A characterization of bi-invariant Schwartz space multipliers on nilpotent Lie groups

by

JOE W. JENKINS (Albany, N. Y.)

Abstract. A simply connected nilpotent Lie group, N, has a naturally defined Schwartz space, \( S(N) \). A continuous endomorphism on \( S(N) \) that commutes with both the right and left action of \( N \) on \( S(N) \) is called a bi-invariant Schwartz multiplier. It is shown that a bi-invariant Schwartz multiplier is given as convolution by a tempered distribution whose Fourier transform is a smooth, Ad*-invariant function on the dual of the Lie algebra of \( N \), all of whose derivatives have polynomial bounds. This characterization is used to discuss summability methods for the eigenfunction expansion of certain hypoelliptic differential operators on nilmanifolds, and to give a criterion for local solvability of invariant differential operators on \( N \).

We recall some well-known facts from the Schwartz theory on Euclidean spaces. Let \( X \) denote a finite-dimensional vector space with a fixed positive-definite inner product, and let \( \mathcal{S}(X) \) denote the Schwartz space on \( X \). We let \( \mathcal{M}(X) \) denote the space of continuous endomorphisms of \( \mathcal{S}(X) \) that commute with the action of \( X \) on \( \mathcal{S}(X) \), i.e., \( \mathcal{M}(X) \) if \( f \rightarrow Ef \) is continuous from \( \mathcal{S}(X) \) to \( \mathcal{S}(X) \) and if for each \( x \in X \), \( f \in \mathcal{M}(X) \), \( E_l(x,f) = E_l(x) \), where \( E_l(x,f) \) is \( f(y-x) \). It follows from the continuity that for \( f \in \mathcal{M}(X) \) the functional \( D_X \) defined on \( \mathcal{S}(X) \) by \( D_X(f) = Ef(0) \) is an element of \( \mathcal{M}(X) \), the space of tempered distributions. The group invariance implies that \( Ef(x) = l_X Ef(0) = E_l(x,f)(0) = \langle D_X, l_X f \rangle = D_X*f(x) \), where \( f(y) = f(-y) \). Conversely, if \( D \in \mathcal{M}(X) \), then one can easily see that \( D \) is a mapping of \( \mathcal{S}(X) \) into the smooth functions on \( X \) that commutes with translation. A natural question arises: For which \( D \) is \( D \in \mathcal{M}(X) \)?

The answer is given in terms of the Fourier transform. For \( f \in \mathcal{S}(X) \), \( \hat{f} \) is the function defined on \( X^* \), the dual space of \( X \), by

\[
\hat{f}(\xi) = \frac{1}{2\pi} \int_X f(x) e^{-2\pi i \langle x, \xi \rangle} dx.
\]

The mapping \( f \rightarrow \hat{f} \) establishes an isomorphism between \( \mathcal{S}(X) \) and \( \mathcal{S}(X^*) \), and allows one to define, for \( D \in \mathcal{M}(X) \), the element \( \hat{D} \) in \( \mathcal{M}(X^*) \) by \( \langle \hat{D}, \hat{f} \rangle = \langle D, f \rangle \). In [Sc], Schwartz proves that for \( D \in \mathcal{M}(X) \), \( E \in \mathcal{M}(X) \) if, and only if, \( \hat{D} \) is a smooth function on \( X^* \) which has polynomial bounds for all derivatives. Furthermore, in this case \( (D \ast f)(\xi) = \hat{D}(\xi)f(\xi) \). In this note we announce analogues of these results for nilpotent Lie groups.

This research was supported in part by a grant from the National Science Foundation.
Let $N$ denote a connected, simply connected nilpotent Lie group, with Lie algebra $n$. The exponential mapping $\exp: n \to N$ is a diffeomorphism, and in terms of the corresponding coordinates the left and right translations on $N$ are polynomial mappings. Thus, if $\mathcal{S}(N)$ denotes the image under composition with $\exp$ of $\mathcal{S}(n)$, the right and left actions of $N$ on $\mathcal{S}(N)$ are continuous endomorphisms. $\mathcal{S}(N)$ is topologized so that composition with $\exp$ is an isomorphism from $\mathcal{S}(n)$ to $\mathcal{S}(N)$. We denote by $\mathcal{S}^*(N)$ the dual of $\mathcal{S}(N)$, the space of tempered distributions on $N$.

For $f \in \mathcal{S}(N)$, the Fourier transform of $f$, $\hat{f}$, is defined on $n^*$, the dual of $n$, by

$$\hat{f}(\xi) = \int n \exp(X) e^{-2\pi i\langle \xi, X \rangle} dX.$$ 

One has that $f \to \hat{f}$ is an isomorphism from $\mathcal{S}(N)$ onto $\mathcal{S}(n^*)$. For $D \in \mathfrak{S}(N)$, $\hat{D}$ is defined on $\mathfrak{S}(n^*)$ by $\langle \hat{D}, \hat{F} \rangle = \langle D, F \circ \log \rangle$, where $\log$ denotes the inverse of exp and, for $F \in \mathfrak{S}(n^*)$ and $X \in n$,

$$\hat{F}(X) = \int n F(\xi) e^{-2\pi i\langle \xi, X \rangle} d\xi.$$ 

Let $Ad^*$ denote the coadjoint representation of $N$ on $n^*$. A tempered distribution $D$ on $n^*$ is said to be $Ad^*$-invariant if $\langle D, F \circ Ad^* \rangle = \langle D, F \rangle$ for all $F \in \mathfrak{S}(n^*)$. A tempered distribution $D$ on $N$ is said to be bi-invariant if $\langle D, r_y f \rangle = \langle \hat{D}, r_{\xi} \hat{F} \rangle$ for all $f \in \mathcal{S}(N)$, where $r_y f(y) = f(yx)$ and $r_{\xi} \hat{F}(\xi) = \hat{F}(\xi^*)$ for all $y, \xi \in N$. A straightforward computation shows that an element $D \in \mathfrak{S}(N)$ is $Ad^*$-invariant if, and only if, $\hat{D}$ is $Ad^*$-invariant.

