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Equivalent norms on separable Asplund spaces

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Abstract. Every Banach space E with separable dual has an equivalent norm such that the dual norm on E^* shares many properties of the norm of l^1. For example, the unit ball of E^* is the closed convex circled hull of a sequence of strongly exposed points (with a uniformity in the strong exposition), similarly to the ball of l^1 which is the closed convex circled hull of the unit vectors. As an application we show e.g. that for given $\varepsilon > 0$, there is an equivalent norm on E such that for every proper subspace $F \subsetneq E^*$ the ball $\varrho B(F)$ of radius ϱ of F is not weak-star dense in the unit ball of E^* whenever $\varrho < 2-\varepsilon$. This is similar to the situation for the case $E^* = l^1$, where this result holds true even for $\varepsilon = 0$. In particular, the mentioned result solves affirmatively a question of D. van Dulst and I. Singer [2].

Introduction. We consider the problem of renorming separable Asplund spaces to have some "smoothness properties".

In particular, we are interested in the characteristics of subspaces of dual spaces. The notion of characteristic was introduced by J. Dixmier [1]. Let us recall the definition. The *characteristic* of a subspace X of a dual space E^* is the greatest number $r = r(X)$ such that the unit ball of X is $\sigma(E^*, E)$ dense in the ball of E^* of radius r. Obviously $0 \leqslant r(X) \leqslant 1$.

D. van Dulst and I. Singer studied the influence of Kadec–Klee norms on the characteristics [2]. If E is a Banach space with separable dual and if the norm of E is a Kadec–Klee norm then any proper subspace of E^* has a characteristic strictly less than one. Therefore every Banach space E with separable dual has an equivalent norm such that any proper subspace of E^* has a characteristic strictly less than one.

D. van Dulst and I. Singer raised the following problem. Let E be a nonreflexive Banach space with separable dual. Does there exist an equivalent norm on E such that for some constant $\alpha < 1$, we have $r(X) \leqslant \alpha$ for all proper subspaces X of E^*? In this note we shall settle this question affirmatively: It is possible to renorm a Banach space E with separable dual so that the number $\chi(E)$ is strictly less than one, where $\chi(E)$ is the supremum of the characteristics for all proper subspaces of E^* (Corollary 8).

Let us note that for a nonreflexive Banach space E, we have $\chi(E) \geqslant 1/2$ [2].

G. Godefroy and P. Saphar proved that if E is an M-ideal of its bidual then $\chi(E) = 1/2$ [5]. And it was shown by B. V. Godun and M. I. Kadets that if E is a quasireflexive Banach space then there exists an equivalent norm such that $\chi(E) = 1/2$ [6]. Our renorming technique provides the following result: If E is a separable Asplund space then for every $\varepsilon > 0$, E has a (Fréchet-differentiable) renorming satisfying $\chi(E) \leqslant \frac{1}{2}+\varepsilon$ (Theorem 9).

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Notation, definitions. Let E be a Banach space. We denote by $B(E)$ (resp. $S(E)$) the closed unit ball (resp. the unit sphere) of E. We denote by E^* the dual of E. $[x_n]$ denotes the norm-closed linear span of a sequence (x_n). We also use the notion of ω^*-basic sequence [7]. A sequence $(x_n^*) \subset E^*$ is called *ω^*-basic* provided that there is a sequence $(x_n) \subset E$ so that (x_n, x_n^*) is biorthogonal and for each y in the ω^*-closure of the space generated by (x_n^*),

$$\sum_{i=1}^{n} y(x_i)\, x_i^* \xrightarrow{\omega^*} y.$$

A. Renormings of separable Asplund spaces. The following properties have been used by J. Lindenstrauss [8], J. Partington [9] and the second author [10] in the study of norm attaining operators.

Definition 1. (a) The Banach space X has *property* α if there are $0 \leqslant \lambda < 1$ and a family $\{(x_\alpha, y_\alpha)\}_{\alpha\in I} \in X \times X^*$, with $\|x_\alpha\| = \|y_\alpha\| = \langle x_\alpha, y_\alpha\rangle = 1$, such that:

(i) For $\beta \neq \alpha$, $|\langle x_\alpha, y_\beta\rangle| \leqslant \lambda$.
(ii) The unit ball of X is the norm-closed circled convex hull of $\{x_\alpha\}_{\alpha\in I}$.

