 1 \). Set \( y = x_r + \sum_{n=1}^{\infty} [x_r - P_n(x)] \).

The series converges to a member of \( U_k \) by a standard argument. Also for each \( j \), \( P_j(y) = P_j(x) + x_r - P_n(x) = x_r \in G_{r+1} \) by orthogonality, so that \( y \in \bigcap_{n=1}^{\infty} P_n^{-1} \{G_n \} \). Therefore, \( (x + U_{n+k}) \cap E_1 \neq \emptyset \), and since \( x \) and \( k \) are arbitrary, \( E_1 \) is dense in \( F \).

Let \( Y_o \) be the finest locally convex topology on \( E_0 = \bigcup_{n=1}^{\infty} E_n \) which induces, for each \( k \), a topology coarser than \( \gamma_k \). Then \( E_k \) is a dense subspace of \( F \) and \( (E_0, \gamma_0) \) is an (LF)-space. Clearly, \( \gamma_o \) is finer than \( \gamma_{k_0} \).

To see the reverse, let \( V \) be an absolutely convex closed neighborhood of 0 in \( (E_0, \gamma_0) \). Let \( k_o \) be fixed. Now, \( V \cap E_{k_0} \) is a \( \gamma_{k_0} \)-neighborhood of 0 so that there exist \( p_0 \geq k_0 \) and a neighborhood \( U_0 \) of 0 in \( (E, \gamma) \) such that

\[
E_{k_0} \cap U_0 \cap \bigcap_{n=k_0}^{p_0} P_n^{-1} [0] \subseteq V.
\]

Choose \( k_1 > p_0 \). Now, \( V \cap E_{k_1} \) is a \( \gamma_{k_1} \)-neighborhood of 0 so that there exist \( p_1 \geq k_1 \) and a \( \gamma_{k_1} \)-neighborhood \( U_1 \) of 0 such that

\[
E_{k_1} \cap \bigcap_{n=k_1}^{p_1} P_n^{-1} [0] \subseteq V.
\]

One easily sees that

\[
P = \left( \sum_{n=k_0}^{p_0} P_n \right) x_{k_0}
\]

is a \( \gamma \)-continuous projection of \( E_{k_0} \) into \( E_{k_0} \).

By orthogonality, \( P \) is idempotent. For each \( n, k_0 \leq n \leq p_0 \),

\[
P_n[E_{k_0}] = P_n \left( \bigcap_{i=k_0}^{\infty} P^{-1}_i \{ G_i \} \right) \subseteq P_n \left( \bigcap_{i=k_0}^{p_0} P^{-1}_i \{ G_i \} \right) = G_{n} \cap E_{k_0}
\]

so that \( P[E_{k_0}] \subseteq E_{k_0} \).

Furthermore, \( P^{-1} [0] = \bigcap_{n=k_0}^{p_0} P^{-1}_n [0] \cap E_{k_0} \) by orthogonality. Then

\[
U_0 \cap P^{-1} [0] \subseteq V \quad \text{and since } E_{k_0} \subseteq E_{k_1} \text{ and } k_1 > p_0,
\]

\[
U_1 \cap P[E_{k_0}] \subseteq E_{k_1} \cap U_1 \cap \bigcap_{n=k_1}^{p_1} P_n^{-1} [0] \subseteq V.
\]

Since \( P^{-1} [0] \) and \( P[E_{k_0}] \) are topological complements in \( (E_{k_0}, \gamma_{k_0}) \),

\[
W = \frac{1}{2} (U_0 \cap P^{-1} [0]) + \frac{1}{2} (U_1 \cap P[E_{k_0}])
\]

is a \( Y_{k_0} \)-neighborhood of 0 and \( W \subseteq \frac{1}{2} V + \frac{1}{2} V = V \). Hence, \( Y_{k_0} = Y_{k_0} \) for \( k = 1, 2, \ldots \), since \( k_0 \) was arbitrary.