$\mathfrak{S}^*(N)$ denote the space of continuous endomorphisms on $\mathcal{S}(N)$ that commute with both right and left translations by elements of $N$. As in the Euclidean case, for each $E \in \mathfrak{S}^*(N)$ there is a $D \in \mathfrak{S}(n^*)$ such that $Ef = Df$, where, as before, $D \circ f(x) = \langle D, \xi \hat{f}(\xi) \rangle \hat{f}(\xi) = f(y^{-1})$. If $D \in \mathfrak{S}^*(N)$ we denote by $E_D$ the mapping defined on $\mathcal{S}(N)$ by $E_Df = D \circ f$.

**Theorem A.** For $D$ in $\mathfrak{S}^*(N)$, $E_D \in \mathfrak{S}^*(N)$ if, and only if, $D$ is a smooth $Ad^*$-invariant function on $n^*$ with polynomial bounds on all derivatives.

Let $\mathfrak{A}^*(n^*)$ denote the space of smooth $Ad^*$-invariant functions defined on $n^*$ with polynomial bounds on all derivatives. For integers $i, j \geq 0$, we define seminorms $v_{ij}$ on $\mathfrak{A}^*(n^*)$ by

$$v_{ij}(\theta) = \sup_{|\xi| \leq j, \xi \neq 0} |\theta(\xi)/(1 + ||\xi||^i)|^2,$$

where $d = \dim(n)$, $\theta_0^i = \theta_0^i \ldots \theta_0^i \theta_0$, $\theta_0$, $\ldots$, $\theta_0$ are directional derivatives with respect to some basis of $n^*$, and $|| \|$ is a norm on $n^*$. The topology on $\mathfrak{A}^*(n^*)$ is determined by saying the sequence $\{\theta_j\}$ converges to zero if for each $j$ there is an $n$ such that $v_{ij}(\theta_n) \to 0$. The space $\mathcal{S}^*(N)$ is topologized by saying a sequence $\{E_k\}$ converges to zero if for each $f \in \mathcal{S}(N)$, $E_k f \to 0$ in $\mathcal{S}(N)$.

**Theorem B.** The mapping $\mathcal{M}^*(N) \to \mathfrak{P}^*(n^*)$: $E \to \hat{D}_E$ is a homeomorphism and an algebra isomorphism, the products being composition in $\mathcal{M}^*(N)$ and pointwise multiplication in $\mathfrak{P}^*(n^*)$.

For $\eta \in \mathfrak{a}^*$, let $\pi_{\eta}$ denote the irreducible unitary representation of $\eta$ corresponding to the $Ad^*$-orbit of $\eta$ by the Kirillov theory. For $\theta \in \mathfrak{P}^*(n^*)$, let $D_{\eta}$ be the tempered distribution on $N$ with Fourier transform $\theta$. For $\eta \in \mathfrak{a}^*$, we have $\pi_{\eta}(D_{\eta} \ast f) = \theta(\eta) \pi_{\eta}(f)$.

As an application of these results, consider the question of local solvability. Recall that a left-invariant differential operator $L$ on $N$ is said to be locally solvable if there is an open set $U \subset N$ such that $C^\infty(U) \subset L(C^\infty(U))$.

Let $\omega(\xi)$ denote the Ad*-orbit in $n^*$ that contains $\xi$ and, having fixed a norm on $n^*$, set $|\omega(\xi)| = \inf |\langle \xi, \xi \rangle|: \xi \in \omega(\xi)$. There is a linear subspace $V \subset n^*$ and a Zariski open subset $V_0 \subset V$ such that the elements in $V_0$ parametrize an open dense set of orbits in $n^*$. Representations corresponding to elements of $V_0$ are said to be in general position.

Suppose that $N$ contains a discrete cocompact subgroup $\Gamma$. Then $L^2(\Gamma \backslash N)$ is a direct sum of subspaces $\mathfrak{M}_\eta$ such that the restriction of $\mathfrak{A}^*(N)$ to right translation is a finite multiple of $\pi_{\eta}$. We denote by $(\Gamma \backslash N)_\eta$ the elements of $\Gamma$ appearing in this decomposition that are in general position.

**Theorem D.** Let $L$ be a left-invariant differential operator on $N$. Suppose that for each $\eta \in \mathfrak{a}^*(\Gamma \backslash N)_\eta$, $(\eta_\eta(L) \ast f)$ has a bounded right inverse $A_\eta$ on $\mathfrak{M}_\eta$, and that the norm of $A_\eta$ is bounded by a polynomial in $|\omega(\xi)|$. Then $L$ is locally solvable.

Although Theorems A and B are stated in terms of convolution between elements of $\mathcal{S}(N)$ and $\mathfrak{P}^*(n^*)$, their proofs require the introduction of somewhat more general spaces. Let $\mathfrak{h}$ be a subspace of the center of $n$, and let $\lambda \in \mathfrak{h}^*$. We define the unitary character $\chi_\lambda$ on $H = \exp(\mathfrak{h})$ by $\chi_\lambda(\exp(x)) = e^{\pi i \langle \mathfrak{h}, x \rangle}$, and denote by $\mathcal{S}(N/H, \chi_\lambda)$ the space of all smooth functions $f$ defined on $N$ such that $f(\exp(x)) = \chi_\lambda(f)(x)$ for all $x \in N, \lambda \in H$, and such that $f \circ \exp_\mathfrak{h} \in \mathcal{S}(\mathfrak{h})$, where $t$ is a complement to $\mathfrak{h}$ in $n$. The topology of $\mathcal{S}(N/H, \chi_\lambda)$ is defined by requiring that the mapping $f \to f \circ \exp_\mathfrak{h}$ be a homeomorphism. Define $P_\chi$: $\mathcal{S}(N) \to \mathcal{S}(N/H, \chi_\lambda)$ by

$$P_\chi f(\exp \mathfrak{x}) = \int_{\mathfrak{h}} f(\exp(x + Y)) \chi_\lambda(\exp(-Y)) dY.$$ 

