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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Metrizable [normable] (LF)-spaces and two classical problems in Fréchet [Banach] spaces

by

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Abstract. It is well known that no strict (LF)-space is metrizable. Also, no (generalized) (LB)-space is metrizable ([17], [18]). In the early 1970's, isolated examples of metrizable and/or normable (LF)-spaces were given by Roelcke [23; p. 269 ff.], De Wilde [5; p. 84], and Saxon [13]. This paper gives a construction for an abundance of metrizable and normable (LF)-spaces: a Fréchet space F has a dense subspace which is an (LF)-space if either

(i) F splits into infinitely many parts each of which has a separable quotient, or
(ii) F has a separable quotient which splits into infinitely many parts.

Note that (ii) is satisfied by every non-Banach Fréchet space, from a result of Eidelheit [4]. Thus every non-Banach Fréchet space is the completion of some (LF)-space (Valdivia and Pérez Carreras [22]), and the same is true for every (infinite-dimensional) Banach space provided the splitting and separable quotient problems have affirmative solutions.

No (LF)-space is both complete and metrizable, since by the Open Mapping Theorem, none is Baire, nor even a (db)-space [17]. In [11], Robertson, Tweddle and Yeomans introduced (db)-spaces, observing (with no distinguishing examples) that

$$\text{unordered Baire-like} \Rightarrow \text{(db)} \Rightarrow \text{Baire-like}.$$

An (LF)-space is metrizable if and only if it is Baire-like [18]. Hence, metrizable (LF)-spaces are precisley those (LF)-spaces which distinguish between Baire-like and (db)-spaces.

Eidelheit [4] showed that every non-Banach Fréchet space E has a quotient which is isomorphic to the (separable) space ω of all scalar sequences. [Choose a sequence $\{f_i\}$ in E' satisfying Bemerkung 3, p. 144, so

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that by Satz 2 of [4], the continuous linear operator $x \mapsto (f_i(x))$ from E into ω $(=(s)$ in [4]) is surjective.] Whether every infinite-dimensional Banach space has a (Hausdorff, infinite-dimensional) separable quotient is a long-standing classical problem, considered e.g. in Rosenthal [12], Lacey [9] and Saxon and Wilansky [19]. Likely, it has been considered since 1932. (In [17], p. 77, we proved that every (LF)-space has a separable quotient.) Here, we show that a given Banach space has a separable quotient if and only if it has a dense subspace which, with a topology finer that the relative topology, is a normable (LF)-space.

Several results pertaining to the quotients of Fréchet and (LF)-spaces are obtained. The paper concludes with a discussion of a number of open questions.

1. Definitions and preliminary results. In general, we assume the terminology and notations in Horváth [7]. We recall from [11], [15], [17], [19] and [21] that φ denotes a fixed $\aleph_0$-dimensional (real or complex) vector space endowed with the finest locally convex topology and that a locally convex space E is

1) *Baire* if E is not the union of a sequence of nowhere dense sets;
2) *unordered Baire-like* if E is not the union of a sequence of nowhere dense absolutely convex sets (cf. [10], [21]);
3) a *(db)-space* if E is not the union of an increasing sequence of subspaces none of which is both dense and barrelled;
4) *Baire-like* if E is not the union of an increasing sequence of nowhere dense absolutely convex sets;
5) *quasi-Baire* if E is barrelled, and is not the union of an increasing sequence of nowhere dense subspaces;
6) an *S_σ-space* if it *is* the union of a strictly increasing sequence of closed subspaces.

Clearly,

Baire $\Rightarrow$ unordered Baire-like $\Rightarrow$ (db) $\Rightarrow$ Baire-like $\Rightarrow$ quasi-Baire,

quasi-Baire $\Rightarrow$ barrelled,

quasi-Baire $\Rightarrow$ not S_σ.

(Note: none of these implication arrows is reversible.)

By a *Fréchet space* we always mean a complete metrizable locally convex space. We shall often use the following special form of Pták's open mapping theorem (p. 299, Proposition 2 in Horváth [7]):

A continuous linear map from a Fréchet space onto a barrelled space is open.

If G is a subspace of a locally convex space (E, τ), $\tau|_G$ denotes the relative topology on G. For a subset $A \subseteq E$, $\mathrm{sp}(A)$ will denote the linear span of A. A continuous linear map P from E into E such that $P(P(x)) = P(x)$ $(x \in E)$ is a *projection* on E. A sequence $\{P_i\}$ of projections on E is an *orthogonal sequence of projections* if $P_i(P_j(x)) = 0$ for all $i \neq j$. A Fréchet space E *splits* if there exist infinite-dimensional closed subspaces M and N such that $M \cap N = \{0\}$ and $M + N = E$. We denote this situation by writing $E = M \oplus N$. We say that E *splits into infinitely many parts* $\{M_n\}$ $(n = 1, 2, \ldots)$ if there exist sequences $\{M_n\}$, $\{N_n\}$ in E such that

$$E = M_1 \oplus N_1, \quad N_1 = M_2 \oplus N_2, \quad N_2 = M_3 \oplus N_3, \ldots$$

Proposition 1. *A Fréchet space E splits into infinitely many parts if and only if there exists a sequence of orthogonal projections with infinite-dimensional ranges.*

Proof. Given $\{M_n\}$ and $\{N_n\}$, define the projections $\{P_n\}$ on E by letting P_i be the identity on M_i and zero on N_i and M_j $(1 \leqslant j < i)$ for each i. The reverse implication is obvious. ∎

A topological space X is said to be *continuously included* in a topological space Y provided $X \subset Y$ and the identity map on X to Y is continuous; i.e., the topology on X is finer than the topology induced by Y.

The next two lemmas can be viewed as special cases of the general result on completeness of inverse limits ([7], Proposition 2.11.3, p. 153). We gratefully accept the referee's very elegant and concise proof.