Again, fixing \( k_0 \), let \( U \) be an absolutely convex closed neighborhood of 0 in \( (F, \gamma) \) such that \( U \cap E_{k_0} \subseteq V \). By Lemma 3, then, for each \( p \geq k_0 \), \( U \cap E_p \) is the closure of \( U \cap E_{k_0} \) in \( (E_p, \gamma_{p_k}) \) since \( E_{k_0} \) is dense in \( (E_p, \gamma_{p_k}) \), and thus \( U \cap E_p \) is the closure of \( U \cap E_{k_0} \) in \( (E_p, \gamma_{O_k}) \) so that \( U \cap E_p \) is contained in the \( Y_o \)-closed set \( V \). Therefore,

\[
U \cap E_0 \subset U \cap \bigcup_{p \geq k_0} E_p \subset V.
\]

and \( V \) is a \( Y_{k_0} \)-neighborhood of 0; i.e., \( Y_{k_0} = Y_0 \). The conclusion of the theorem follows, setting \( F_0 = E_0 \). \( \blacksquare \)
Corollary 6. The familiar Banach spaces \( l_p \) \((1 \leq p < \infty)\), \( c_0 \), \( C[0, 1] \) and \( L_p[0, 1] \) \((p \geq 1)\) and the familiar (nuclear) Fréchet spaces \( s \) and \( \omega \) all have dense subspaces which, with relative topology, are (LF)-spaces. Indeed so do all Fréchet spaces with an unconditional basis. (See Corollary 8.)

Proof. If \( \{x_i\}^\infty_{i=1} \) is an unconditional basis for a Fréchet space \( E \), then letting \( \{S_k\}^\infty_{k=1} \) be any partition of \( \{1, 2, \ldots\} \) into infinite disjoint sets and, for each \( k \), defining

\[
P_k(\sum_{i=1}^\infty a_i x_i) = \sum_{j \in S_k} a_j x_j
\]

for each \( x = \sum_{i=1}^\infty a_i x_i \) in \( E \), we see that \( \{P_k\} \) is a sequence of orthogonal projections, and each infinite-dimensional subspace \( P_k[E] \) admits a separable quotient (by the trivial subspace \( \{0\} \)). Hence Theorem 4 applies.

Exactly the same technique yields the result for \( l_\infty \), since each \( P_k[l_\infty] \) is isomorphic to \( l_\infty \), which is known to have a separable quotient (see [19; Sec. 2.1–2.5]).

For \( C[0, 1] \), choose an infinite sequence \( \{[a_n, b_n]\}^\infty_{n=1} \) of disjoint nondegenerate subintervals of \([0, 1]\) and choose \( \{[c_n, d_n]\}^\infty_{n=1} \) such that \( a_n < c_n < d_n < b_n \) for each \( n \). Define projections \( P_n \colon C[0, 1] \rightarrow C[0, 1] \) by

\[
\begin{cases}
0 & \text{for } x \notin [a_n, c_n] \text{ or } [d_n, b_n], \\
\frac{f(t)}{t} & \text{for } t < a_n, \quad \frac{f(t)}{t} & \text{for } t \notin (a_n, b_n) \text{ or } (c_n, d_n), \\
l_{\text{linear}}(x) & \text{on } [a_n, c_n] \text{ or } [d_n, b_n].
\end{cases}
\]

Each \( P_n[C[0, 1]] \) is isomorphic to \( C[0, 1] \), thus is infinite-dimensional and separable, and \( \|P_n\| = 1 \). Theorem 4 applies. [Note. \( C[0, 1] \) has a basis, but not an unconditional basis (Singer [20]).]

For \( L_p[0, 1] \) \((p \geq 1)\), the projections as in the above paragraph yield the result.

Corollary 7. There are lot of nonisomorphic normable (LF)-spaces.

4. More on quotients. The next, in conjunction with Theorem 4 (Main result), shows that if a Fréchet space \( F \) has a separable quotient which splits into infinitely many parts, then \( F \) has a dense subspace which is an (LF)-space.