$P_\chi$ is an open surjection and thus its adjoint $P_\chi^*$ is an isomorphism of $\mathcal{S}^*(N/H, \chi_\lambda)$ into $\mathcal{S}^*(N)$. Let $\check{\mathfrak{h}}$ be the annihilator of $\mathfrak{h}$ in $n^*$. For $\lambda \in \mathfrak{h}^*$ (identified with a subspace of $n^*$), there is a natural Schwartz space on $\check{\mathfrak{h}} + \lambda, \mathcal{S}(\check{\mathfrak{h}} + \lambda)$, given
by composing elements of $\mathcal{S}(h^\pm)$ with translation by $-\lambda$. Considering $\mathcal{S}^+(N/H, \chi_\lambda)$ and $\mathcal{S}(h^\pm + \lambda)$ as subspaces of $\mathcal{S}^*(N)$ and $\mathcal{S}(n^\pm)$ respectively, the Fourier transform is defined on these spaces and $f \mapsto \hat{f}$ is an isomorphism of $\mathcal{S}(N/H, \chi_\lambda)$ onto $\mathcal{S}(h^\pm + \lambda)$ and of $\mathcal{S}(h^\pm + \lambda)$ onto $\mathcal{S}(N/H, \chi_\lambda)$. Also, for $D \in \mathcal{S}(n^\pm)$, $(P^*_D D) = R^*_D D$, where $R_3: \mathcal{S}(n^\pm) \rightarrow \mathcal{S}(h^\pm + \lambda)$ is restriction. Thus $(P^*_D D)$ is supported on $h^\pm - \lambda$ and has no normal derivatives.

For $f \in \mathcal{S}(N/H, \chi_\lambda)$ and $D \in \mathcal{S}(N/H, \chi_\lambda)$, the convolution $D * f$ is defined by setting $D * f(x) = (D, Df)$ for each $x \in N$. Suppose now that $D \in \mathcal{S}(N)$ and $f \in \mathcal{S}(N)$. One can use Abelian Fourier analysis to study the mapping defined on $\chi$, the center of $N$, by $Y \mapsto D * f(\exp(X + Y))$. If this mapping is in $\mathcal{S}(\chi)$, then

$$D * f(\exp(X)) = \int_{\mathcal{S}} P_D (D * f)(\exp(X)) d\lambda,$$

for appropriately normalized Lebesgue measure $d\lambda$. Furthermore, $P_D (D * f) = D_1 \ast P_1 f$, where $D_1$ is the element of $\mathcal{S}(N/H, \chi_\lambda)$ whose Fourier transform is the restriction to $h^\pm + \lambda$ of $D$. Thus, convolution between elements of $\mathcal{S}^*(N)$ and $\mathcal{S}(N)$ decomposes into convolution between elements of $\mathcal{S}^*(N/H, \chi_\lambda)$ and $\mathcal{S}(N/H, \chi_\lambda)$ in such a way that smoothness and growth conditions on $D, D \in \mathcal{S}(N/H, \chi_\lambda)$, are inherited by $D_1, D_1 \in \mathcal{S}(N/H, \chi_\lambda)$. One then proceeds by induction on the dimension of $N/H$. Of course, this requires maintaining considerable control of the various seminorm estimates that appear in the decompositions.

Remarks. The sufficiency of the condition in Theorem A was first proved by R. Howe in [He], and indeed, the ideas presented there are the foundation of this work. Theorem C was proved for the case where $\theta$ is a polynomial by A. Kirillov in [K]. In [CG], L. Corwin and F. Greenleaf proved Theorem D with the additional assumption that all the representations in general position were induced from a common normal subgroup. One-sided Schwartz multipliers have been studied by L. Corwin in [C].

Acknowledgements. The author is particularly indebted to R. Howe for many discussions and suggestions concerning this work. Thanks are also due to C. Benson and G. Ratcliffe for their many contributions improving an earlier version of this paper.

1. Preliminaries. Let $N$ denote a connected, simply connected nilpotent Lie group with Lie algebra $n$. Let $[X, Y]$ denote the Lie bracket of elements $X, Y \in \mathfrak{n}$. Denote by $\text{ad}$ the adjoint representation of $\mathfrak{n}$ on $\mathfrak{n}$, i.e. $\text{ad}(X)(Y) = [X, Y]$. The rank of $\mathfrak{n}, r$, is the smallest integer $s$ such that $\text{ad}(X)^s = 0$ for all $X \in \mathfrak{n}$.

The exponential mapping, denoted by $\exp$, is a diffeomorphism of $\mathfrak{n}$ onto $N$. For $X, Y \in \mathfrak{n}$, define $C(X, Y)$ by $\exp(C(X, Y)) = \exp(X) \exp(Y)$. The Campbell–Hausdorff formula (cf. [S]) gives

$$C(X, Y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot \sum_{i=0}^{n-1} \tau(X^{p_1}, Y^{q_1}, \ldots, X^{p_n}, Y^{q_n})}{A(p_1, q_1, \ldots, p_n, q_n)},$$

where the second sum runs over the integers $p_i, q_i \geq 0$ with $p_i + q_i \geq 1$ for $i = 1, \ldots, n$ and

$$A(p_1, q_1, \ldots, p_n, q_n) = \sum_{i=1}^{n} (p_i + q_i) p_1 q_1 \cdots p_n q_n,$$