(b) The Banach space X has *property* β if there are $0 \leqslant \lambda < 1$ and a family $\{(x_\alpha, y_\alpha)\}_{\alpha\in I} \in X \times X^*$, with $\|x_\alpha\| = \|y_\alpha\| = \langle x_\alpha, y_\alpha\rangle = 1$, such that:

(i) For $\beta \neq \alpha$, $|\langle x_\alpha, y_\beta\rangle| \leqslant \lambda$.
(ii) For $x \in X$, $\|x\| = \sup\{|\langle x, y_\alpha\rangle| : \alpha \in I\}$.

The typical example of a space with property α is l^1; on the other hand, the spaces c_0, l^∞ and finite-dimensional spaces with polyhedral unit balls have property β.

The duality between properties α and β has been studied in [10]: X has property α iff X^* with the dual norm has property β, where in addition to (i) and (ii) of Definition 1(b) the family $(y_\alpha)_{\alpha\in I} \subseteq X^{**}$ is in X. A typical example is the pair $X = l^1$ and $X^* = l^\infty$.

The example of the pair $X = c_0$ and $X^* = l^1$ leads us to introduce a new property which we shall call property β in the strong sense:

Definition 2. X has *property* β *in the strong sense* if X has property β and in addition the unit ball of X^* is the norm-closed circled convex hull of $(y_\alpha)_{\alpha\in I}$.

Clearly X has property β in the strong sense iff X^* has property α, where the family $(y_\alpha)_{\alpha\in I} \subseteq X^{**}$ in Definition 1(a) may be chosen in X. Note that in general the unit ball of X^* is the ω^*-closed convex hull of $(y_\alpha)_{\alpha\in I}$. Also note that a separable Banach space with property β in the strong sense has a separable dual.

Of course, property β in the strong sense implies property β, while the converse is not true. Moreover, if X is a reflexive Banach space then X has property β in the strong sense iff X has property β iff X^* has property α.

In [9], it is shown that every Banach space has an equivalent renorming satisfying property β. On the other hand, every WCG space may be equivalently renormed to have property α [10].

In this section we show that every Banach space with separable dual may be equivalently renormed to have property β in the strong sense.

The next lemma is a consequence of the results of [7].

Lemma 3. *Let X be a separable Banach space. Then X^* contains an ω^*-basic normalized sequence which is not equivalent to the usual basis of l^1.*

Proof. By [7], we may find an ω^*-basic normalized sequence $(x_n^*)_{n=1}^\infty$ in X^* and an associated biorthogonal sequence $(x_n)_{n=1}^\infty$ in X.

If (x_n^*) is not equivalent to the l^1-basis we are done. If (x_n^*) is equivalent to the usual basis of l^1, write $(x_n)_{n=1}^\infty$ and $(x_n^*)_{n=1}^\infty$ as a double-indexed sequence $((x_k^l, x_k^{*l})_{l=1}^{2^k})_{k=1}^\infty$, where the correspondence is given by $n = 2^k+l-2$. Define, for $k \in N$ and $1 \leqslant m \leqslant k$,

$$z_k^m = \sum_{l=1}^{2^k} r_m\big((l-1/2)/2^k\big)\, x_k^l,$$

$$z_k^{*m} = 2^{-k} \sum_{l=1}^{2^k} r_m\big((l-1/2)/2^k\big)\, x_k^{*l},$$

where r_m denotes the mth Rademacher function on $[0, 1]$. Clearly the double sequence $((z_k^m, z_k^{*m})_{m=1}^k)_{k=1}^\infty$ is biorthogonal and the sequence $((\|z_k^{*m}\|)_{m=1}^k)_{k=1}^\infty$ is bounded from above and below. Hence letting $e_k^{*m} = z_k^{*m}/\|z_k^{*m}\|$, we obtain an ω^*-basic normalized sequence which is not equivalent to the usual basis of l^1 as, by Khinchin's inequality, the sequence $(\|k^{-1/2}\sum_{l=1}^k e_k^{*l}\|)_{k=1}^\infty$ stays bounded.

Finally, relabel $((e_k^{*m})_{m=1}^k)_{k=1}^\infty$ as $(e_n^*)_{n=1}^\infty$, and the lemma is proved.

The next proposition uses the idea employed in [9] and [10].