Lemma 1. *Let (F, Υ) and (G, τ) be Fréchet spaces and Q be a continuous linear map from F into G. Further, suppose that (G_1, τ_1) is a Fréchet space continuously included in (G, τ). Then $F_1 = Q^{-1}[G_1]$ is a Fréchet space under the topology Υ_1 having as a base of neighborhoods of 0 the set $\{U \cap Q^{-1}[V]$: U and V are neighborhoods of 0 in (F, Υ) and (G_1, τ_1) respectively$\}$.*

Proof. $(F_1, \Upsilon_1) \xrightarrow{(I,Q)} (F, \Upsilon) \times (G_1, \tau_1)$ is a topological isomorphism onto a closed subspace. ∎

Lemma 2. *Let (F_n, Υ_n) be a sequence of Fréchet spaces, each of which is continuously included in some Hausdorff space (F, Υ). Then $F_0 = \bigcap_{n=1}^{\infty} F_n$ is a Fréchet space given the topology Υ_0 with a subbase of neighborhoods of 0 the set*

$$\Sigma = \{F_0 \cap U: U \text{ is a } \Upsilon_n\text{-neighborhood of } 0 \text{ for some } n\}.$$

Proof. Again,

$$(F_0, \Upsilon_0) \xrightarrow{\overline{U_n; n\in N}} \prod_{n=1}^{\infty} (F_n, \Upsilon_n)$$

is a topological isomorphism onto a closed subspace. ■

2. Some basic properties of (LF)-spaces. Let (E, τ) be a locally convex Hausdorff space. If there exists a strictly increasing sequence $\{(E_n, \tau_n)\}_{n=1}^{\infty}$ of Fréchet spaces such that $E = \bigcup_{n=1}^{\infty} E_n$, each (E_n, τ_n) is continuously included in (E_{n+1}, τ_{n+1}) and τ is the finest locally convex Hausdorff topology for which (E_n, τ_n) is continuously included in (E, τ) for each n, then (E, τ) is said to be an *(LF)-space*, $\{(E_n, \tau_n)\}_{n=1}^{\infty}$ is an *inductive sequence* which *defines* the (LF)-space (E, τ) and we write

$$(E, \tau) = \underset{\longrightarrow}{\lim} (E_n, \tau_n).$$

Dieudonné, Schwartz, Grothendieck and Köthe pioneered the study of (LF)-spaces (cf. [3], [6], [8]). Note that an absolutely convex set U in E is a τ-neighborhood of 0 if and only if $U \cap E_n$ is a τ_n-neighborhood of 0 for each n, and a linear function f from (E, τ) into a locally convex space F is continuous if and only if $f|_{E_n}$ is continuous for each n. If the inclusion mappings are bicontinuous onto their images (i.e., $\tau_{n+1}|_{E_n} = \tau_n$ for each n), we say that $\{(E_n, \tau_n)\}_{n=1}^{\infty}$ is a *strict inductive sequence* and (E, τ) is a *strict (LF)-space*. If each E_n is a Banach space, (E, τ) is an *(LB)-space* (a *strict (LB)-space* if further $\{(E_n, \tau_n)\}_{n=1}^{\infty}$ is strict). Two inductive sequences $\{(E_n^{(1)}, \tau_n^{(1)})\}_{n=1}^{\infty}$, $\{(E_n^{(2)}, \tau_n^{(2)})\}_{n=1}^{\infty}$ in E (defining two possibly different Hausdorff topologies on E) are said to be *equivalent* if each member of either sequence is continuously included in some member of the other; i.e., if $i \in \{1, 2\}$ and n is arbitrary, there is some k such that $E_n^{(i)} \subset E_k^{(3-i)}$ and $\tau_k^{(3-i)}|_{E_n^{(i)}} \leqslant \tau_n^{(i)}$. One easily sees that equivalent inductive sequences define the same (LF)-space (E, τ).

THEOREM 1 (Grothendieck's Equivalence Theorem). *Let* $(E, \tau^{(i)}) = \underset{\longrightarrow}{\lim} (E_n^{(i)}, \tau_n^{(i)})$ *for* $i = 1, 2$. *The following statements are equivalent*:

(a) $\{(E_n^{(1)}, \tau_n^{(1)})\}_{n=1}^{\infty}$ *is equivalent to* $\{(E_n^{(2)}, \tau_n^{(2)})\}_{n=1}^{\infty}$.

(b) $\tau^{(1)} = \tau^{(2)}$.

(c) *The infimum of* $\tau^{(1)}$ *and* $\tau^{(2)}$ *is Hausdorff.*

Proof. Clearly (a) ⇒ (b) ⇒ (c).

Now (c) ⇒ (b) by Grothendieck [6; p. 17, Thm. B 2] and (b) ⇒ (a) is essentially Cor. 3 to Thm. A, *loc. cit.* ■

Note. One easily proves (c) ⇒ (a) directly by using Lemma 2, the fact that every Fréchet space is (db), and Pták's open mapping theorem.

COROLLARY 1. *Let (E, τ) be a Hausdorff locally convex space. Then there is at most one topology on E finer than τ which makes E an (LF)-space.*

EXAMPLE 1. *There exists a strict (LB)-space E with a nonstrict inductive sequence of non-Banach spaces which defines E.*

Let (s) denote the nonnormable nuclear Fréchet space of all rapidly decreasing sequences. Clearly (s) is continuously included in the Banach space l_1. Let

$$E_n = \underbrace{l_1 \times \ldots \times l_1}_{n \text{ factors}} \times \{0\} \times \{0\} \times \ldots,$$

$$F_n = \underbrace{l_1 \times \ldots \times l_1}_{n \text{ factors}} \times \{s\} \times \{0\} \times \{0\} \times \ldots$$

and let Υ_n and τ_n be the product topologies on E_n, F_n respectively. Now, E_n is continuously included in F_n and F_n is continuously included in E_{n+1} so that $\{(E_n, \Upsilon_n)\}_{n=1}^{\infty}$ and $\{(F_n, \tau_n)\}_{n=1}^{\infty}$ are equivalent inductive sequences in the strict (LB)-space $(E, \tau) = \underset{\longrightarrow}{\lim} (E_n, \Upsilon_n) = \underset{\longrightarrow}{\lim} (F_n, \tau_n)$, with the former a strict inductive sequence of Banach spaces, the latter a nonstrict inductive sequence of non-Banach spaces. ■