$$\tau(X^{p_1}, Y^{q_1}, \ldots, X^{p_n}, Y^{q_n}) = \left[ \ldots \left[ [X^{p_1}, Y^{q_1}] \right] X^{p_2} Y^{q_2} \right] \ldots X^{p_n} Y^{q_n},$$

where, by definition,

$$[X^p, Y^q] = \left[ \ldots \left[ [X, X], X \right] \ldots X \right] Y, \ldots, Y,$$

for appropriately normalized Lebesgue measure $d\lambda$. Furthermore, $P_D (D * f) = D_1 \ast P_1 f$, where $D_1$ is the element of $\mathcal{S}(N/H, \chi_\lambda)$ whose Fourier transform is the restriction to $h^\pm + \lambda$. Thus, convolution between elements of $\mathcal{S}(N)$ and $\mathcal{S}(N)$ decomposes into convolution between elements of $\mathcal{S}(N/H, \chi_\lambda)$ and $\mathcal{S}(N/H, \chi_\lambda)$ in such a way that smoothness and growth conditions on $D, D \in \mathcal{S}(N/H, \chi_\lambda)$, are inherited by $D_1, D_1 \in \mathcal{S}(N/H, \chi_\lambda)$. One then proceeds by induction on the dimension of $N/H$. Of course, this requires maintaining considerable control of the various seminorm estimates that appear in the decompositions.

Remarks. The sufficiency of the condition in Theorem A was first proved by R. Howe in [He], and indeed, the ideas presented there are the foundation of this work. Theorem C was proved for the case where $\theta$ is a polynomial by A. Kirillov in [K]. In [CG], L. Corwin and F. Greenleaf proved Theorem D with the additional assumption that all the representations in general position were induced from a common normal subgroup. One-sided Schwartz multipliers have been studied by L. Corwin in [C].

Acknowledgements. The author is particularly indebted to R. Howe for many discussions and suggestions concerning this work. Thanks are also due to C. Benson and G. Ratcliffe for their many contributions improving an earlier version of this paper.

1. Preliminaries. Let $N$ denote a connected, simply connected nilpotent Lie group with Lie algebra $n$. Let $[X, Y]$ denote the Lie bracket of elements $X, Y \in \mathfrak{n}$. Denote by $\text{ad}$ the adjoint representation of $\mathfrak{n}$ on $\mathfrak{n}$, i.e. $\text{ad}(X)(Y) = [X, Y]$. The rank of $\mathfrak{n}, r$, is the smallest integer $s$ such that $\text{ad}(X)^s = 0$ for all $X \in \mathfrak{n}$.

The exponential mapping, denoted by $\exp$, is a diffeomorphism of $\mathfrak{n}$ onto $N$. For $X, Y \in \mathfrak{n}$, define $C(X, Y)$ by $\exp(C(X, Y)) = \exp(X) \exp(Y)$.
The other observations follow from the commutation relations
\[ [\varphi(n) \circ \log, \varphi^{(b)}(n)] \subset \varphi^{(b+n-2)}(n). \]

Henceforth in this section, \( \mathfrak{h} \), \( \mathfrak{l} \) and \( \mathfrak{m} \) will denote subspaces of \( n \) such that the center of \( \mathfrak{g} \), \( \mathfrak{s} \), is the direct sum of \( \mathfrak{h} \) and \( \mathfrak{l} \) and such that \( n = \mathfrak{m} \oplus \mathfrak{l} \oplus \mathfrak{h} \). Let \( \langle , \rangle \) be a positive-definite inner product on \( n \) for which \( \mathfrak{m} \), \( \mathfrak{l} \) and \( \mathfrak{h} \) are mutually orthogonal, and let \( \| \cdot \| \) denote the corresponding Euclidean norm. Pick an orthonormal basis \( \{ X_1, \ldots, X_d \} \) of \( n \) such that \( \{ X_n, X_{n+1}, \ldots, X_d \} \) is a basis for \( \mathfrak{h} \), \( \{ X_1, X_{n+1}, \ldots, X_{2n-1} \} \) is a basis for \( \mathfrak{l} \), and with \( \{ X_1, \ldots, X_{n-1} \} \) a basis for \( \mathfrak{m} \). Let \( \{ X^*_1, \ldots, X^*_d \} \) be the dual basis in \( n^* \). There is a unique norm on \( n^* \) for which \( \{ X^*_1, \ldots, X^*_d \} \) is an orthonormal set.

By restriction, \( \langle , \rangle \) defines an inner product on any subspace of \( n \). If \( p \) is an ideal in \( n \), we define an inner product on \( n/p \) by requiring that the projection from \( n \) the orthogonal complement to \( p \) in \( n \) to \( n/p \) be an isometry. In this manner, the inner product on \( n \) can be used to define inner products on any subquotient of \( n \).

Given the inner product \( \langle , \rangle \) on \( n \) or \( n^* \), there is a natural extension of \( \langle , \rangle \) to \( S(V) \), again denoted by \( \langle , \rangle \), for any subspace \( V \) of \( n \) or \( n^* \). It is obtained by requiring that the homogeneous components of \( S(V) \) be orthogonal and that, for \( X, Y \in V \), it satisfies \( \langle X', Y' \rangle = \langle X, Y \rangle \). By using the symmetrization mapping \( \sigma \), the inner product on \( n \) pulls back to an inner product on \( \mathfrak{g} \), i.e. for \( L, L' \in \mathfrak{g}(n) \), set \( \langle L, L' \rangle = \langle \sigma(L), \sigma(L') \rangle \).

For \( X \in \mathfrak{g}, \text{ad}(X) \) is an endomorphism on \( \mathfrak{g} \) and hence has an operator norm, \( \| \text{ad}(X) \| \). Also, the mapping \( X \mapsto \text{ad}(X) \) of \( n \) into \( \text{End}(\mathfrak{g}) \) has an operator norm \( \| \text{ad} \| \).