Proposition 4. *Let X be a separable Banach space, D a norm-separable ω^*-compact subset of X^* and $0 < \lambda < 1$. There exist bounded families $((x_{i,j})_{i=1}^\infty)_{j=1}^\infty$ in X and $((x_{i,j}^*)_{i=1}^\infty)_{j=1}^\infty$ in X^* with $\langle x_{i,j}, x_{i,j}^*\rangle = 1$ such that:*

(i) *For $(i, j) \neq (k, l)$, $|\langle x_{i,j}, x_{k,l}^*\rangle| \leqslant \lambda$.*

(ii) *If C denotes the norm-closed circled convex hull of* $((x_{i,j}^*)_{i=1}^{\infty})_{j=1}^{\infty}$, *then C is ω^*-compact and C contains D.*

Proof. Without loss of generality, we may assume D to be circled and convex. Apply Lemma 3 to find a normalized ω^*-basic sequence $(e_n^*)_{n=1}^{\infty}$ in X^* which is not equivalent to the usual basis of l^1. It is easy to see that we may rearrange it into a double-indexed sequence $((e_{i,j}^*)_{j=1}^{\infty})_{i=1}^{\infty}$ in such a way that each of the sequences $(e_{i,j}^*)_{j=1}^{\infty}$ is not equivalent to the usual basis of l^1.

By changing signs if necessary, we may therefore assume that for each $i\in N$ and $\varepsilon>0$ there are scalars $(\mu_j)_{j=1}^M$, $0\leqslant\mu_j\leqslant 1$, $\sum_{j=1}^M\mu_j=1$, such that

$$\Big\|\sum_{j=1}^{M}\mu_j e_{i,j}^*\Big\|<\varepsilon.$$

Let $((e_{i,j})_{j=1}^{\infty})_{i=1}^{\infty}$ be a bounded sequence biorthogonal to $((e_{i,j}^*)_{j=1}^{\infty})_{i=1}^{\infty}$ and $(z_i^*)_{i=1}^{\infty}$ a dense sequence in D. Let $M=\sup\{\|e_{i,j}\|:\ i,j\in N\}$, $N=\sup\{\|z^*\|:\ z^*\in D\}$ and $K>(\lambda^{-1}+1)MN$. For $i,j\in N$ define $x_{i,j}$ and $x_{i,j}^*$ in the following way:

$$x_{i,j}^*=z_i^*+Ke_{i,j}^*,\qquad x_{i,j}=e_{i,j}/\langle e_{i,j},\,x_{i,j}^*\rangle.$$

Clearly the sequences $((x_{i,j})_{j=1}^{\infty})_{i=1}^{\infty}$ and $((x_{i,j}^*)_{j=1}^{\infty})_{i=1}^{\infty}$ are bounded since

$$|\langle e_{i,j},\,x_{i,j}^*\rangle|\geqslant|\langle e_{i,j},\,Ke_{i,j}^*\rangle|-|\langle e_{i,j},\,z_i^*\rangle|\geqslant K-MN>\lambda^{-1}MN.$$

And for $i,j\in N$, one has $\langle x_{i,j},\,x_{i,j}^*\rangle=1$.

On the other hand, if $(i,j)\neq(k,l)$

$$|\langle x_{i,j},\,x_{k,l}^*\rangle|\leqslant|\langle x_{i,j},\,z_k^*\rangle|+K|\langle x_{i,j},\,e_{k,l}^*\rangle|$$
$$<MN/(\lambda^{-1}MN)+0\leqslant\lambda.$$

The set C contains D. Indeed, for given $i\in N$ and $\varepsilon>0$ choose $(\mu_j)_{j=1}^M$ as above. Then

$$\Big\|z_i^*-\sum_{j=1}^{M}\mu_j x_{i,j}^*\Big\|=K\Big\|\sum_{j=1}^{M}\mu_j e_{i,j}^*\Big\|<K\varepsilon.$$

Hence C contains each z_i^* and therefore D.

The set C is ω^*-compact. Indeed, let $(w_\alpha^*)_{\alpha\in I}=(\sum_{i,j}\mu_{\alpha,i,j}x_{i,j}^*)_{\alpha\in I}$ be a net of convex combinations of $(x_{i,j}^*)_{i,j=1}^{\infty}$ converging weak-star to some $w^*\in X^*$. We have to show that w^* is in C. We may assume that, for every (i,j), the net $(\mu_{\alpha,i,j})_{\alpha\in I}$ converges to some $\mu_{i,j}$. Let

$$\mu=\sum_{i,j}\mu_{i,j}x_{i,j}^*.$$

Clearly $0\leqslant\mu\leqslant 1$ and we may find a finite subset $F\subset N\times N$ such that $\mu-\sum_{(i,j)\in F}\mu_{i,j}x_{i,j}^*<\varepsilon$.

