the derivative of a, and with the involution $a^*(z) = \bar{a}(\bar{z})$. The extreme points of P_A can be identified with $[-1,1]$, the set of all fixed points of $z \to \bar{z}$. By Corollary 3.1 the function a defined by $a(z) := 1-z$ is contained in the *cone* $\bar{A}_+$, but it is not in A_+: Let $a(z) = 1-z = \sum_{i=1}^{N} a_i^*(z)\, a_i(z)$ be a combination of positive elements. Differentiating both sides and using the continuity of a' on D we obtain $a'(1) = -1 = 0$, a contradiction.

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On the integrability and L^1-convergence of sine series*

by

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Abstract. We study sine series $(*)\sum_{k=1}^{\infty} a_k \sin kx$ with coefficients belonging to one or two of the classes $\tilde{C}$, $\tilde{B}V$, and $\tilde{V}_p$ introduced in this paper. Among other things, we prove that if $\{a_k\} \in \tilde{C} \cap \tilde{B}V$, then $(*)$ is the Fourier series of some function $f \in L^1(0, \pi)$. Furthermore, if $\{a_k\} \in \tilde{C} \cap \tilde{B}V$, or $\{a\} \in \tilde{V}_p$ and $f \in L^1(0, \pi)$, then the condition $(**)$ $a_n \log n \to 0$ is necessary and sufficient for the $L^1(0, \pi)$-convergence of the partial sums $s_n(x)$ of series $(*)$. Criterion $(**)$ has been known so far only in the case of cosine series. Our results generalize those obtained by Telyakovskii [9] for sine series, while our new classes are the counterparts of those introduced by Garrett and Stanojević [5] as well as by Bojanic and Stanojević [2] for cosine series.

1. Introduction. We will study the sine series

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1.1}$$

where $\{a_k\}$ is a sequence of real numbers in the class $\tilde{B}V$ defined as follows.

DEFINITION 1. A null sequence $\{a_k\}$ belongs to $\tilde{B}V$ if

$$\sum_{k=1}^{\infty} k\,|\Delta b_k| < \infty \tag{1.2}$$

where

$$b_k := a_k/k, \quad \Delta b_k := b_k - b_{k+1} \quad (k = 1, 2, \ldots).$$

We do not require any monotonicity of the sequences $\{a_k\}$ and $\{b_k\}$.

Following an idea of Kano [6], we represent the partial sums $s_n(x)$ of series (1.1) in the form

$$s_n(x) = \sum_{k=1}^{n} a_k \sin kx = -\sum_{k=1}^{n} b_k (\cos kx)'$$

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where prime denotes derivative. By summation by parts,

(1.3) $$s_n(x) = -\sum_{k=1}^{n} \Delta b_k D_k'(x) - b_{n+1} D_n'(x) \quad (n = 1, 2, \ldots)$$

where

$$D_n(x) = \tfrac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}$$

is the Dirichlet kernel. It is easy to see that

(1.4) $$|D_n'(x)| \leqslant Cn/x^2 \quad \text{uniformly in } n \text{ and } x.$$

Here and in the sequel, C denotes positive absolute constants not necessarily the same at different occurrences.

Now a routine calculation gives that if $\{a_k\} \in \tilde{B}V$, then at every x the series $\sum_{k=1}^{\infty} \Delta b_k D_k'(x)$ converges absolutely and $b_{n+1} D_n'(x) \to 0$ $(n \to \infty)$. Consequently, we can write

(1.5) $$\sum_{k=1}^{\infty} a_k \sin kx = -\sum_{k=1}^{\infty} \Delta b_k D_k'(x) =: f(x), \quad \text{say}.$$

Remark 1. We recall that the class BV of sequences of bounded variation is defined as follows. A null sequence $\{a_k\}$ belongs to BV if

$$\sum_{k=1}^{\infty} |\Delta a_k| < \infty \quad \text{where} \quad \Delta a_k := a_k - a_{k+1}.$$

However, it seems to us that $\tilde{B}V$ is a more appropriate notion for sine series than BV.

We note that the difference between BV and $\tilde{B}V$ is merely a slight one. For instance, if F denotes the class of sequences $\{a_k\}$ such that

(1.6) $$\sum_{k=1}^{\infty} |a_k|/k < \infty,$$

then we have $\tilde{B}V \cap F = BV \cap F$.

Indeed, the inclusion $BV \cap F \subset \tilde{B}V$ follows from the inequality

(1.7) $$k|\Delta b_k| \leqslant |\Delta a_k| + \frac{|a_{k+1}|}{k+1}.$$

On the other hand, the inequality

(1.8) $$|\Delta a_k| \leqslant k|\Delta b_k| + \frac{|a_{k+1}|}{k+1}$$

implies $\tilde{B}V \cap F \subset BV$.