**Lemma 1.3.** Let \( p \) be an ideal of \( n \) and let \( \mathfrak{g} \) be the orthogonal complement to \( p \) in \( n \). There is a constant \( C \) that bounds the function \( M \) defined on \( n \times p \) by
\[ M(X, Y) = (1 + \|X\|)(1 + \|Y\|)(1 + \|C(X, Y)\|). \]
It follows that for any nonnegative integers \( r, s \) there is an integer \( t \) and a constant \( C \) such that for \( X \in \mathfrak{g} \) and \( Y \in p \),
\[ (1 + \|X\|)^r(1 + \|Y\|)^s \leq C(1 + \|C(X, Y)\|)^t. \]

**Proof.** Assume there exist \( \{ X_n \} \subset \mathfrak{g} \) and \( \{ Y_n \} \subset p \) such that \( M(X_n, Y_n) \to \infty \). Then \( s_n = \|X_n\| \to \infty \) and \( t_n = \|Y_n\| \to \infty \). Let \( p' \), \( i = 1, \ldots, m \), be orthogonal subspaces of \( p \) such that \( p = \bigoplus p' \) and
\[ [p', p'] = \bigoplus_{k \geq 1} p'^k. \]

Let \( Y'_n = \sum Y_n \) and \( C(X_n, Y_n) = X_n + \sum C(X_n, Y_n) \), where \( Y'_n \) and \( C(X_n, Y_n) \) are elements of \( p' \).
Since $(1 + \|X\|)(1 + \|Y\|)/|s\cdot t| \to 1$, \(\|C(X_n, Y_n)/|s\cdot t| \to 0\). Thus \(X_n(t_n^i s_n^i)^{1/2} \to 0\), which implies that \(t_n^i s_n^i \to \infty\). Now \(C^1(X_n, Y_n) = Y_n^1\) and for \(i \geq 2, C(X_n, Y_n)\) is a sum of terms involving \(Y_n^i\) and Lie products of \(X_n, Y_n^i, \ldots, Y_n^{i-1}\), with constants depending on \(n\). Since \(\|C(X_n, Y_n)/|s\cdot t|^{1/2}\) \(\to 0\) for each \(i\), by induction one sees that \(\|Y_n^i/|s\cdot t|^i\) \(\to 0\). This implies that \(s_n^i t_n^i \to \infty\), a contradiction.

Similarly, as one can show very easily, for each \(r \geq 0\) there is a constant \(C\) and an integer \(s\) such that

\[
(1 + \|C(X, Y)\|^3) \leq C(1 + \|X\|^2)(1 + \|Y\|^2)^3.
\]

2. Schwartz spaces. Although one can define the Schwartz space on \(N\) in terms of the Schwartz space on \(\mathbb{R}^n\) and \(\exp\), or even go further, suppress \(N\) altogether by introducing a second group structure on \(\mathbb{R}^n\) via the Campbell-Hausdorff formula, for our purposes it is necessary to make estimates that involve the action of \(N\) on Schwartz functions. Thus, it is more convenient to define the Schwartz space explicitly in terms of \(N\).

The Schwartz space on \(N\), denoted by \(\mathcal{S}(N)\), is the space of all smooth functions defined on \(N\) for which the seminorm

\[
\left( \int (1 + \|X\|^2)^p |\partial_X f(\exp(X))| dX \right)^{1/p}
\]

is finite for each \(p \geq 0\), for each \(1 \leq q \leq \infty\), and for each \(L \in \mathcal{U}(\mathbb{R})\). The topology on \(\mathcal{S}(N)\) is generated by these seminorms.

Recall that if, \(I\) and \(J\) are orthogonal subspaces of \(\mathbb{R}\) such that \(\mathbb{R} = M \oplus I \oplus J\), and the center of \(G\) is given by \(I \oplus J\). For each \(\lambda \in \mathbb{R}\), define the unitary character \(\chi_\lambda\) on \(H = \exp(iI)\) by

\[
\chi_\lambda(\exp(X)) = e^{iX \cdot \chi_\lambda}.
\]

Define \(\mathcal{S}(N/H, \chi_\lambda)\) to be the space of smooth functions defined on \(N\) that satisfy

\[
f(\exp(X + Y)) = \chi_\lambda(\exp(Y)) f(\exp(X)), \quad X, Y \in \mathbb{R} \oplus I, \quad Y \in \mathbb{R} \oplus J,
\]

and for which the seminorm

\[
\left( \int (1 + \|X\|^2)^p |\partial_X f(\exp(X))| dX \right)^{1/p}
\]

is finite for each \(p \geq 0\), all \(1 \leq q \leq \infty\), and all \(L \in \mathcal{U}(\mathbb{R})\). (It should be noted that the measures on \(\mathbb{R}\) and \(\mathbb{R} \oplus I\) are the Lebesgue measures induced by the inner products, and that they are carried by the exponential map to Haar measures on the groups \(N\) and \(N/H\). It should also be noted that because \(I\) is a central ideal, if \(f\) is a smooth function that satisfies (2.1) and \(L \in \mathcal{U}(\mathbb{R})\) then \(\partial_X f\) again satisfies (2.1).)

The space \(\mathcal{S}(N/H, \chi_\lambda)\) has the topology generated by the seminorms.

given in (2.2). There are three different generating families of these seminorms that we will use. They are denoted by \(\|f\|_{s,q}\), where \(s = 1, 2, \infty\), and \(p, q\) are nonnegative integers, and are defined by

\[
\|f\|_{s,q} = \left( \int (1 + \|X\|^2)^p |\partial_X f(\exp(X))|^q dX \right)^{1/q}
\]

for \(s = 1, 2\), and where the sum is over an orthonormal basis of \(\mathcal{U}(\mathbb{R})\). The seminorms in (2.3) do, of course, depend on the choice of orthonormal basis, but only up to equivalence.