For $\alpha\in I$, let

$$t_\alpha^*=\sum_{(i,j)\in F}\mu_{\alpha,i,j}x_{i,j}^*,$$
$$u_\alpha^*=\sum_{(i,j)\in N^2\setminus F}\mu_{\alpha,i,j}z_i^*,$$
$$v_\alpha^*=K\sum_{(i,j)\in N^2\setminus F}\mu_{\alpha,i,j}e_{i,j}^*.$$

One has $w_\alpha^*=t_\alpha^*+u_\alpha^*+v_\alpha^*$. We may assume that $(t_\alpha^*)_{\alpha\in I}$, $(u_\alpha^*)_{\alpha\in I}$ and $(v_\alpha^*)_{\alpha\in I}$ converge weak-star to t^*, u^*, v^* respectively. Clearly,

$$t^*=\sum_{(i,j)\in F}\mu_{i,j}x_{i,j}^*\in\mu C.$$

On the other hand, for $\delta>0$, there is $\alpha_0\in I$ such that for $\alpha\geqslant\alpha_0$

$$\sum_{(i,j)\in N^2\setminus F}\mu_{\alpha,i,j}<(1-\mu)+\delta,\qquad\text{hence}\qquad u_\alpha^*\in\big((1-\mu)+\delta\big)D.$$

Since D is ω^*-compact and is contained in C and δ is arbitrary, $u^*\in(1-\mu)C$, whence $t^*+u^*\in C$.

Since the ω^*-basic sequence $(e_n^*)_{n=1}^{\infty}$ ω^*-converges to zero ([7], Remark III.2),

$$v^*=K\sum_{(i,j)\in N^2\setminus F}\mu_{i,j}e_{i,j}^*,$$

and therefore $\|v^*\|\leqslant K\varepsilon$, so that $\operatorname{dist}(w^*,C)\leqslant\|v^*\|\leqslant\varepsilon K$.

This shows that C is ω^*-compact and we have proved the proposition. ■

Corollary 5. *Every Banach space X with separable dual may be equivalently renormed to have property β in the strong sense. In fact, the constant λ appearing in Definition 1 may be chosen arbitrarily in* $]0,1[$.

Proof. Given $\lambda\in]0,1[$ apply Proposition 4 with D the unit ball of X^* to find bounded families $(x_{i,j})_{i,j=1}^{\infty}$ in X and $(x_{i,j}^*)_{i,j=1}^{\infty}$ in X^* with $\langle x_{i,j},\,x_{i,j}^*\rangle=1$ satisfying (i) and (ii). For $x\in X$, define $|||x|||=\sup\{|\langle x,\,x_{i,j}^*\rangle|:\ i,j\in N\}$. This is an equivalent norm on X such that $(X,|||\cdot|||)$ has property β in the strong sense. ■

B. Applications

1. A smoothness property. Let us recall the definition of almost uniform smoothness introduced in [3]. Let X be a Banach space. $\mathscr{D}(X)$ is the set of the points of the unit sphere where the norm is smooth; for every $x\in\mathscr{D}(X)$, we denote by f_x the support-mapping at the point x.

Definition 6. We say that X is *almost uniformly smooth* (a.u.s.) if there exists a family $(A_\varepsilon)_{0<\varepsilon<1}$ of subsets of $\mathscr{D}(X)$ such that:

(i) $\forall \varepsilon \in]0, 1[, \exists \delta(\varepsilon) > 0$:

$$[y \in B(X^*),\ x \in A_\varepsilon \text{ and } y(x) \geqslant 1-\delta(\varepsilon)] \Rightarrow \|y-f_x\| \leqslant \varepsilon.$$

(ii) $\forall \varepsilon \in]0, 1[$: $B(X^*) = \overline{\text{conv}}\,\{f_x\colon x \in A_\varepsilon\} + \varepsilon B(X^*)$.

EXAMPLES. $c_0(\Gamma)$ is a.u.s. for any Γ. Every superreflexive Banach space is a.u.s. for every equivalent norm. If X and Y are superreflexive Banach spaces then the space of compact operators from X to Y is a.u.s.