EXAMPLE 2. One can easily modify Example 1 (e.g. replace l_1 by l_2 and (s) by l_1) to obtain a strict (LB)-space with a nonstrict inductive sequence of Banach spaces. ■

COROLLARY 2. *If $(E, \tau) = \underset{\longrightarrow}{\lim} (E_n, \tau_n)$ is a strict (LF)-space and also an (LB)-space, then (E, τ) is a strict (LB)-space. In fact, if $\{(E_n, \tau_n)\}_{n=1}^{\infty}$ is a strict inductive sequence, then each (E_n, τ_n) is a Banach space ($n = 1, 2, \ldots$).*

Proof. By hypothesis, there exists a (necessarily equivalent) defining sequence $\{(B_n, \Upsilon_n)\}_{n=1}^{\infty}$ of Banach spaces so that for any n, there exist k and p with $E_n \subset B_k \subset E_p$ where $\tau_p|_{B_k} \leqslant \Upsilon_k$ and $\Upsilon_k|_{E_n} \leqslant \tau_n$ yielding $\tau_p|_{E_n} \leqslant \Upsilon_k|_{E_n} \leqslant \tau_n$. But by strictness, $\tau_p|_{E_n} = \tau_n$ and therefore τ_n is the norm topology $\Upsilon_k|_{E_n}$. ■

It is well known that all (LF)-spaces are barrelled, and we make the observation that no (LF)-space is a (db)-space. For, no E_n can be both dense and barrelled in (E, τ) by Pták's open mapping theorem applied to the identity map from (E_n, τ_n) onto $(E_n, \tau|_{E_n})$, since $E_n \subsetneq E_{n+1}$ for each n. By a similar argument, no (LB)-space is Baire-like; consider an increasing sequence of multiples of the unit balls of $\{E_n\}_{n=1}^{\infty}$. Every strict (LF)-space is S_σ and thus not quasi-Baire.

In [18], we partitioned the class of all (LF)-spaces into three mutually disjoint nonempty classes:

An (LF)-space (E, τ) is of *type* (*i*) or simply an *(LF)$_i$-space* if it satisfies the condition (i) below ($i = 1, 2, 3$):

(1) (E, τ) has a defining sequence none of whose members is dense in (E, τ);

(2) (E, τ) is nonmetrizable and has a defining sequence each of whose members is dense in (E, τ);

(3) (E, τ) is metrizable.

In [18] we have also shown that the (LF)-space (E, τ) is of

type (1) if and only if it contains a complemented copy of φ,
if and only if it contains a closed $\aleph_0$-codimensional subspace,
if and only if it is not quasi-Baire;

type (2) if and only if it contains φ but not φ complemented,
if and only if it is quasi-Baire but not Baire-like;

type (3) if and only if it does not contain φ,
if and only if it is Baire-like (but not (db)).

Hence we see that

$(LF)_1$-spaces are *precisely* the class of (LF)-spaces which distinguish between barrelled and quasi-Baire spaces;

$(LF)_2$-spaces are *precisely* the class of (LF)-spaces which distinguish between quasi-Baire and Baire-like spaces;

$(LF)_3$-spaces are *precisely* the class of (LF)-spaces which distinguish between Baire-like and (db)-spaces.

We note that each of these distinguishing classes of (LF)-spaces is indeed rich: every strict (LF)-space is of type (1); every (LB)-space with a defining sequence of dense subspaces, e.g., the space l_{p^-} of Example 4 below is of type (2); in Section 4, we achieve a primary purpose of this paper by constructing a large class of metrizable (and normable) (LF)-spaces. We give here a quick and concrete

EXAMPLE 3. Let ω denote the Fréchet space of all scalar sequences with the product topology. The Banach space l_p $(p \geq 1)$ is densely and continuously included in ω. Let

$$E_n = \underbrace{\omega \times \ldots \times \omega}_{n \text{ factors}} \times l_p \times l_p \times \ldots \quad (n = 1, 2, \ldots)$$

Then given the product topology, E_n is a strictly increasing sequence of Fréchet spaces, with E_n continuously included in E_{n+1}. One easily sees that $E = \bigcup_{n=1}^{\infty} E_n$ is a dense subspace of the Fréchet space $F = \omega \times \omega \times \ldots$ which, with the relative topology, is a (metrizable) (LF)-space. Hence ω contains a dense (LF)-subspace, since ω is isomorphic to F. ∎

On the one hand, we establish in this paper the abundance of metrizable (LF)-spaces while, on the other hand, (LB)-spaces and strict (LF)-spaces are always nonmetrizable. We now give the promised Example 4, including a nonmetrizable (LF)-space which is neither an (LB)-space nor a strict (LF)-space.

EXAMPLE 4. Let $p > 1$ and choose N such that $p - 1/(N+1) > 1$. Let l_{p^-} denote the (LB)-space

$$\varinjlim_n l_{p-1/(N+n)}.$$

(Note that l_{p^-} is independent of the choice of N). The (LF)-space $\omega \times l_{p^-}$ has $(\omega \times l_{p-1/(N+n)})_{n=1}^{\infty}$ as a defining sequence. Since it contains a copy of l_{p^-}, it is not metrizable, and is, in fact, an $(LF)_2$- space. Thus, it is not an $(LF)_1$-space, and is not a strict (LF)-space. Since there is no Hausdorff vector topology on ω strictly coarser than the product topology, there is no coarser norm topology on ω. Thus by Theorem 1, $\omega \times l_{p^-}$ is not an (LB)-space. ∎

In [18], we show that $\varphi \times l_{p^-}$ is an (LB)-space which is a nonstrict $(LF)_1$-space.