Various conditions are known in the literature which ensure that series (1.1) is a Fourier series (see e.g. [4], [6]–[9], and [10, Vol. 1, pp. 185–186]). Recently, Telyakovskii [9] introduced a class $\tilde{S}$ as follows. A null sequence $\{a_k\}$ belongs to $\tilde{S}$ if there exists a nonincreasing sequence $\{B_k\}$ of numbers such that

(1.9) $$|\Delta b_k| \leqslant B_k \quad \text{for all } k, \quad \sum_{k=1}^{\infty} kB_k < \infty.$$

Theorem A (Telyakovskii [9]). *If $\{a_k\} \in \tilde{S}$, then series (1.1) is the Fourier series of some function $f \in L^1(0, \pi)$.*

2. Main results. We introduce another new class $\tilde{C}$ of coefficient sequences for sine series.

Definition 2. A null sequence $\{a_k\}$ belongs to $\tilde{C}$ if for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n, and such that for all n,

(2.1) $$\int_0^{\delta} \Big|\sum_{k=n}^{\infty} \Delta b_k D_k'(x)\Big| dx \leqslant \varepsilon.$$

It is clear that (2.1) implies, for $1 \leqslant n \leqslant N$,

(2.2) $$\int_0^{\delta} \Big|\sum_{k=n}^{N} \Delta b_k D_k'(x)\Big| dx \leqslant 2\varepsilon.$$

Conversely, by virtue of Fatou's lemma, it follows from (2.2) that (2.1) holds true with 2ε in place of ε. Thus, conditions (2.1) and (2.2) as required to be satisfied for all $\varepsilon > 0$ are equivalent.

Definition 3. Motivated by (1.3), the sums

(2.3) $$u_n(x) := s_n(x) + b_{n+1} D_n'(x) \quad (n = 1, 2, \ldots)$$

are called the *modified sums* of series (1.1).

It will turn out that these modified sums $u_n(x)$ exhibit nicer convergence behavior in comparison with that of the ordinary partial sums $s_n(x)$.

According to (1.3),

(2.4) $$u_n(x) = -\sum_{k=1}^{n} \Delta b_k D_k'(x).$$

Another representation for $u_n(x)$ is

$$u_n(x) = \sum_{k=1}^{n} \Big(\sum_{j=k}^{n} \Delta a_j\Big) \sin kx + a_{n+1} \tilde{K}_n(x)$$

where

$$\tilde{K}_n(x) = \frac{1}{n+1} \sum_{k=1}^{n} \tilde{D}_k(x) = \sum_{k=1}^{n} \Big(1 - \frac{k}{n+1}\Big) \sin kx$$

is the conjugate Fejér kernel, while

$$\tilde{D}_n(x) = \sum_{k=1}^{n} \sin kx = \tfrac{1}{2}\cot\tfrac{1}{2}x - \frac{\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}$$

is the conjugate Dirichlet kernel.

We note that in the case of cosine series an analogous class C as well as analogous modified cosine sums were defined by Garrett and Stanojević [5].

THEOREM 1. *If* $\{a_k\} \in \tilde{B}V$, *then*

$$\|u_n - f\| \to 0 \quad (n \to \infty) \quad \textit{if and only if} \quad \{a_k\} \in \tilde{C}. \tag{2.5}$$

In this paper, $\|\cdot\|$ denotes the $L^1(0, \pi)$-norm:

$$\|g\| := \int_0^\pi |g(x)|\,dx.$$

COROLLARY 1. *If* $\{a_k\} \in \tilde{C} \cap \tilde{B}V$, *then series* (1.1) *is the Fourier series of some function* $f \in L^1(0, \pi)$.

In Section 3, we will prove that the class $\tilde{S}$ is a subset of $\tilde{C} \cap \tilde{B}V$ (see Lemma 7 there). Thus, Theorem A is a special case of Corollary 1.

COROLLARY 2. *If* $\{a_k\} \in \tilde{C} \cap \tilde{B}V$, *then*

$$\|s_n - f\| \to 0 \quad \textit{if and only if} \quad a_n \log n \to 0 \quad (n \to \infty). \tag{2.6}$$

Combining Corollary 2 and Lemma 7 results in a natural continuation of Theorem A.