If \(V\) is a subspace of \(\mathbb{R}^n\) or \(\mathbb{R}^n\), the Schwartz space of \(V, \mathcal{S}(V)\), is defined as usual, i.e. \(\mathcal{S}(V)\) is the space of all smooth functions defined on \(V\) for which the seminorms

\[
\left( \int (1 + \|X\|^2)^p |\partial_X f(\exp(X))| dX \right)^{1/p}
\]

are finite, for each \(p \geq 0\), \(r \geq 1\), and each \(Y \in \mathcal{S}(\mathbb{R}^n)\), and the topology is generated by these seminorms. Similarly to the above, there are analogous families of seminorms \(\|f\|_{s,q}\).

The following lemma shows that the exponential mapping induces an isomorphism between \(\mathcal{S}(N)\) and \(\mathcal{S}(\mathbb{R})\).

**Lemma 2.5.** For \(p \geq 0\) and \(q \geq 1\), there is a constant \(C_{p,q}\) such that for each \(f \in \mathcal{S}(\mathbb{R})\),

\[
\|f\|_{p,q} \leq C_{p,q} \|f \circ \exp\|_{p+q(q-1),pq}^{-1}
\]

and for \(f \in \mathcal{S}(\mathbb{R})\),

\[
\|f\|_{p,q} \leq C_{p,q} \|f \circ \exp\|_{p+q(q-1)(2q-3)+...+q-1,q-1}^{-1}
\]

The proof is an immediate consequence of Lemma 1.2.

If \(f \in \mathcal{S}(\mathbb{R})\) and \(p \geq 0\), \(q \geq 1\), there exist constants \(A, B, C, p_i, q_i, i = 1, 2, 3\), such that

\[
\|f\|_{p,q} \leq A_1 \|f\|_{p_1,q_1} \leq B_2 \|f\|_{p_2,q_2} \leq C_\infty \|f\|_{p_3,q_3}.
\]

The first inequality is a Sobolev inequality, while the second and third inequalities are established using the Schwarz inequality. Using Lemma 2.5, one can prove analogous inequalities on \(\mathcal{S}(N)\). By restricting to the exponential map \(\exp(x)\), one gets an isometry from \(\mathcal{S}(N/H, \chi_\lambda)\) to \(\mathcal{S}(N/H)\). This, combined with the mappings between \(\mathcal{S}(\mathbb{R})\) and \(\mathcal{S}(N/H)\) induced by the exponential, establishes (2.6) for the spaces \(\mathcal{S}(N/H, \chi_\lambda)\).

Let \(\mathfrak{h}^i\) denote the annihilator of \(\mathfrak{h}\) in \(\mathfrak{g}^i\). For \(\lambda \in \mathfrak{h}^i\), \(\mathcal{S}(\mathfrak{h}^i + \lambda)\) is the space of functions \(f\) defined on \(\mathfrak{h}^i + \lambda\) such that the function \(f_i\) given by \(f_i(\eta)\)
\[
= f(\eta + \lambda) \text{ is in } \mathcal{S}(h^+) \text{. The seminorms on } \mathcal{S}'(h^+) \text{ are pulled back to } \mathcal{S}'(h^+ + \lambda) \text{ by the mapping } f \to f_\lambda \text{ and generate the topology.}
\]

For \( f \in \mathcal{S}(N) \), define \( \tilde{f} \) on \( n^* \) by
\[
\tilde{f}(\xi) = \int f(\exp(X)) e^{-2\pi i \langle \xi, X \rangle} d\xi \text{.}
\]

Since, by Lemma 2.5, the mapping \( f \to f \circ \exp \) is an isomorphism from \( \mathcal{S}(N) \) to \( \mathcal{S}(n) \), the usual theory establishes that \( f \to \tilde{f} \) is an isomorphism from \( \mathcal{S}(N) \) to \( \mathcal{S}(n^*) \). Likewise, for \( F \in \mathcal{S}(n^*) \), \( \tilde{F} \circ \log \circ \exp \in \mathcal{S}(N) \), where, for \( X \in \mathfrak{n} \),
\[
\tilde{F}(X) = \int F(\xi) e^{-2\pi i \langle \xi, X \rangle} d\xi \text{.}
\]

Note that for \( F \in \mathcal{S}(n^*) \)
\[
(\tilde{F} \circ \log \circ \exp)(\xi) = F(-\xi) = \tilde{F}(\xi) \text{,}
\]
while for \( f \in \mathcal{S}(N) \)
\[
(\tilde{f} \circ \log \circ \exp)(\xi) = f(-\xi) = \tilde{f}(\xi) \text{.}
\]

Let \( \mathcal{S}^*(N) \) denote the dual space of \( \mathcal{S}(N) \). For \( \tilde{f} \in \mathcal{S}^*(N) \), \( \hat{\tilde{f}} \) is defined in \( \mathcal{S}^*(n^*) \) by \( \langle \hat{\tilde{f}}, F \rangle = \langle \hat{f}, \tilde{F} \rangle \). Similarly, for \( \tilde{F} \in \mathcal{S}^*(n^*) \), \( \hat{\tilde{F}} \) is defined on \( \mathcal{S}(N) \) by \( \langle \hat{\tilde{F}}, f \rangle = \langle \hat{F}, \tilde{f} \rangle \). From (2.7) and (2.8) one has \( \langle \hat{f}, \tilde{f} \rangle = \langle f, \tilde{f} \rangle \).