THEOREM 2. *Let* $(E, \Upsilon) = \varinjlim (E_n, \Upsilon_n)$ *be an* $(LF)_i$-*space* $(1 \leq i \leq 3)$ *with a closed subspace* M. *If* $E_n + M = E$ *for some* n, *then the quotient space* E/M *is a Fréchet space*; *otherwise*, E/M *is an* $(LF)_j$-*space for some* $j \geq i$.

Proof. *Case* 1. Suppose for some n, $E_n + M = E$. Then if Q is the (continuous) quotient map from E onto E/M, we have $Q[E_n] = E/M$. Since $\Upsilon|_{E_n} \leq \Upsilon_n$, $Q|_{E_n}$ is a continuous surjection from the Fréchet space (E_n, Υ_n) onto the barrelled space E/M; therefore by Pták's open mapping theorem, E/M is a Fréchet space.

Case 2. Suppose $E_n + M \subsetneq E$ for each n. Then $Q[E_n] \subsetneq E/M$ for each n. Thus for some subsequence $\{(F_p, \eta_p)\}_{p=1}^{\infty}$ of $\{E_n\}_{n=1}^{\infty}$, if $G_p = Q[F_p]$ for $p = 1, 2, \ldots$, then $\{G_p\}$ is a strictly increasing sequence. If τ denotes the quotient topology on E/M, and τ_p the quotient topology of η_p on G_p, then $Q|_{F_p}$ is continuous from (F_p, η_p) onto $(G_p, \tau|_{G_p})$ and therefore $\tau|_{G_p} \leq \tau_p$.

Let $(E/M, \xi) = \varinjlim (G_p, \tau_p)$. Since $(E, \Upsilon) = \varinjlim (F_p, \eta_p)$ and since $Q|_{(F_p, \eta_p)}$ is continuous onto (G_p, τ_p), it is continuous onto $(G_p, \xi|_{G_p})$, $p = 1, 2, \ldots$ Thus we see that Q is continuous from (E, Υ) onto $(E/M, \xi)$. Also the mapping is clearly open for ξ. Therefore, $\xi = \tau$, since there is only one topology on E/M that makes Q continuous and open; i.e., the quotient space E/M is an $(LF)_j$-space for some j, $1 \leq j \leq 3$. If F_p is dense in E, then so is G_p in E/M, by continuity and surjectivity of Q, and E/M is metrizable whenever E is. Thus $j \geq i$. ∎

By relaxing the requirement that inductive sequences must be *strictly* increasing, one could regard Fréchet spaces as the remaining class of (LF)-spaces of *type* (4), in respect of the above theorem. Every (LF)-space of types

(1), (2) or (3) has a (Hausdorff, infinite-dimensional) separable quotient [17]; in [18] we prove that every (LF)-space of type (3) has a quotient which is a separable infinite-dimensional Fréchet space. Also in [18], we prove that the cartesian product of an $(LF)_i$-space with an $(LF)_j$-space is an $(LF)_k$-space where $k = \min(i, j)$; an infinite product of (LF)-spaces is never an (LF)-space; the Hausdorff inductive limit of an increasing sequence of (LF)-spaces is an (LF)-space; a countable-codimensional subspace of an (LF)-space is an (LF)-space if and only if it is closed, and not contained in any member of the defining sequence.

COROLLARY 3 (Köthe's Open Mapping Theorem [7; vol. II, p. 43]). *If f: $F \to G$ is a continuous linear surjection from an (LF)-space F onto an (LF)-space G, then f is open.*

Proof. Let $\bar{f}$ denote the (continuous) associated injection from F/M onto G, where $M = f^{-1}[0]$. Note that f is open if and only if $\bar{f}$ is an isomorphism. Since G is a non-Fréchet barrelled space, F/M cannot be a Fréchet space by Pták's open mapping theorem. Thus, F/M is an (LF)-space by Theorem 2. Since $\bar{f}$ is continuous, it carries a finer (LF)-topology onto G so that by Corollary 1, the two topologies on G coincide. I.e., $\bar{f}$ is an isomorphism, and f is open. ∎

Note. If G is *not* an (LF)-space, the conclusion can fail, even under the hypothesis that F is an $(LF)_3$-space, and G is metrizable and barrelled, by Example 2 of [17].

The following is an alternative version of Theorem 2.

THEOREM 2A. *Let $\{(F_n, \Upsilon_n)\}_{n=1}^{\infty}$ be a defining sequence for an (LF)-space (F_0, Υ_0) and let M be a closed subspace of F_0 with $M \subseteq F_1$. Then (G_0, τ_0) is an (LF)-space defined by the sequence $\{(G_n, \tau_n)\}_{n=1}^{\infty}$, where $G_i = (F_i, \Upsilon_i)/M$ and τ_i is the quotient topology, $i = 0, 1, 2, \ldots$*

COROLLARY 4. *If F_0 is a metrizable (LF)-space and M is a complete subspace, then the quotient F_0/M is a metrizable (LF)-space.*

Proof. Let (F_i, Υ_i) be as in Theorem 2A. The Fréchet space M is a (db)-space. That is to say, one of the covering spaces $M \cap F_N = E_N$ is both dense and barrelled in M. Clearly, $\Upsilon_N|_{E_N}$ is finer than $\Upsilon_0|_{E_N}$ so that $(E_N, \Upsilon_N|_{E_N})$ is a Fréchet space (by Lemma 2) mapped continuously onto $(E_N, \Upsilon_0|_{E_N})$, a dense barrelled subspace of M. Hence by Pták's open mapping theorem, the spaces are isomorphic so that E_N is a dense complete subspace of M; i.e., $M \subseteq F_N$. Since $\{(F_n, \Upsilon_n)\}_{n \geq N}$ is a defining sequence for (F_0, Υ_0), the theorem applies. ∎

COROLLARY 5. *No metrizable (LF)-space is complete.*

Proof. If F_0 is Fréchet, then by Corollary 4, F_0/F_0 is an (LF)-space contradicting the fact that the defining sequences are strictly increasing. ∎

Remark. Some nonmetrizable (e.g., all strict) (LF)-spaces are complete (Köthe [8; vol. I, p. 225]).