COROLLARY 3. *If* $\{a_k\} \in \tilde{S}$, *then the equivalence relation in* (2.6) *holds true.*

We recall that if $\{a_k\} \in \tilde{S}$, then by Lemma 2 we have (1.6). If, in addition, $\{a_k\}$ is a nonincreasing sequence of positive numbers, then it follows from (1.6) that $a_n \log n \to 0$ $(n \to \infty)$ and we get back the following classical result.

THEOREM B ([10, Vol. 1, p. 185]). *If a null sequence* $\{a_k\}$ *is nonincreasing, then* (1.1) *is the Fourier series of some function* $f \in L^1(0, \pi)$ *if and only if condition* (1.6) *is satisfied. Moreover, if* (1.6) *is satisfied, then*

$$\|s_n - f\| \to 0 \quad (n \to \infty).$$

PROBLEM. We are unable either to prove that $\{a_k\} \in \tilde{S}$ implies $a_n \log n \to 0$ $(n \to \infty)$ or to construct a counterexample.

We introduce further classes $\tilde{V}_p$ of sequences as follows.

DEFINITION 4. Let $p > 0$. A sequence $\{a_k\}$ belongs to $\tilde{V}_p$ if

$$n^{-1} \sum_{k=1}^{n} k^{2p} |\Delta b_k|^p \to 0 \quad (n \to \infty) \tag{2.7}$$

(recall that $b_k = a_k/k$).

Condition (2.7) is certainly satisfied if

$$\sum_{k=1}^{\infty} k^{2p-1} |\Delta b_k|^p < \infty. \tag{2.8}$$

This immediately follows by applying the so-called Kronecker lemma (see e.g. [1, p. 68]). A particular case of (2.8) in the case $p \geqslant 1$ is the one where $\{a_k\} \in \tilde{B}V$ and $k^2 |\Delta b_k| \leqslant C$ $(k = 1, 2, \ldots)$.

On the other hand, in Section 3 we will prove that $\tilde{S} \subset \tilde{V}_p$ for any $p > 0$ (see Lemma 8 there).

Remark 2. In the case of cosine series analogous classes V_p were defined by Bojanic and Stanojević [2] as follows. A sequence $\{a_k\}$ belongs to V_p if

$$n^{-1} \sum_{k=1}^{n} k^p |\Delta a_k|^p \to 0 \quad (n \to \infty).$$

Now, it follows easily that if

$$n^{-1} \sum_{k=1}^{n} |a_{k+1}|^p \to 0 \quad (n \to \infty),$$

in particular, if $a_n \to 0$ $(n \to \infty)$, then actually $V_p = \tilde{V}_p$ for any $p > 0$. In fact, the inclusion $V_p \subset \tilde{V}_p$ follows from (1.7), while $\tilde{V}_p \subset V_p$ follows from (1.8) in routine ways.

THEOREM 2. *If* $\{a_k\} \in \tilde{V}_p$ *for some* $p > 1$ *and* $f \in L^1(0, \pi)$, *then*

$$\|u_n - f\| \to 0 \quad (n \to \infty). \tag{2.9}$$

We can draw two corollaries.

COROLLARY 4. *If* $\{a_k\} \in \tilde{V}_p \cap \tilde{B}V$ *for some* $p > 1$, *then*

$$f \in L^1(0, \pi) \quad \textit{if and only if} \quad \{a_k\} \in \tilde{C}.$$

COROLLARY 5. *If* $\{a_k\} \in \tilde{V}_p$ *for some* $p > 1$ *and* $f \in L^1(0, \pi)$, *then the equivalence relation in* (2.6) *holds true.*

Using Hölder's inequality, it is easily shown that the class $\tilde{V}_p$ is wider when p is closer to 1. Hence, without loss of generality we may assume that $1 < p \leqslant 2$ in all subsequent considerations.

Before proving our theorems and the corollaries to them in Section 4, we will cite a few known and prove a few new auxiliary lemmas in Section 3.

3. Auxiliary results

LEMMA 1. *If* $\{B_k\}$ *is a null sequence of positive numbers such that*

$$\sum_{k=1}^{\infty} k^2 |\Delta B_k| < \infty, \tag{3.1}$$

then

$$n^2 B_n \to 0 \quad (n \to \infty), \tag{3.2}$$

$$\sum_{k=1}^{\infty} kB_k < \infty. \tag{3.3}$$

In the case when $\{B_k\}$ is nonincreasing, conditions (3.1) *and* (3.3) *are equivalent.*

Proof. (3.2) is clear from the estimate

$$n^2 B_n = n^2 \sum_{k=n}^{\infty} \Delta B_k \leqslant \sum_{k=n}^{\infty} k^2 |\Delta B_k|.$$

Summation by parts gives

$$\sum_{k=1}^{n} k^2 \Delta B_k = \sum_{k=1}^{n} (2k-1) B_k - n^2 B_{n+1}, \tag{3.4}$$

whence

$$\sum_{k=1}^{n} kB_k \leqslant \sum_{k=1}^{n} k^2 \Delta B_k + n^2 B_{n+1}.$$

This yields (3.3).