Theorem 5. Let \( Q \colon F \rightarrow G \) be a continuous linear surjection of a Fréchet space \( (F, \gamma) \) onto a Fréchet space \((G, \tau)\). \( G \) has a dense subspace \( G_0 \) which, with the relative topology, is an (LF)-space if and only if \( F \) has a dense subspace \( F_0 \) which, with the relative topology, is an (LF)-space containing \( Q^{-1}[0] \).

Proof. Suppose \( F_0 \) is a dense subspace of \( F \) such that \( (F_0, Y_F) \) is an (LF)-space with \( F_0 \supseteq Q^{-1}[0] \). Let \( M = Q^{-1}[0] \), and \( G_0 = Q[F_0] \). Then \( M \)

is a subspace of \( F_0 \), so that the quotient topology of \( F_0/M \) is induced by that of \( F/M \). Since by the Open Mapping Theorem, \( Q \colon F/M \rightarrow G \) defined by \( Q(x + M) = Q(x), x \in F \), is an isomorphism, its restrictions are also, showing that \( (G_0, \tau_{G_0}) \) is isomorphic to \( F_0/M \), an (LF)-space by Corollary 4. Now, \( F_0 \) is dense in \( F \) so \( Q[F_0] = G_0 \) is dense in \( Q[F] = G \), completing the "if" part of the proof.

Conversely, suppose \( G \) has a dense subspace \( G_0 \) such that \( (G_0, \tau_{G_0}) = \lim \{G_n, \tau_n\}^\infty_{n=1} \) \((G_n, \tau_n) \) are Fréchet spaces, \( n = 1, 2, \ldots \). Define \( F = Q^{-1}[G_0] \) and give the topology \( Y_F \) on \( F \) as in Lemma 1 with a base of neighborhoods of \( 0 \) the set \( \{Q^{-1}[V] \cap U \mid U \subseteq V \text{ and } V \text{ are neighborhoods of } 0 \text{ in } (G_n, \tau_n) \text{ and } (F, \gamma) \text{ respectively} \} \) \((n = 1, 2, \ldots)\). Then \( \{F_n, Y_F\}^\infty_{n=1} \) is a sequence of Fréchet spaces strictly increasing since \( G_n \)

\( \supseteq \).

Let \( F_0 = \bigcup_{n=1}^\infty F_n = \bigcup_{n=1}^\infty Q^{-1}[G_n] = Q^{-1}\left( \bigcup_{n=1}^\infty G_n \right) = Q^{-1}[G_0] \).

\( F_0 \) is dense in \( F \), trivially, for \( Q \) is open and \( G_0 \) is dense in \( G \).

Now, \( \gamma_{F_0} \) is clearly coarser than \( \gamma_F \) for each \( n \), so \( \gamma_{F_0} \) is coarser than \( \gamma_Y \).

On the other hand, let \( W \) be an absolutely convex \( Y_0 \)-neighborhood of \( 0 \) in \( F_0 \). For each \( n, W \cap F_n \) is a \( Y_0 \)-neighborhood of \( 0 \), therefore contains \( U_n \cap Q^{-1}[V] \) for some absolutely convex neighborhoods \( U_n, V_n \) of \( 0 \) in \( F, (G_n, \tau_n) \), respectively. Then we easily check that

\[
Q[W] \supseteq Q[U_n \cap Q^{-1}[V]] = Q[U_n] \cap V_n
\]

and this set is a neighborhood of \( 0 \) in \((G_n, \tau_n) \) since \( Q \) is open and \( \tau_{G_n} \) is coarser than \( \tau_n \). Since the absolutely convex set \( Q[W] \) intersects with each \( G_n \) in a \( \tau_n \)-neighborhood of \( 0 \), \( Q[W] \cap G_0 \) is a neighborhood of \( 0 \) in the inductive limit topology, hence by hypothesis is a \( Y_0 \)-neighborhood of \( 0 \). Since \( Q \) is continuous, \( Q^{-1}[Q[W]] \) is a \( Y_{F_0} \)-neighborhood of \( 0 \). Now, \( Q^{-1}[Q[W]] = Q^{-1}[0] + W \) and, for any fixed \( n, U_n \cap Q^{-1}[V] \subseteq W \) implies that \( W \supseteq U_n \cap Q^{-1}[0] \) \((0 \in V) \) so that we have