3. Constructing metrizable (LF)-spaces. We use the following theorem, of independent interest, in the main result. A former Banach space version appeared in [19]. For a non-Banach Fréchet space E the stated equivalence merely says that conditions (i)–(v) hold, since by the result of Eidelheit [4], E has a quotient isomorphic to ω.

THEOREM 3 (see Corollary 8). *Let (E, Υ) be a Fréchet space. The following statements are equivalent:*

(i) *E has a separable (infinite-dimensional) quotient (by a closed subspace).*
(ii) *E has a dense S_σ-subspace.*
(iii) *E has a dense nonbarrelled subspace.*
(iv) *E has a dense non-(db)-subspace.*
(v) *E densely, properly and continuously includes a Fréchet space M (M can be chosen so as to contain any specified countable subset of E).*

Note. The equivalence of (iii) and (v) is essentially due to Bennett and Kalton [2; p. 512, Prop. 1].

Proof. (i) ⇒ (ii). If M is a closed subspace of E and E/M is infinite-dimensional and separable, then there exists a linearly independent sequence $\{x_n\}$ in E such that $M \cap \mathrm{sp}(\{x_n\}) = \{0\}$ and $M + \mathrm{sp}(\{x_n\})$ is dense in E. Then $\bigcup_{n=1}^{\infty} [M + \mathrm{sp}(\{x_1, \ldots, x_n\})]$ is a dense S_σ-subspace of E.

(ii) ⇒ (iii). This is clear since by [1] or [14], metrizable barrelled spaces are Baire-like (cf. Section 1).

Trivially (iii) ⇒ (iv).

(iv) ⇒ (iii). Suppose M is a dense subspace and M is the union of an increasing sequence $\{M_n\}$ of subspaces none of which is both dense in M and barrelled. Then if M is nonbarrelled, we are through. If M is barrelled, then M is quasi-Baire by [14], so that some M_n is dense in M, hence dense and nonbarrelled in E.

Thus (iii) and (iv) are equivalent.

(iii) ⇒ (v). Suppose N is any dense nonbarrelled subspace of E, and let C be any countable subset of E. Then $\mathrm{sp}(N \cup C)$ is nonbarrelled (see [16]) so there exists a closed absolutely convex set V such that $M = \mathrm{sp}(V) \supseteq N \cup C$, and V is not a $\Upsilon|_M$-neighborhood of 0. The collection $\{k^{-1} V \cap U$: k is a positive integer and U is a closed neighborhood of 0 in $(E, \Upsilon)\}$ forms a base of Υ-complete neighborhoods of 0 for a metrizable topology Υ_0 on M. By [7; Prop. 5, p. 207], (M, Υ_0) is complete, and thus is a Fréchet space continuously included in E. M is a dense proper subspace, since it contains N and since V is not a neighborhood of 0 on the barrelled space E.

Clearly, (v) ⇒ (iii) by the Open Mapping Theorem.

(iii) ⇒ (i). If E is Banach, the implication is given by Saxon and Wilansky [19]; if E is non-Banach, (i) holds by Eidelheit [4], since ω is separable. ∎

We note without proof the following elementary lemma.

LEMMA 3. *Let G be a dense subset of a topological vector space E and let U be a closed absolutely convex neighborhood of 0 in E. Then $\overline{U \cap G} = U$.*

THEOREM 4 (MAIN RESULT). *Let (F, Υ) be a Fréchet space with a sequence $\{P_n\}_{n=1}^{\infty}$ of orthogonal projections such that each of the (necessarily closed) subspaces $P_n[F]$ has a separable (Hausdorff, infinite-dimensional) quotient. Then F contains a dense subspace F_0 which, with the relative topology, is a metrizable (LF)-space.*

(In short, *a Fréchet space has a dense (LF)-subspace if it splits into infinitely many parts, each of which has a separable quotient.*)

Proof. By (i) ⇒ (v) of Theorem 3, for each n, there exists a dense proper subspace G_n of $P_n[F]$ which, with a topology τ_n finer than the relativization of Υ, is a Fréchet space. Set $F_n = P_n^{-1}[G_n]$ and let $E_k = \bigcap_{n \geq k} F_n$ for all n, k. Then by Lemmas 1 and 2, each E_k is a Fréchet space with the topology Υ_k having a base of neighborhoods of 0 $\{E_k \cap U \cap (\bigcap_{n=k}^{p} P_n^{-1}[V_n])$: $p \geq k$, U is a neighborhood of 0 in (F, Υ) and V_n is a τ_n-neighborhood of 0 in G_n for $k \leq n \leq p\}$. Clearly, $E_k \subseteq E_{k+1}$ and Υ_{k+1} induces on E_k a topology coarser than Υ_k.

Now, there exists $x \in P_k[F] \setminus G_k$ and since the projections P_n are orthogonal, $x \in E_{k+1} \setminus E_k$: E_k is properly contained in E_{k+1}. To see that E_1 is dense in F, let $\{U_p\}_{p=1}^{\infty}$ be a neighborhood base of 0 for Υ such that each U_k is absolutely convex and closed, and $U_p + U_p \subseteq U_{p-1}$ for $p \geq 2$ and let $x \in F$. For an arbitrary positive integer k, choose $x_p \in G_p$ such that $[x_p - P_p(x)] \in U_{k+p}$ for $p \geq 1$. Set

$$y = x + \sum_{i=1}^{\infty} [x_i - P_i(x)].$$

The series converges to a member of U_k by a standard argument. Also for each j, $P_j(y) = P_j(x) + x_j - P_j(x) = x_j \in G_j$, by orthogonality, so that $y \in \bigcap_{j=1}^{\infty} P_j^{-1}[G_j] = E_1$. Therefore, $(x + U_k) \cap E_1 \neq \emptyset$, and since x and k are arbitrary, E_1 is dense in F.