The motivation for the introduction of this class was the following: among the a.u.s. Banach spaces a nice characterization of dual spaces is available [4]: an a.u.s. Banach space X is isometric to a dual space iff one has $r_X(B) = r_{X^{**}}(B)$, for every bounded subset B of X, where $r_X(B)$ denotes the Chebyshev radius of B in X,

$$r_X(B) = \inf_{x\in X}\{\sup_{y\in B}\|x-y\|\}.$$

Let X be a Banach space with property β in the strong sense. There exist $\lambda \in [0, 1[$ and a family $(x_\alpha, y_\alpha)_{\alpha\in I} \subset X\times X^*$ as in Definition 2. Note that the points $(y_\alpha)_{\alpha\in I}$ are uniformly strongly exposed by ω^*-continuous functionals: indeed, let $\varepsilon > 0$, choose $\delta(\varepsilon) = (1-\lambda)\varepsilon/2$. Let $x^* \in B(X^*)$ and $\alpha \in I$. Without loss of generality, we may assume that there exist scalars μ_i ($i = 1, \ldots, n$) such that

$$\sum_{i=1}^{n} |\mu_i| = 1, \qquad x^* = \sum_{i=1}^{n} \mu_i y_{\alpha_i}, \qquad \alpha_1 = \alpha.$$

One has

$$1-(1-\lambda)\varepsilon/2 \leqslant \sum_{i=1}^{n} \mu_i \langle x_\alpha, y_{\alpha_i}\rangle$$
$$\leqslant |\mu_1| + \sum_{i=2}^{n} |\mu_i|\,|\langle x_\alpha, y_{\alpha_i}\rangle| \leqslant |\mu_1| + \lambda(1-|\mu_1|).$$

Therefore $1-\varepsilon/2 \leqslant |\mu_1|$ and $\|x^*-y_\alpha\| \leqslant \varepsilon$.

Hence if X has property β in the strong sense then X is almost uniformly smooth. Note that in this case the family $(A_\varepsilon)_{0<\varepsilon<1}$ is just reduced to one set $A = \{x_\alpha\}_{\alpha\in I}$.

Summing up we obtain the subsequent consequence of Corollary 5:

COROLLARY 7. *Every Banach space with separable dual may be equivalently renormed to be almost uniformly smooth.*

2. Characteristics of subspaces of dual Banach spaces. The following corollary solves in the affirmative a problem raised by D. van Dulst and I. Singer [2].

COROLLARY 8. *Let X be a Banach space with separable dual. Then there exists an equivalent norm on X such that $\chi(X)$ is strictly less than one.*

Proof. Indeed, it has been shown in [3] that $\chi(X)$ is strictly less than one for an almost uniformly smooth Banach space X. ■

By making direct use of the renorming technique of Corollary 5, we can give a more precise version of this result. It is known that if X is a quasireflexive Banach space then there exists on X an equivalent norm such that $\chi(X) = 1/2$ [6]. We now prove that it is possible to renorm a Banach space X with separable dual so that $\chi(X)$ is close to 1/2.

THEOREM 9. *Let X be a Banach space with separable dual and $\delta > 0$. There is an equivalent norm on X such that $\chi(X) \leqslant 1/2+\delta$.*

Proof. Let $|||\cdot|||$ be the renorming of Corollary 5 with $\lambda < \eta/8$, where $\eta = 4\delta(1+2\delta)^{-1}$. Let $(x_n)_{n=1}^{\infty} \subset X$ and $(x_n^*)_{n=1}^{\infty} \subset X^*$ be two sequences satisfying Definition 2 for the above λ. We have to show that for every hyperplane $Z \subseteq X^*$ there are $x^* \in X^*$, $|||x^*||| \leqslant 1$, and an ω^*-neighbourhood V of x^* such that, for $z^* \in Z\cap V$, $|||z^*||| \geqslant 2-\eta$ [2].

Let $g \in X^{**}$, $|||g||| = 1$, such that $Z = \ker g$. By the property (ii) of Definition 1(a) for the norm $|||\cdot|||$ we may find $n_0 \in N$ such that $\langle g, x_{n_0}^*\rangle > 1-\lambda$.

Let V be the ω^*-neighbourhood of $x_{n_0}^*$ defined by

$$V = \{x^* \in X^*\colon \langle x_{n_0}, x^*\rangle > 1-\lambda\}.$$

Now let us suppose that there is $x^* \in Z\cap V$ with $|||x^*||| < 2-\eta$. As the convex combinations of $(x_n^*)_{n=1}^{\infty}$ are dense in the unit ball of $(X, |||\cdot|||)^*$ we then could find $N \geqslant n_0$ and real scalars $(\mu_n)_{n=1}^{N}$ with

$$\sum_{n=1}^{N} |\mu_n| < 2-\eta \tag{1}$$

such that, for $y^* = \sum_{n=1}^{N} \mu_n x_n^*$, we get $|\langle g, y^*\rangle| < \lambda$ and $y^* \in V$. Clearly $|||y^*||| < 2-\eta$. Let

$$y_1^* = \mu_{n_0} x_{n_0}^*, \qquad y_2^* = \sum_{\substack{n=1\\ n\neq n_0}}^{N} \mu_n x_n^*.$$

