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The socle and finite-dimensionality of a semiprime Banach algebra

by

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Abstract. All finite-dimensional semiprime Banach algebras are semisimple.

The purpose of this paper is to give a characterization of the elements of the socle of a semiprime Banach algebra. If A is a semiprime Banach algebra we prove that $\text{soc } A \cap \text{rad } A = \{0\}$, and $t \in \text{soc } A$ if and only if $\dim(tAt) < +\infty$ (i.e. tAt has finite dimension). This extends a result of Alexander in [1] concerning semisimple Banach algebras, and is used to prove that the elements of $\text{soc } A$ are algebraic and that A is finite-dimensional if and only if $A = \text{soc } A$ (and in this case A is forced to be semisimple). This completes Tullo's assertion in Theorem 5 of [8]. We also give a different proof of Tullo's result.

An element s of A is called *single* if whenever $asb = 0$ for some a, b in A , at least one of as or sb is zero. We say that an element t of A acts *compactly* if the map $a \rightarrow tat$ ($A \rightarrow A$) is compact. If the algebra A has no minimal ideals we define $\text{soc } A = \{0\}$.

In general, notation and terminology used are as in [3]. All the algebras and subspaces considered will be over the complex field.

Single elements that act compactly have proved to have a close connection with the elements of the minimal ideals of the algebra.

More precisely, with a slight modification (see e.g. [5] or [6]) in the proofs of Theorem 4 and Corollary 5 in [4] one can easily deduce Theorem 1 and Corollary 2 below (see also [7] for an alternative approach).

THEOREM 1. *Let s and t be nonzero compactly acting single elements of a semiprime Banach algebra A , and $s, t \notin \text{rad } A$. Then:*

- (i) *There exist minimal idempotents e and f such that $s = se$ and $t = ft$.*
- (ii) *The dimension of tAs is at most 1.*

From Theorem 1 we find that if $s \notin \text{rad } A$ and s is a compactly acting

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single element of A then As (sA) is a minimal left (right) ideal of A . The converse is also valid and so we have

COROLLARY 2. *The socle of A consists of all finite sums of compactly acting single elements of A that are not in $\text{rad } A$.*

PROPOSITION 3. *Let A be a semiprime Banach algebra. Then $\text{soc } A \cap \text{rad } A = \{0\}$.*

Proof. Suppose on the contrary that there exists a nonzero element t in $\text{soc } A \cap \text{rad } A$. Then $t \notin \text{lan}(\text{soc } A)$ ([3], p. 162, Lemma 4). This implies that $ty \neq 0$ for some $y \in \text{soc } A$ and so from Corollary 2, $ts \neq 0$ for some compactly acting single element s of A which is not in $\text{rad } A$. As is then a left minimal ideal of A and since $\{0\} \neq Ats \subseteq As$ we have $As = Ats \subset \text{soc } A \cap \text{rad } A$, which is a contradiction (since $\text{rad } A$ does not contain nonzero idempotents).

COROLLARY 4. *Let A be a semiprime Banach algebra. Then:*

- (i) $\text{soc } A \subset \text{lan}(\text{rad } A) = \text{ran}(\text{rad } A)$.
- (ii) $\text{rad } A \subset \text{lan}(\text{soc } A) = \text{ran}(\text{soc } A)$.

Proof. This follows immediately from Proposition 3 above and Lemma 4, p. 162 of [3].

LEMMA 5. *Let A be an algebra that has proper left ideals, let X be a finite-dimensional subspace of A , and let t be a nonzero element of A . If $X \cap L \neq \{0\}$ for every left ideal L of A contained in At , then At contains a left minimal ideal of A .*

Proof. We distinguish two cases:

Case 1: $X \cap L = X \cap L'$ for any two left ideals $L, L' \subset At$. Since $X \cap L \neq \{0\}$ for every left ideal $L \subset At$, it is clear that the intersection of all such L 's is nonzero and therefore it is a minimal left ideal of A .

Case 2: There exist two left ideals L_1, L_2 of A contained in At such that $\{0\} \neq X \cap L_2 \subsetneq X \cap L_1$. Then from the assumptions we have $\{0\} \neq X \cap L$ for every left ideal $L \subset L_2 \subset At$.

Therefore, either $X \cap L = X \cap L_2$ for all such L 's, which leads to case 1, or there exists a left ideal $L_3 \subset L_2$ such that

$$\{0\} \neq X \cap L_3 \subsetneq X \cap L_2 \subsetneq X \cap L_1.$$

Since X has finite dimension we deduce the existence of a finite sequence L_1, \dots, L_k of left ideals of A contained in At such that

$$\{0\} \neq X \cap L_k \subsetneq \dots \subsetneq X \cap L_2 \subsetneq X \cap L_1 \quad \text{and} \quad X \cap L_k = X \cap L \neq \{0\}$$

for every left ideal $L \subsetneq L_k$. The intersection of such L 's is clearly a minimal left ideal of A .