Let Υ_0 be the finest locally convex topology on $E_0 = \bigcup_{k=1}^{\infty} E_k$ which induces, for each k, a topology coarser than Υ_k. Then E_0 is a dense subspace of F and (E_0, Υ_0) is an (LF)-space. Clearly, Υ_0 is finer than $\Upsilon|_{E_0}$.

To see the reverse, let V be an absolutely convex closed neighborhood

of 0 in (E_0, Υ_0). Let k_0 be fixed. Now, $V \cap E_{k_0}$ is a Υ_{k_0}-neighborhood of 0 so that there exist $p_0 \geq k_0$ and a neighborhood U_0 of 0 in (F, Υ) such that

$$E_{k_0} \cap U_0 \cap (\bigcap_{n=k_0}^{p_0} P_n^{-1}[0]) \subseteq V.$$

Choose $k_1 > p_0$. Now, $V \cap E_{k_1}$ is a Υ_{k_1}-neighborhood of 0 so that there exist $p_1 \geq k_1$ and a Υ-neighborhood U_1 of 0 such that

$$E_{k_1} \cap U_1 \cap (\bigcap_{n=k_1}^{p_1} P_n^{-1}[0]) \subseteq V.$$

One easily sees that

$$P = (\sum_{n=k_0}^{p_0} P_n)|_{E_{k_0}}$$

is a Υ-continuous projection of E_{k_0} into E_{k_0}.

[By orthogonality, P is idempotent. For each n, $k_0 \leq n \leq p_0$,

$$P_n[E_{k_0}] = P_n(\bigcap_{i=k_0}^{\infty} P_i^{-1}[G_i]) \subseteq P_n[P_n^{-1}[G_n]] = G_n \subset E_{k_0}$$

so that $P[E_{k_0}] \subseteq E_{k_0}$.]

Furthermore, $P^{-1}[0] = \bigcap_{n=k_0}^{p_0} P_n^{-1}[0] \cap E_{k_0}$ by orthogonality. Then $U_0 \cap P^{-1}[0] \subseteq V$ and since $E_{k_0} \subseteq E_{k_1}$ and $k_1 > p_0$,

$$U_1 \cap P[E_{k_0}] \subseteq E_{k_1} \cap U_1 \cap (\bigcap_{m=k_1}^{p_1} P_m^{-1}[0]) \subseteq V.$$

Since $P^{-1}[0]$ and $P[E_{k_0}]$ are topological complements in $(E_{k_0}, \Upsilon|_{E_{k_0}})$,

$$W = \tfrac{1}{2}(U_0 \cap P^{-1}[0]) + \tfrac{1}{2}(U_1 \cap P[E_{k_0}])$$

is a $\Upsilon|_{E_{k_0}}$-neighborhood of 0 and $W \subseteq \frac{1}{2}V + \frac{1}{2}V = V$. Hence, $\Upsilon|_{E_k} = \Upsilon_0|_{E_k}$ for $k = 1, 2, \ldots$, since k_0 was arbitrary.

Again, fixing k_0, let U be an absolutely convex closed neighborhood of 0 in (F, Υ) such that $U \cap E_{k_0} \subseteq V$. By Lemma 3, then, for each $p \geq k_0$, $U \cap E_p$ is the closure of $U \cap E_{k_0}$ in $(E_p, \Upsilon|_{E_p})$ since E_{k_0} is dense in $(E_p, \Upsilon|_{E_p})$, and thus $U \cap E_p$ is the closure of $U \cap E_{k_0}$ in $(E_p, \Upsilon_0|_{E_p})$ so that $U \cap E_p$ is contained in the Υ_0-closed set V. Therefore,

$$U \cap E_0 = U \cap (\bigcup_{p \geq k_0} E_p) \subseteq V,$$

and V is a $\Upsilon|_{E_0}$-neighborhood of 0; i.e., $\Upsilon|_{E_0} = \Upsilon_0$. The conclusion of the theorem follows, setting $F_0 = E_0$. ∎

Corollary 6. *The familiar Banach spaces* l_p $(1 \leqslant p \leqslant \infty)$, c_0, $C[0, 1]$ *and* $L_p[0, 1]$ $(p \geqslant 1)$ *and the familiar (nuclear) Fréchet spaces* (s) *and* ω *all have dense subspaces which, with relative topology, are (LF)-spaces. Indeed so do all Fréchet spaces with an unconditional basis.* (See Corollary 8.)

Proof. If $\{x_i\}_{i=1}^{\infty}$ is an unconditional basis for a Fréchet space E, then letting $\{S_k\}_{k=1}^{\infty}$ be any partition of $\{1, 2, \ldots\}$ into infinite disjoint sets and, for each k, defining

$$P_k\Big(\sum_{i=1}^{\infty} a_i x_i\Big) = \sum_{j \in S_k} a_j x_j$$

for each $x = \sum_{i=1}^{\infty} a_i x_i$ in E, we see that $\{P_k\}_{k=1}^{\infty}$ is a sequence of orthogonal projections, and each infinite-dimensional subspace $P_k[E]$ admits a separable quotient (by the trivial subspace $\{0\}$). Hence Theorem 4 applies.

Exactly the same technique yields the result for l_∞, since each $P_k[l_\infty]$ is isomorphic to l_∞, which is known to have a separable quotient (see [19; Sec. 2.1–2.5]).