The spaces \( \mathcal{S}(N/H, \chi_\lambda) \) and \( \mathcal{S}'(h^+ + \lambda) \) may be considered as subspaces of \( \mathcal{S}(N) \) and \( \mathcal{S}^*(n^*) \) respectively, and thus the above definition of Fourier transform applies. We have

**Lemma 2.9.** For \( f \in \mathcal{S}(N/H, \chi_\lambda) \), the distribution \( \tilde{f} \) is absolutely continuous with respect to the Lebesgue measure on \( n^* \), and has density given by
\[
\tilde{f}(\xi + \lambda) = \int f(\exp(Y)) e^{-2\pi i \langle \xi + \lambda, Y \rangle} dY \text{.}
\]

For \( F \in \mathcal{S}(h^+ + \lambda) \),
\[
\hat{F}(\xi) = \int F(\xi + \lambda) e^{-2\pi i \langle \xi, \lambda, X \rangle} d\xi \text{.}
\]

It follows that
\[
\langle \hat{f}, f \rangle = \int \langle \hat{f}(\exp(X)), f(\exp(X)) \rangle dX \text{,}
\]
and
\[
\langle \hat{F}, F \rangle = \int \langle \hat{F}(\xi + \lambda), F(\xi + \lambda) \rangle d\xi \text{.}
\]

**Proof.** Let \( f \in \mathcal{S}(N/H, \chi_\lambda) \) and \( \varphi \in \mathcal{S}(n) \). Then
\[
\langle \tilde{f}, \varphi \rangle = \int f(\exp(X)) \varphi(\exp(X)) dX \text{ = } \int f(\exp(X)) \varphi(\xi) e^{-2\pi i \langle \xi, X \rangle} d\xi dX \text{.}
\]

This establishes (2.10). Since \( f \circ \exp_{\mathfrak{m} \mathfrak{g}_0} \in \mathcal{S}(m \mathfrak{g}_0) \), it follows that \( \tilde{f} \in \mathcal{S}(h^+ + \lambda) \).

The proof of 2.11 is immediate from the definitions, as well as the fact that \( f \in \mathcal{S}(N/H, \chi_{-\lambda}) \). This and the previous inclusion establish the equalities (2.12) and (2.13).

For \( \lambda \in \mathfrak{h}^* \), define \( P_{\lambda} : \mathcal{S}(N) \to \mathcal{S}(N/H, \chi_{\lambda}) \) by
\[
P_{\lambda} f(\exp(X)) = \int f(\exp(X + Y)) e^{-2\pi i \langle \xi, Y \rangle} dY \text{.}
\]

Let \( f \in \mathcal{S}(N/H, \chi_{\lambda}) \) and define \( f_1 \in \mathcal{S}(m \mathfrak{g}_0) \) by \( f_1(X) = f(\exp(X)) \) for \( X \in \mathfrak{m} \mathfrak{g}_0 \). Let \( f_2 \in \mathcal{S}(h^+) \) such that \( f_2(\lambda) = 1 \). Then \( f = f_1 \otimes f_2 \circ \exp \in \mathcal{S}(N) \) and \( P_{\lambda} f = f \). Note that if \( f \to f \) in \( \mathcal{S}(N/H, \chi_{\lambda}) \), this construction yields a sequence \( f_n \to f \) in \( \mathcal{S}(N) \) such that \( f_n = P_{\lambda} f_n \).

Let \( \mathcal{S}^*(N/H, \chi_{\lambda}) \) denote the dual space of \( \mathcal{S}(N/H, \chi_{\lambda}) \). Then \( P_{\lambda}^* \), the adjoint of \( P_{\lambda} \), is a continuous injection of \( \mathcal{S}^*(N/H, \chi_{\lambda}) \) into \( \mathcal{S}^*(N) \). Furthermore, the range of \( P_{\lambda}^* \) is the annihilator of the kernel of \( P_{\lambda} \).

Let \( D \in \mathcal{S}(N/H, \chi_\lambda) \), \( f \in \mathcal{S}(N/H, \chi_\lambda_0) \), and \( x \in \mathfrak{h} \). Define
\[
\langle f, \chi_\lambda \rangle(X) = \langle f(\exp(X)), \chi_\lambda \rangle(X) = \langle f(\exp(X)), \chi_\lambda \rangle \text{ for } X \in \mathfrak{h} \text{.}
\]

**Lemma 2.14.** \( P_{\lambda}^* \) is an isomorphism from \( \mathcal{S}^*(N/H, \chi_\lambda) \) to \( \mathcal{S}^*(N) \).

**Proof.** It is easy to see that for \( D \in \mathcal{S}^*(N/H, \chi_\lambda) \), \( P_{\lambda}^* (D) \in \mathcal{S}^*(N) \). For the other inclusion, we recall some facts from the Schwartz theory on \( \mathbb{R}^d \). Specifically, if \( D \in \mathcal{S}^*(\mathbb{R}^d) \) and \( f \in \mathcal{S}(\mathbb{R}^d) \) then \( D \ast f \) is defined on \( \mathbb{R}^d \) by \( (D \ast f)(X) = \langle D, I \rangle f(X - Y) \), where \( I \) is the identity operator. \( D \ast f \) is again a tempered distribution on \( \mathbb{R}^d \).

Suppose now that \( D \in \mathcal{S}^*_\lambda (N) \) and define \( D \circ \exp \in \mathcal{S}^*(N) \) by \( D \circ \exp (f) = \langle D, f \circ \exp \rangle \). One easily checks that \( (D \circ \exp) \ast f = (D \ast f) \circ \exp \). Thus, if \( g \in \ker(P_{\lambda}) \),
\[
\langle D \circ \exp \ast f, g \circ \exp \rangle = \int \langle D \circ \exp, I_{\lambda} \rangle(f \circ \exp) g(\exp(X + Y)) dY dX = 0.
\]

Since \( f \) was arbitrary in \( \mathcal{S}(N) \), this implies that \( \langle D, g \rangle = 0 \).