\[
(*) \quad 2W = W + W \supseteq (U_n \cap Q^{-1}[0]) + W \supseteq \frac{1}{2} U_n \cap [Q^{-1}[0] + (W \cap \frac{1}{2} U)]
\]

as is easily shown. [Let \( y \) be in the right-hand side, where \( y = z + w, y \in \frac{1}{2} U_n, z \in Q^{-1}[0] \) and \( w \in W \cap \frac{1}{2} U_n \). Then \( z = y - w \in \frac{1}{2} U_n, \frac{1}{2} U_n = U_n, \) so \( z \in U_n \cap Q^{-1}[0] \) and \( y = z + w \in (U_n \cap Q^{-1}[0]) + W \) left-hand side.] But \( W \cap \frac{1}{2} U_n \) is a \( Y_0 \)-neighborhood of \( 0 \) in \( F_0 \), so the preceding argument shows that

\[
Q^{-1}[Q[W \cap \frac{1}{2} U_n]] = (W \cap \frac{1}{2} U) + Q^{-1}[0]
\]
It is a venerable, long-standing open question to whether \((S1)\) and/or \((S2)\) is valid \([17, 19]\). We have shown that \((S3)\) holds for many Banach spaces. By Theorem 6, \((S3)\) implies \((S1)\), so half of \((S6)\) is always valid. Also \((S1) \Leftrightarrow (S4)\) (Theorem 6). Obviously \((S3) \Rightarrow (S4)\). Also \((S2) \Rightarrow (S5)\), for if \(E\) is separable, then \(E\) splits infinitely often via \((S2)\) and Theorem 4 applies. Moreover, by Theorems 4 and 5, \((S2) \Rightarrow (S6)\). It is now apparent that \((S1)\) and \((S2)\) \(\Rightarrow (S3)\). If \((S5)\) holds for all \(E\), then so does \((S6)\) by Theorems 5 and 6.

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References

Homogeneous Besov spaces on locally compact Vilenkin groups

by

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Abstract. In this paper we shall show the equivalence of various characterizations of the homogeneous Besov spaces defined on certain topological groups $G$ that are the locally compact analogue of the compact groups introduced by Vilenkin in 1947. We then apply some of the results to study the regular extension to $G \times Z$ of the distributions belonging to such Besov spaces.

1. Introduction. For $\alpha > 0$ and $0 < p, q \leq \infty$ there exist a large number of equivalent characterizations of the Besov or generalized Lipschitz spaces $B^p_{\alpha}$ on $\mathbb{R}^n$. For early results, subject to the restrictions $\alpha > 0$ and $1 \leq p, q \leq \infty$, see the papers by Besov [2] and Taibleson [13]–[15]. For additional results, see [11] or [20], whereas for the atomic decomposition of Besov spaces on $\mathbb{R}^n$, see [6]. In [12] Ricci and Taibleson considered the harmonic extension to the upper half-plane $\mathbb{R}^2_+$ of functions belonging to certain Besov spaces on $\mathbb{R}$. They introduced a class of function spaces, called $A^p_q$, on $\mathbb{R}^2_+$ and showed that the boundary values of the functions in $A^p_q$ can be identified as linear functionals on certain Besov spaces. In [3] Bui extended their results to $\mathbb{R}^n$. These papers were the motivation for the present paper in which we consider this circle of ideas in the context of a certain class of topological groups instead of $\mathbb{R}$ or $\mathbb{R}^n$.

We now summarize the content of this paper. In the remainder of this section we describe the topological groups $G$ that will be considered here and we give a brief outline of the distribution theory on these groups. In Section 2 we introduce the inhomogeneous and homogeneous Besov spaces on $G$. We present several equivalent (quasi-) norms for these spaces and state a duality theorem. In that section we also compare the inhomogeneous and the

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