If $\{B_k\}$ is nonincreasing, then from (3.4) it follows that

$$\sum_{k=1}^{n} k^2 |\Delta B_k| = \sum_{k=1}^{n} k^2 \Delta B_k \leqslant 2 \sum_{k=1}^{n} kB_k,$$

completing the proof of the equivalence of (3.1) and (3.3).

Lemma 2 (Telyakovskii [9]). *Let*

$$B_k := \max_{j \geqslant k} |\Delta b_j| \quad (k = 1, 2, \ldots).$$

Then $\sum_{k=1}^{\infty} kB_k < \infty$ is equivalent to the following two conditions:

$$\sum_{k=1}^{\infty} (\max_{j \geqslant k} |\Delta a_j|) < \infty, \qquad \sum_{k=1}^{\infty} |a_k|/k < \infty. \tag{3.5}$$

We note that this equivalence remains valid if the second condition in (3.5) is replaced by the stronger one

$$\sum_{k=1}^{\infty} (\max_{j \geqslant k} |a_j/j|) < \infty.$$

Lemma 3. *If $T(x)$ is an even trigonometric polynomial of order n, then*

$$\|T'\| \leqslant n \|T\|.$$

This is S. Bernstein's inequality in the $L^1(0, \pi)$-metric (see e.g. [10, Vol. 2, p. 11]).

Lemma 4 (Bojanic–Stanojević [2]). *Let $\{c_k\}$ be a sequence of real numbers. Then for any $1 < p \leqslant 2$ and $n \geqslant 1$*

$$\Big\|\sum_{k=n}^{2n-1} c_k D_k\Big\| \leqslant C_p n^{1-1/p} \Big(\sum_{k=n}^{2n-1} |c_k|^p\Big)^{1/p} \tag{3.6}$$

where the constant C_p depends only on p.

The following special case, known as the Sidon–Fomin inequality, can easily be deduced from Lemma 4.

Lemma 5. *Let $\{c_k\}$ be a sequence of real numbers such that $|c_k| \leqslant 1$ for all k. Then there exists a constant C such that for any $n \geqslant 1$*

$$\Big\|\sum_{k=0}^{n} c_k D_k\Big\| \leqslant Cn.$$

Now we will prove a counterpart of inequality (3.6) in the case where $D_k'(x)$ is used instead of $D_k(x)$.

Lemma 6. *Let $\{c_k\}$ be a sequence of real numbers. Then for any $1 < p \leqslant 2$ and $n \geqslant 1$*

$$\Big\|\sum_{k=1}^{n} c_k D_k'\Big\| \leqslant \tilde{C}_p n^{1-1/p} \Big(\sum_{k=1}^{n} k^p |c_k|^p\Big)^{1/p} \tag{3.7}$$

where the constant $\tilde{C}_p$ depends only on p.

Proof. Without loss of generality, we may assume that n is of the form $n = 2^m - 1$ with some $m \geqslant 1$. Let $j \geqslant 1$. Applying first Bernstein's inequality, then inequality (3.6) yields

$$\Big\|\sum_{k=2^{j-1}}^{2^j-1} c_k D_k'\Big\| \leqslant 2^j \Big\|\sum_{k=2^{j-1}}^{2^j-1} c_k D_k\Big\| \leqslant 2^j C_p 2^{(j-1)(1-1/p)} \Big(\sum_{k=2^{j-1}}^{2^j-1} |c_k|^p\Big)^{1/p}$$
$$\leqslant 2^{2-1/p} C_p 2^{j(1-1/p)} \Big(\sum_{k=2^{j-1}}^{2^j-1} k^p |c_k|^p\Big)^{1/p}.$$

Continuing by making use of the triangle inequality, then Hölder's inequality with the exponents p and q, $1/p + 1/q = 1$, we get

$$\Big\|\sum_{k=1}^{2^m} c_k D_k'\Big\| \leqslant \sum_{j=1}^{m} \Big\|\sum_{k=2^{j-1}}^{2^j-1} c_k D_k'\Big\|$$
$$\leqslant 2^{2-1/p} C_p \sum_{j=1}^{m} 2^{j(1-1/p)} \Big(\sum_{k=2^{j-1}}^{2^j-1} k^p |c_k|^p\Big)^{1/p}$$

$$\leqslant 2^{2-1/p} C_p\Big(\sum_{j=1}^{m} 2^{j(1-1/p)q}\Big)^{1/q}\Big(\sum_{j=1}^{m}\sum_{k=2^{j-1}}^{2^j-1} k^p|c_k|^p\Big)^{1/p}$$
$$\leqslant \tilde{C}_p 2^{m(1-1/p)}\Big(\sum_{k=1}^{2^m-1} k^p|c_k|^p\Big)^{1/p},$$

which is (3.7) for $n = 2^m-1$.