For $C[0, 1]$, choose an infinite sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ of disjoint nondegenerate subintervals of $[0, 1]$ and choose $\{[c_n, d_n]\}_{n=1}^{\infty}$ such that $a_n < c_n < d_n < b_n$ for each n. Define projections P_n: $C[0, 1] \to C[0, 1]$ by

$$(P_n(f))(t) = \begin{cases} f(t) & \text{for } c_n \leqslant t \leqslant d_n, \\ 0 & \text{for } t \notin (a_n, b_n), \\ \text{linear on } [a_n, c_n] \text{ and } [d_n, b_n]. \end{cases}$$

Each $P_n[C[0, 1]]$ is isomorphic to $C[0, 1]$, thus is infinite-dimensional and separable, and $\|P_n\| = 1$. Theorem 4 applies. [Note. $C[0, 1]$ has a basis, but not an unconditional basis (Singer [20]).]

For $L_p[0, 1]$ $(p \geqslant 1)$, the projections as in the above paragraph yield the result. ∎

Corollary 7. *There are lots of nonisomorphic normable (LF)-spaces.*

4. More on quotients. The next theorem, in conjunction with Theorem 4 (Main result), shows that if a Fréchet space F has a separable quotient which splits into infinitely many parts, then F has a dense subspace which is an (LF)-space.

Theorem 5. *Let* Q: $F \to G$ *be a continuous linear surjection of a Fréchet space* (F, Υ) *onto a Fréchet space* (G, τ). G *has a dense subspace* G_0 *which, with the relative topology, is an (LF)-space if and only if* F *has a dense subspace* F_0 *which, with the relative topology, is an (LF)-space containing* $Q^{-1}[0]$.

Proof. Suppose F_0 is a dense subspace of F such that $(F_0, \Upsilon|_{F_0})$ is an (LF)-space with $F_0 \supseteq Q^{-1}[0]$. Let $M = Q^{-1}[0]$, and $G_0 = Q[F_0]$. Then M

is a subspace of F_0, so that the quotient topology of F_0/M is induced by that of F/M. Since by the Open Mapping Theorem, $\bar{Q}$: $F/M \to G$ defined by $\bar{Q}(x+M) = Q(x)$, $x \in F$, is an isomorphism, its restrictions are also, showing that $(G_0, \tau|_{G_0})$ is isomorphic to F_0/M, an (LF)-space by Corollary 4. Now, F_0 is dense in F so $Q[F_0] = G_0$ is dense in $Q[F] = G$, completing the "if" part of the proof.

Conversely, suppose G has a dense subspace G_0 such that $(G_0, \tau|_{G_0}) = \varinjlim \{(G_n, \tau_n)\}_{n=1}^{\infty}$ (where (G_n, τ_n) are Fréchet spaces, $n = 1, 2, \ldots$). Define $F_n = Q^{-1}[G_n]$ and give the topology Υ_n on F_n as in Lemma 1 with a base of neighborhoods of 0 the set $\{Q^{-1}[V] \cap U$: V and U are neighborhoods of 0 in (G_n, τ_n) and (F, Υ) respectively$\}$. Then $\{(F_n, \Upsilon)\}_{n=1}^{\infty}$ is a sequence of Fréchet spaces strictly increasing since $\{G_n\}_{n=1}^{\infty}$ is.

Let $(F_0, \Upsilon) = \varinjlim (F_n, \Upsilon_n)$. Since

$$F_0 = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} Q^{-1}[G_n] = Q^{-1}\Big[\bigcup_{n=1}^{\infty} G_n\Big] = Q^{-1}[G_0],$$

F_0 is dense in F, trivially, for Q is open and G_0 is dense in G.

Now, $\Upsilon|_{F_n}$ is clearly coarser than Υ_n for each n, so $\Upsilon|_{F_0}$ is coarser than Υ_0. On the other hand, let W be an absolutely convex Υ_0-neighborhood of 0 in F_0. For each n, $W \cap F_n$ is a Υ_n-neighborhood of 0, therefore contains $U_n \cap Q^{-1}[V_n]$ for some absolutely convex neighborhoods U_n, V_n of 0 in F, (G_n, τ_n), respectively. Then we easily check that

$$Q[W] \supseteq Q[U_n \cap Q^{-1}[V_n]] = Q[U_n] \cap V_n,$$

and this set is a neighborhood of 0 in (G_n, τ_n) since Q is open and $\tau|_{G_n}$ is coarser than τ_n. Since the absolutely convex set $Q[W]$ intersects with each G_n in a τ_n-neighborhood of 0, $Q[W] = Q[W] \cap G_0$ is a neighborhood of 0 in the inductive limit topology, hence by hypothesis is a $\Upsilon|_{G_0}$-neighborhood of 0. Since Q is continuous, $Q^{-1}[Q[W]]$ is a $\Upsilon|_{F_0}$-neighborhood of 0. Now, $Q^{-1}[Q[W]] = Q^{-1}[0] + W$ and, for any fixed n, $U_n \cap Q^{-1}[V_n] \subseteq W$ implies that $W \supseteq U_n \cap Q^{-1}[0]$ $(0 \in V_n)$ so that we have

$$(*) \qquad 2W = W + W$$
$$\supseteq (U_n \cap Q^{-1}[0]) + W \supseteq \tfrac{1}{2}U_n \cap [Q^{-1}[0] + (W \cap \tfrac{1}{2}U_n)]$$

as is easily shown. [Let y be in the right-hand side, where $y = z + w$, $y \in \frac{1}{2}U_n$, $z \in Q^{-1}[0]$ and $w \in W \cap \frac{1}{2}U_n$. Then $z = y - w \in \frac{1}{2}U_n - \frac{1}{2}U_n = U_n$ so $z \in U_n \cap Q^{-1}[0]$ and $y = z + w \in (U_n \cap Q^{-1}[0]) + W =$ left-hand side.] But $W \cap \frac{1}{2}U_n$ is a Υ_0-neighborhood of 0 in F_0, so the preceding argument shows that

$$Q^{-1}[Q[W \cap \tfrac{1}{2}U_n]] = (W \cap \tfrac{1}{2}U_n) + Q^{-1}[0]$$

is a $\Upsilon|_{F_0}$-neighborhood of 0; therefore, so is its intersection with $\frac{1}{2}U_n$. Thus (*) shows that $2W$, and therefore W, is a $\Upsilon|_{F_0}$-neighborhood of 0. That is, $\Upsilon_0 = \Upsilon|_{F_0}$, completing the proof. ∎