We have $|\langle x_{n_0}, y_2^*\rangle| \leqslant 2\lambda$, whence $\langle x_{n_0}, y_1^*\rangle > (1-\lambda)-2\lambda = 1-3\lambda$. Therefore

$$\mu_{n_0} > 1-3\lambda. \tag{2}$$

It follows that $\langle g, y_1^*\rangle > (1-3\lambda)(1-\lambda)$. As $|\langle g, y^*\rangle| < \lambda$, we get

$$|\langle g, y_2^*\rangle| > (1-3\lambda)(1-\lambda)-\lambda.$$

Hence

(3) $$\sum_{\substack{n=1\\ n\neq n_0}}^{n} |\mu_n| > (1-3\lambda)(1-\lambda)-\lambda.$$

(2) and (3) give

$$\sum_{n=1}^{N} |\mu_n| > \big((1-3\lambda)(1-\lambda)-\lambda\big)+1-3\lambda$$
$$\geqslant 2-\lambda(8-3\lambda) > 2-8\lambda.$$

It follows by the choice of λ that

$$\sum_{n=1}^{N} |\mu_n| > 2-\eta,$$

in contradiction with (1). This completes the proof of the theorem. ∎

COROLLARY 10. *Let X be a nonreflexive Banach space with separable dual and $0<\delta<2$. Then there exists an equivalent norm $|||\cdot|||$ on X such that there exists no projection p from $X\oplus \boldsymbol{R}x^{**}$ onto X with $x^{**}\in X^{**}\setminus X$ and $\ker p = \boldsymbol{R}x^{**}$ of norm $|||p|||\leqslant 2-\delta$.*

Proof. Let $|||\cdot|||$ be the renorming of Theorem 9 with $\lambda<\delta/(2(2-\delta))$. One has $\chi(X)\leqslant 1/2+\lambda$. Observe that $\chi(X)$ may also be computed by the following formula:

$$\chi(X)^{-1} = \inf_{x^{**}\in X^{**}\setminus X} \{|||p|||: p \text{ is the projection of } X\oplus \boldsymbol{R}x^{**} \text{ onto } X \text{ of kernel } \boldsymbol{R}x^{**}\}.$$

Hence for any projection p from $X\oplus \boldsymbol{R}x^{**}$ onto X with $x^{**}\in X^{**}\setminus X$ and $\ker p = \boldsymbol{R}x^{**}$, one has $|||p|||>2-\delta$. ∎

Remarks. 1) The assumption that X is an Asplund space is essential in Corollary 8 and Theorem 9. Indeed, if X is a separable Banach space then there exists a $\sigma(X^{**}, X^{*})$ dense sequence (x_n) in X^{**} so $r[x_n]=1$. Therefore if X is a separable Banach space with nonseparable dual then for every equivalent norm $\chi(X)$ is just one.

2) Since the Fréchet-differentiable norms of a Banach space with separable dual are dense in the set of all norms, it is possible to renorm a Banach space X with separable dual in such a way that the equivalent norm is Fréchet-differentiable and satisfies $\chi(X)\leqslant 1/2+\delta$.

References

[1] J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. 15 (1948), 1057–1071.

[2] D. van Dulst and I. Singer, *On Kadec–Klee norms on Banach spaces*, Studia Math. 54 (1975), 205–211.

[3] C. Finet, *Basic sequences and smooth norms in Banach spaces*, ibid. 89 (1988), 1–9.

[4] —, *Une classe d'espaces de Banach à prédual unique*, Quart. J. Math. Oxford 35 (1984), 403–414.

[5] G. Godefroy and P. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, to appear.

[6] B. V. Godun and M. I. Kadets, *Norming subspaces, biorthogonal systems and predual Banach spaces*, Sibirsk. Mat. Zh. 23 (1982), 44–48 (in Russian).

[7] W. B. Johnson and H. P. Rosenthal, *On ω*-basic sequences and their applications to the study of Banach spaces*, Studia Math. 43 (1972), 77–92.

[8] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. 1 (1963), 139–148.

[9] J. R. Partington, *Norm attaining operators*, ibid. 43 (1982), 273–276.

[10] W. Schachermayer, *Norm attaining operators and renormings of Banach spaces*, ibid. 44 (1983), 201–212.

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