Lemma 7. *$\tilde{S} \subset \tilde{C} \cap \tilde{B}V$.*

Proof. It is plain that $\tilde{S} \subset \tilde{B}V$.

In order to prove $\tilde{S} \subset \tilde{C}$ we take a sequence $\{a_k\}$ in $\tilde{S}$ and set

$$c_k := \Delta b_k/B_k, \qquad E_k(x) := \sum_{j=1}^{k} c_j D_j'(x) \qquad (k = 1, 2, \ldots).$$

Clearly $|c_k| \leqslant 1$ for all k. By summation by parts, for $N > n$,

$$\sum_{k=n}^{N} \Delta b_k D_k'(x) = \sum_{k=n}^{N} B_k c_k D_k'(x)$$
$$= -B_{n+1} E_n(x) + \sum_{k=n+1}^{N} \Delta B_k E_k(x) + B_{N+1} E_N(x).$$

Hence

$$\Big\|\sum_{k=n}^{N} \Delta b_k D_k'\Big\| \leqslant B_{n+1}\|E_n\| + \sum_{k=n+1}^{N} \Delta B_k \|E_k\| + B_{N+1}\|E_N\|. \tag{3.8}$$

Since $D_j(x)$ is an even trigonometric polynomial of order j, applying Lemma 3 gives

$$\|E_k\| \leqslant k\Big\|\sum_{j=1}^{k} c_j D_j\Big\|,$$

then applying the Sidon–Fomin lemma gives

$$\|E_k\| \leqslant Ck^2 \qquad (k = 1, 2, \ldots).$$

Substituting this into (3.8) yields

$$\Big\|\sum_{k=n}^{N} \Delta b_k D_k'\Big\| \leqslant C\Big[n^2 B_{n+1} + \sum_{k=n+1}^{\infty} k^2 \Delta B_k + N^2 B_{N+1}\Big].$$

Given any $\varepsilon > 0$, by (1.9) and Lemma 1, the last inequality implies that

$$\Big\|\sum_{k=n}^{N} \Delta b_k D_k'\Big\| \leqslant \varepsilon/2 \quad \text{if } n \text{ and } N \text{ are large enough,} \tag{3.9}$$

say $N \geqslant n > n_0$. Finally, using the obvious inequality

$$|D_k'(x)| = \Big|\sum_{j=1}^{k} j \sin jx\Big| \leqslant k(k+1)/2, \tag{3.10}$$

for any $1 \leqslant n \leqslant N$ we can estimate as follows:

$$\int_0^{\delta}\Big|\sum_{k=n}^{N} \Delta b_k D_k'(x)\Big|\,dx \leqslant \int_0^{\delta}\Big|\sum_{k=n}^{n_0} \Delta b_k D_k'(x)\Big|\,dx \tag{3.11}$$
$$+\int_0^{\pi}\Big|\sum_{k=n_0+1}^{N} \Delta b_k D_k'(x)\Big|\,dx \leqslant \tfrac{1}{2}\delta\sum_{k=1}^{n_0} k(k+1)|\Delta b_k| + \varepsilon/2 \leqslant \varepsilon$$

provided δ is small enough. This proves that $\{a_k\} \in \tilde{C}$.

Lemma 8. *$\tilde{S} \subset \tilde{V}_p$ for any $p > 0$.*

Proof. We assume $\{a_k\} \in \tilde{S}$ and will prove $\{a_k\} \in \tilde{V}_p$. To this end, set

$$\delta_j := \Big[2^{(p-1)j}\sum_{k=2^j}^{2^{j+1}-1} |\Delta b_k|^p\Big]^{1/p} \qquad (j = 0, 1, \ldots).$$

Owing to (1.9)(i) and the nonincreasing property of $\{B_k\}$, we conclude that $\delta_j \leqslant 2^j B_{2^j}$, whence

$$\sum_{j=0}^{\infty} 2^j \delta_j \leqslant \sum_{j=0}^{\infty} 2^{2j} B_{2^j}. \tag{3.12}$$