LEMMA 6. *Let t be a nonzero element of a semiprime Banach algebra A such that tAt has finite dimension. Then $t \in \text{soc } A$.*

In particular, there exist a finite set $\{e_1, \dots, e_n\}$ of minimal idempotents and $t_1, \dots, t_{n-1} \in A$ such that $t = te_1 + t_1 e_2 + \dots + t_{n-1} e_n$.

Proof. We put $X = tAt$, and suppose that $\dim(tAt) = n$. Then from Lemma 5 the left principal ideal At contains a minimal idempotent $e_1 = a_1 t$, for some $a_1 \in A$. We put $t_1 = t - te_1$. Then

$$t_1 At_1 = (t - te_1) A (t - te_1) \subset t A (t - te_1) = t A (t - ta_1 t) \subset tAt.$$

However, the inclusion $tAt(1 - e_1) \subset tAt$ is strict because $te_1 = ta_1 t$ is in tAt , is nonzero and cannot be of the form $tat(1 - e_1)$. Hence

$$\dim(t_1 At_1) < \dim(tAt).$$

If $t_1 \neq 0$, by a similar argument we can find a minimal idempotent e_2 so that

$$\dim(t_2 At_2) < \dim(t_1 At_1)$$

for $t_2 = t_1 - t_1 e_2$.

A finite number of repetitions of this process gives an element

$$t^* = te_1 + t_1 e_2 + \dots + t_{k-1} e_k \in \text{soc } A,$$

for some $k \leq n$, such that $(t - t^*)A(t - t^*) = \{0\}$. Hence $t = t^*$, since A is semiprime.

THEOREM 7. *Let A be a semiprime Banach algebra. Then $t \in \text{soc } A$ if and only if the dimension of tAt is finite.*

Proof. If $t \in \text{soc } A$, then $t = s_1 + \dots + s_n$ is a finite sum of compactly acting single elements not in $\text{rad } A$ (Corollary 2) and therefore $tAt \subset \sum_{i,j=1}^n s_i A s_j$. Since $s_i A s_j$ has dimension at most 1 (Theorem 1) it follows that $\dim(tAt) < +\infty$.

Conversely, if $\dim(tAt) < +\infty$ then $t \in \text{soc } A$ from Lemma 6. In particular, $t = 0$ if and only if $tAt = \{0\}$, since A is semiprime.

COROLLARY 8. *Let A be a semiprime Banach algebra. If A has finite dimension then:*

- (i) $A = \text{soc } A$.
- (ii) A is semisimple.

Proof. If A has finite dimension, then tAt has finite dimension for every $t \in A$. Hence $A \subset \text{soc } A$. Since the reverse inclusion is also valid we have $A = \text{soc } A$.

From Proposition 3 above and Corollary 20, p. 126 of [3] we now have $\text{rad } A = \text{rad}(\text{soc } A) = \text{soc } A \cap \text{rad } A = \{0\}$, i.e. A is semisimple.

COROLLARY 9. *Let A be a semiprime Banach algebra. Then every element of $\text{soc } A$ is algebraic.*

Proof. If $t \in \text{soc } A$, then $\dim(tAt) = n < +\infty$. Hence the elements t^4, \dots, t^{n+4} are linearly dependent and therefore there are $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ not all zero with $\lambda_1 t^3 + \lambda_2 t^4 + \dots + \lambda_{n+1} t^{n+4} = 0$.

COROLLARY 10 (Tullo). *Let A be a semiprime Banach algebra and $A = \text{soc } A$. Then A is semisimple and finite-dimensional.*

Proof. A is semisimple as in the proof of Corollary 8. Also, every element of A is algebraic (Corollary 9) and therefore A is finite-dimensional (Corollary 1 in [2]).

We summarize the results concerning the finite-dimensionality of algebras in the following

THEOREM 11. *Let A be a semiprime Banach algebra. Then the following conditions are equivalent:*

- (i) A has finite dimension.
- (ii) $A = \text{soc } A$.

Moreover, if (i) (and therefore (ii)) is valid, then A is semisimple.

Remark. If A is a semiprime Banach algebra and $A = \overline{\text{soc } A}$, then A is compact (Lemma 12, p. 177 of [3]) and therefore A has a discrete structure space (Theorem 18, p. 180 of [3]).

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