COROLLARY 8 (Eidelheit [4], Valdivia and Pérez Carreras [22], Saxon and Narayanaswami). *If E is a non-Banach Fréchet space, then conditions* (i)–(v) *of Theorem* 3 *hold, and moreover E has a dense (LF)-subspace.*

Proof. By Eidelheit [4], E has a quotient isomorphic to ω, so (i)–(v) hold by Theorem 3. Also ω has a dense (LF)-subspace by either Example 3 or Corollary 6. Thus, so does E by Theorem 5. ∎

THEOREM 6. *Let (E, Υ) be a Banach space. Conditions* (i)–(v) *of Theorem* 3 *are equivalent to the following condition*:

(vi) *There exists a dense subspace E_0 and a topology Υ_0 on E_0 finer than $\Upsilon|_{E_0}$ such that (E_0, Υ_0) is a normable (LF)-space.*

Proof. (vi) ⇒ [(i)–(v)]. Suppose (E_0, Υ_0) exists as in (vi). Then there is a strictly increasing sequence $\{(E_n, \Upsilon_n)\}_{n=1}^{\infty}$ of Fréchet spaces such that $E_0 = \bigcup_{n=1}^{\infty} E_n$ where each Υ_n is finer than $\Upsilon_0|_{E_n}$, hence finer than $\Upsilon|_{E_n}$. Thus, either $\bigcup_{n=1}^{\infty} \bar{E}_n$ is a dense S_σ-subspace of E [(ii)] or one of the Fréchet spaces (E_n, Υ_n) is a dense (and necessarily proper) subspace of E [(v)].

(i) ⇒ (vi). Assume M is a closed subspace of E such that E/M is a separable (infinite-dimensional) Banach space. One readily sees that E/M must densely and continuously include a copy of the Banach space l_1. [Let $\{x_i, f_i\}_{i=1}^{\infty}$ be any biorthogonal sequence such that $\operatorname{sp}(\{x_i\})$ is dense in E/M; identify the unit vectors in l_1 with small multiples of the x_i's.]

Let Q be the quotient map of E onto E/M. By Lemma 1, $F = Q^{-1}[l_1]$ is a Banach space with a topology μ finer than $\Upsilon|_F$, such that $Q|_F$ is μ-continuous onto l_1. Also, F is dense in E, trivially, since l_1 is dense in E/M and Q is open. By Corollary 6, l_1 has a dense (LF)-subspace G_0. The desired conclusion follows from Theorem 5. ∎

5. Some related open questions. We conclude by relating some open questions in Banach spaces. If E is an arbitrary infinite-dimensional Banach space, the following statements may or may not be true:

(S1): E has a separable quotient.
(S2): Every separable space splits.
(S3): E has a dense subspace which, with the relative topology, is a (normable) (LF)-space.
(S4): E has a dense subspace which, with a topology finer than the relative topology, is a normable (LF)-space.
(S5): If E is separable, (S3) holds.
(S6): (S1) holds if and only if (S3) holds.

It is a venerable, long-outstanding open question as to whether (S1) and/or (S2) is valid ([17], [19]). We have shown that (S3) holds for many Banach spaces. By Theorem 6, (S3) implies (S1), so half of (S6) is always valid. Also (S1) ⇔ (S4) (Theorem 6). Obviously (S3) ⇒ (S4). Also (S2) ⇒ (S5), for if E is separable, then E splits infinitely often via (S2) and Theorem 4 applies. Moreover, by Theorems 4 and 5, (S2) ⇒ (S6). It is now apparent that [(S1) and (S2)] ⇒ (S3). If (S5) holds for all E, then so does (S6) by Theorems 5 and 6.

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Homogeneous Besov spaces on locally compact Vilenkin groups

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Abstract. In this paper we shall show the equivalence of various characterizations of the homogeneous Besov spaces defined on certain topological groups G that are the locally compact analogue of the compact groups introduced by Vilenkin in 1947. We then apply some of the results to study the regular extension to $G \times Z$ of the distributions belonging to such Besov spaces.

1. Introduction. For $\alpha > 0$ and $0 < p, q \leqslant \infty$ there exist a large number of equivalent characterizations of the Besov or generalized Lipschitz spaces B_{pq}^{α} on $\boldsymbol{R}^n$. For early results, subject to the restrictions $\alpha > 0$ and $1 \leqslant p, q \leqslant \infty$, see the papers by Besov [2] and Taibleson [13]–[15]. For additional results, see [11] or [20], whereas for the atomic decomposition of Besov spaces on $\boldsymbol{R}^n$, see [6]. In [12] Ricci and Taibleson considered the harmonic extension to the upper half-plane $\boldsymbol{R}_+^2$ of functions belonging to certain Besov spaces on $\boldsymbol{R}$. They introduced a class of function spaces, called A_{pq}^{α}, on $\boldsymbol{R}_+^2$ and showed that the boundary values of the functions in A_{pq}^{α} can be identified as linear functionals on certain Besov spaces. In [3] Bui extended their results to $\boldsymbol{R}^n$. These papers were the motivation for the present paper in which we consider this circle of ideas in the context of a certain class of topological groups instead of $\boldsymbol{R}$ or $\boldsymbol{R}^n$.

We now summarize the content of this paper. In the remainder of this section we describe the topological groups G that will be considered here and we give a brief outline of the distribution theory on these groups. In Section 2 we introduce the inhomogeneous and homogeneous Besov spaces on G. We present several equivalent (quasi-) norms for these spaces and state a duality theorem. In that section we also compare the inhomogeneous and the

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