Now we apply the following Cauchy type theorem: If $\{c_k\}$ is a nonincreasing sequence of positive numbers, then the series $\sum_{k=1}^{\infty} kc_k$ and $\sum_{j=0}^{\infty} 2^{2j} c_{2^j}$ are convergent or divergent simultaneously. Therefore (1.9)(ii) implies that the series on the right-hand side of (3.12) is convergent. So is the series on the left-hand side of (3.12). In particular,

$$2^j \delta_j \to 0 \qquad (j \to \infty). \tag{3.13}$$

A simple estimate shows that

$$2^j \delta_j \geqslant \tfrac{1}{4}\Big[2^{-j}\sum_{k=2^j}^{2^{j+1}-1} k^{2p}|\Delta b_k|^p\Big]^{1/p},$$

whence, via (3.13),

$$2^{-j}\sum_{k=2^j}^{2^{j+1}-1} k^{2p}|\Delta b_k|^p \to 0 \qquad (j \to \infty).$$

It is not hard to see that this is equivalent to condition (2.7) to be proved (cf. [2, Lemma 2.1]).

In the proofs of Corollaries 2 and 5 we will need the following estimate.

LEMMA 9. *There exist two positive constants C_1 and C_2 such that*

(3.14) $$C_1 n \log(n+1) \leqslant \|D_n'\| \leqslant C_2 n \log(n+1) \quad (n = 1, 2, \ldots,).$$

Proof. The upper estimate in (3.14) immediately follows from Bernstein's inequality (stated in Lemma 3 above) and from the estimate of the Lebesgue constant $L_n = (2/\pi)\|D_n\|$ (see e.g. [10, Vol. 1, p. 67]).

In order to prove the lower estimate in (3.14), we begin with the representation

(3.15) $$D_n'(x) = -\sum_{k=1}^{n} k \sin kx = (n+1)[\tilde{K}_n(x) - \tilde{D}_n(x)]$$
$$= (n+1)\left[\tilde{K}_n(x) - \tfrac{1}{2}\cot\tfrac{1}{2}x + \frac{\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}\right]$$

where $\tilde{D}_n(x)$ and $\tilde{K}_n(x)$ are the conjugate Dirichlet and Fejér kernels, respectively. It is well known that

(3.16) $$|\tilde{K}_n(x) - \tfrac{1}{2}\cot\tfrac{1}{2}x| \leqslant \frac{C}{(n+1)x^2} \quad (0 < x \leqslant \pi)$$

(see e.g. [10, Vol. 1, p. 92]).

By (3.15),

(3.17) $$\|D_n'\| \geqslant (n+1)\left[\int_{\pi/(2n+1)}^{\pi} \frac{|\cos(n+\frac{1}{2})x|}{2\sin\frac{1}{2}x}dx - \int_{\pi/(2n+1)}^{\pi} |\tilde{K}_n(x) - \tfrac{1}{2}\cot\tfrac{1}{2}x|\,dx\right] - \int_0^{\pi/(2n+1)} |D_n'(x)|\,dx.$$

A similar reasoning which leads to a lower estimate of the integral

$$\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})x|}{2\sin\frac{1}{2}x}dx$$

(see e.g. [10, Vol. 1, p. 67]) shows that there is a constant $C > 0$ such that

(3.18) $$\int_{\pi/(2n+1)}^{\pi} \frac{|\cos(n+\frac{1}{2})x|}{2\sin\frac{1}{2}x}dx \geqslant C\log(n+1) \quad (n = 1, 2, \ldots).$$

Hence and from (3.16), (3.17) and (3.10), we get

$$\|D_n'\| \geqslant C(n+1)\log(n+1) - C(2n+1)/\pi - \pi(n+1)/4,$$

and this completes the proof of (3.14).

Remark 3. A more accurate calculation shows that actually

$$\frac{1}{\log(n+1)} \int_{\pi/(2n+1)}^{\pi} \frac{|\cos(n+\frac{1}{2})x|}{2\sin\frac{1}{2}x}dx \to \frac{2}{\pi} \quad (n \to \infty)$$

(cf. (3.18)), which implies in turn that

$$\frac{\|D_n'\|}{n\log(n+1)} \to \frac{2}{\pi} \quad (n \to \infty).$$

4. Proofs of Theorems 1–2 and Corollaries 1–5

Proof of Theorem 1. Sufficiency. Assume $\{a_k\} \in \tilde{C} \cap \tilde{B}V$. By (1.5) and (2.4),

(4.1) $$f(x) - u_n(x) = -\sum_{k=n+1}^{\infty} \Delta b_k D_k'(x).$$

Given any $\varepsilon > 0$, let δ correspond to $\varepsilon/2$ in the definition of the class $\tilde{C}$. Then, by applying (1.4) again, a simple calculation shows that

$$\|u_n - f\| = \left(\int_0^{\delta} + \int_{\delta}^{\pi}\right)\left|\sum_{k=n+1}^{\infty} \Delta b_k D_k'(x)\right| dx$$
$$\leqslant \varepsilon/2 + \sum_{k=n+1}^{\infty} |\Delta b_k| \int_{\delta}^{\pi} |D_k'(x)|\,dx$$
$$\leqslant \varepsilon/2 + C\sum_{k=n+1}^{\infty} k|\Delta b_k| \int_{\delta}^{\pi} dx/x^2$$
$$\leqslant \varepsilon/2 + C\delta^{-1}\sum_{k=n+1}^{\infty} k|\Delta b_k| \leqslant \varepsilon$$

if n is large enough, thanks to the fact that $\{a_k\} \in \tilde{B}V$. This proves the limit relation in (2.5).

Necessity. Now we assume that given any $\varepsilon > 0$, $\|u_n - f\| \leqslant \varepsilon/2$ if n is large enough, say $n \geqslant n_0$. By (4.1), this is equivalent to

(4.2) $$\int_0^{\pi}\left|\sum_{k=n+1}^{\infty} \Delta b_k D_k'(x)\right| dx \leqslant \varepsilon/2 \quad \text{if } n \geqslant n_0.$$

If $1 \leqslant n \leqslant n_0$, then by (3.10)

$$\int_0^{\delta}\left|\sum_{k=n}^{\infty} \Delta b_k D_k'(x)\right| dx \leqslant \tfrac{1}{2}\delta \sum_{k=1}^{n_0} k(k+1)|\Delta b_k| + \varepsilon/2 \leqslant \varepsilon$$

(cf. (3.11)) provided δ is small enough. This is (2.1), which means that $\{a_k\} \in \tilde{C}$.

Proof of Corollary 1. In order to prove that (1.1) is a Fourier series, by a standard argument it suffices to show that for the sum $f(x)$ of series (1.1) we have $f \in L^1(0, \pi)$. According to Theorem 1, this is the case whenever $\{a_k\} \in \tilde{C} \cap \tilde{B}V$.

Proof of Corollary 2. Sufficiency. Keeping (2.3) in mind, by

Theorem 1,

$$\|s_n-f\| \leqslant \|s_n-u_n\|+\|u_n-f\| = |b_{n+1}|\,\|D_n'\|+o(1).$$

Necessity. Similarly, by assumption and again by Theorem 1,

$$|b_{n+1}|\,\|D_n'\| = \|u_n-s_n\| \leqslant \|u_n-f\|+\|f-s_n\| = o(1).$$

In both cases, it remains only to take into account that by Lemma 9

$$(4.3)\qquad |b_{n+1}|\,\|D_n'\| = \frac{|a_{n+1}|}{n+1}\|D_n'\| \sim |a_{n+1}|\log(n+1),$$

where $\sim$ means that the sides are of the same order of magnitude.

Proof of Theorem 2. Denote by $\sigma_n(x)$ the first arithmetic means of series (1.1), i.e.,

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n} s_k(x) = \sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)a_k\sin kx \qquad (n = 1, 2, \ldots).$$

The basic idea is to show that under the conditions of Theorem 2,

$$(4.4)\qquad \|u_n-\sigma_n\| \to 0 \quad (n\to\infty).$$

To this end, we use the representation

$$s_n(x)-\sigma_n(x) = \frac{1}{n+1}\sum_{k=1}^{n} ka_k\sin kx = -\frac{1}{n+1}\sum_{k=1}^{n} kb_k(\cos kx)'$$

whence by summation by parts,

$$s_n(x)-\sigma_n(x) = -\frac{1}{n+1}\sum_{k=1}^{n} k\,\Delta b_k D_k'(x)+\frac{1}{n+1}\sum_{k=1}^{n} b_{k+1}D_k'(x)-b_{n+1}D_n'(x).$$

By (2.3), this can be rewritten as

$$u_n(x)-\sigma_n(x) = -\frac{1}{n+1}\sum_{k=1}^{n} k\Delta b_k D_k'(x)+\frac{1}{n+1}\sum_{k=1}^{n} b_{k+1}D_k'(x),$$

so

$$(4.5)\qquad \|u_n-\sigma_n\| \leqslant \frac{1}{n+1}\Big\|\sum_{k=1}^{n} k\Delta b_k D_k'\Big\|+\frac{1}{n+1}\Big\|\sum_{k=1}^{n} b_{k+1}D_k'\Big\|.$$

We apply Lemma 6 twice in order to obtain

$$(4.6)\qquad \frac{1}{n+1}\Big\|\sum_{k=1}^{n} k\Delta b_k D_k'\Big\| \leqslant \frac{\tilde{C}_p n^{1-1/p}}{n+1}\Big(\sum_{k=1}^{n} k^{2p}|\Delta b_k|^p\Big)^{1/p}$$
$$\leqslant \tilde{C}_p\Big(n^{-1}\sum_{k=1}^{n} k^{2p}|\Delta b_k|^p\Big)^{1/p} = o(1)$$

by (2.7), whereas

$$(4.7)\qquad \frac{1}{n+1}\Big\|\sum_{k=1}^{n} b_{k+1}D_k'\Big\| \leqslant \frac{\tilde{C}_p n^{1-1/p}}{n+1}\Big(\sum_{k=1}^{n} k^p|b_{k+1}|^p\Big)^{1/p}$$
$$\leqslant \tilde{C}_p\Big(n^{-1}\sum_{k=2}^{n+1} k^p|b_k|^p\Big)^{1/p} = o(1)$$

because $kb_k = a_k \to 0$ as $k\to\infty$. Putting (4.5)–(4.7) together yields (4.4).

To complete the proof we have to take into account that

$$\|u_n-f\| \leqslant \|u_n-\sigma_n\|+\|\sigma_n-f\|.$$

The first term on the right tends to zero by (4.4), while the second term tends to zero since $f\in L^1(0,\pi)$, thereby yielding (2.9).

We note that this kind of approach was first applied by Bray [3] for cosine series.

Proof of Corollary 4. Necessity. Assume $\{a_k\}\in\tilde{B}V$. Then, by Corollary 1, $\{a_k\}\in\tilde{C}$ is a sufficient condition for $f\in L^1(0,\pi)$.

Sufficiency. Assume $f\in\tilde{V}_p$ for some $p>1$ and $f\in L^1(0,\pi)$. Then, by Theorem 2, we have (2.9). By combining this with the condition $\{a_k\}\in\tilde{B}V$, Theorem 1 implies $\{a_k\}\in\tilde{C}$.

Proof of Corollary 5. It is essentially a repetition of that of Corollary 2, with Theorem 2 used instead of Theorem 1.

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The socle and finite-dimensionality of a semiprime Banach algebra

by

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Abstract. All finite-dimensional semiprime Banach algebras are semisimple.

The purpose of this paper is to give a characterization of the elements of the socle of a semiprime Banach algebra. If A is a semiprime Banach algebra we prove that $\operatorname{soc} A \cap \operatorname{rad} A = \{0\}$, and $t \in \operatorname{soc} A$ if and only if $\dim(tAt) < +\infty$ (i.e. tAt has finite dimension). This extends a result of Alexander in [1] concerning semisimple Banach algebras, and is used to prove that the elements of $\operatorname{soc} A$ are algebraic and that A is finite-dimensional if and only if $A = \operatorname{soc} A$ (and in this case A is forced to be semisimple). This completes Tullo's assertion in Theorem 5 of [8]. We also give a different proof of Tullo's result.

An element s of A is called *single* if whenever $asb = 0$ for some a, b in A, at least one of as or sb is zero. We say that an element t of A *acts compactly* if the map $a \to tat$ $(A \to A)$ is compact. If the algebra A has no minimal ideals we define $\operatorname{soc} A = \{0\}$.

In general, notation and terminology used are as in [3]. All the algebras and subspaces considered will be over the complex field.

Single elements that act compactly have proved to have a close connection with the elements of the minimal ideals of the algebra.

More precisely, with a slight modification (see e.g. [5] or [6]) in the proofs of Theorem 4 and Corollary 5 in [4] one can easily deduce Theorem 1 and Corollary 2 below (see also [7] for an alternative approach).

THEOREM 1. *Let s and t be nonzero compactly acting single elements of a semiprime Banach algebra A, and $s,t \notin \operatorname{rad} A$. Then:*
(i) *There exist minimal idempotents e and f such that $s = se$ and $t = ft$.*
(ii) *The dimension of tAs is at most* 1.

From Theorem 1 we find that if $s \notin \operatorname{rad} A$ and s is a compactly acting

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