For \( \lambda \in \mathfrak{h}^* \), let \( R_{\lambda} : \mathcal{S}(n^*) \to \mathcal{S}(h^+ + \lambda) \) be the restriction mapping. Then
$R_{\lambda}$, the adjoint of $R_{\lambda}$, is an injection of $\mathcal{S}(\mathfrak{h} + \lambda)$ into the tempered distributions supported on $\mathfrak{h} + \lambda$ and without normal derivatives.

**Lemma 2.15.** For $D \in \mathcal{S}(N/H, \chi_2)$, $(P_{\mathfrak{n}}^* D) = R_{-\lambda}^* B$, where $B$ is defined by the adjoint of the Fourier transform mapping $\mathcal{S}(\mathfrak{h} + \lambda) \to \mathcal{S}(N/H, \chi_2)$.

**Proof.** Let $F \in \mathcal{S}(N)$ and $X \in m \mathfrak{g}$. Then

$$
P_{\mathfrak{h}}(\hat{F} \circ \log)(\exp(X)) = \int F(\eta + \mu) e^{-2\pi i \langle \eta, \mu \rangle} d\mu d\eta = \int F(\eta - \lambda) e^{-2\pi i \langle \eta, \lambda \rangle} d\eta = (R_{-\lambda} F)(X).
$$

An element $D \in \mathcal{S}(N/H, \chi_2)$ is said to be bi-invariant if $\langle D, l_{\eta} f \rangle = \langle D, r_{-\eta} f \rangle$ for all $x \in N$ and $f \in \mathcal{S}(N/H, \chi_2)$, where $r_{-\eta} f(y) = f(yx)$. An element $D \in \mathcal{S}(\mathfrak{h} + \lambda)$ is said to be Ad*-invariant if for all $f \in \mathcal{S}(\mathfrak{h} + \lambda)$, $\langle D, f \circ \text{Ad}^* \rangle = \langle D, f \rangle$.

**Lemma 2.16.** $D \in \mathcal{S}(N/H, \chi_2)$ is bi-invariant if, and only if, $\hat{D} \in \mathcal{S}(\mathfrak{h} + \lambda)$ is Ad*-invariant.

**Proof.** Let $f \in \mathcal{S}(N/H, \chi_2)$ and $x \in N$. Then

$$
|l_{\eta} r_{-\eta} f(x)| = \int f(x \cdot \exp(Y) \cdot x) e^{-2\pi i \langle \eta, -\eta \rangle} dY = \int f(\exp(\text{Ad} x Y)) e^{-2\pi i \langle \eta, -\eta \rangle} dY = \int f(\exp(Y)) e^{-2\pi i \langle \eta, -\eta \rangle} dY = f(\text{Ad}^* x (\xi - \lambda)).
$$

The following lemmas are needed in the proof of Theorem A.

**Lemma 2.17.** Let $f$ be a smooth function on $N$ and suppose that for each pair of nonnegative integers $p, q$ there are constants $l = l_{p,q}$ and $C_{p,q}$ such that for $X \in m \mathfrak{g}$, $Y \in \mathfrak{h}$, and $L \in \mathfrak{u}^\mathfrak{g}$,

(i) $|\xi_L f(\exp(X + Y))| \leq C_{p,q} ||L|| (1 + ||X||^2)^l (1 + ||Y||^2)^q$,

(ii) $\int_{m \mathfrak{g}} ||P_{\mathfrak{n}} q \xi_L f||_{p,0} d\lambda \leq C_{p,q} ||L||$.

Then $f \in \mathcal{S}(N)$.

**Proof.** We will show that $\xi_L f(\exp(X + Y))$ is finite for all $p, q \geq 0$.

Note that by (i), for each fixed $X$ and $L$, the mapping $Y \rightarrow \xi_L f(\exp(X + Y))$ is bounded by $C_{p,q} ||L|| (1 + ||X||^2)^l (1 + ||Y||^2)^q$. Thus the usual Fourier inversion gives

$$
\xi_L f(\exp(X + Y)) = \int \xi_L f(\exp(X + Y)) d\lambda.
$$

Thus,

$$
(1 + ||X||^2)^l \xi_L f(\exp(X + Y)) \leq \int (1 + ||X||^2)^l \xi_L f(\exp(X + Y)) d\lambda \leq \int_{m \mathfrak{g}} ||P_{\mathfrak{n}} q \xi_L f||_{p,0} d\lambda \leq C_{p,q} ||L||.
$$

Therefore,

$$
(1 + ||X + Y||^2)^l \xi_L f(\exp(X + Y)) \leq (1 + ||X||^2)^l (1 + ||Y||^2)^l \xi_L f(\exp(X + Y)) \leq C_{p,q} ||L|| (1 + ||X||^2)^l (1 + ||Y||^2)^l \xi_L f(\exp(X + Y)) \leq C_{p,q} C_{p+1,q} ||L||^2.
$$

**Lemma 2.18.** Given integers $p, q \geq 0$, there exist integers $p', q' \geq 0$ and a constant $C$ such that for all $f \in \mathcal{S}(N)$,

$$
\int_{m \mathfrak{g}} ||P_{\mathfrak{n}} q \xi_L f||_{p',q'} d\lambda \leq C ||f||_{p',q'}.
$$

**Proof.** For $s = 1$,

$$
\int_{m \mathfrak{g}} ||P_{\mathfrak{n}} q \xi_L f||_{p',q'} d\lambda = \int_{m \mathfrak{g}} \sum_{i} (1 + ||X||^2)^i ||\xi_L P_{\mathfrak{n}} f(\exp(X))||_{p',q'} d\lambda (\text{the sum is over an orthonormal basis of } \mathfrak{h}^\mathfrak{h} (n/\mathfrak{h})) \leq \int \sum_{i} (1 + ||X||^2)^i (1 + ||Y||^2)^i (1 - d)^i \xi_L f(\exp(X + Y)) d\lambda \leq C ||f||_{p + 1, q + 2k, q + 2k}.
$$

for $k$ sufficiently large.

3. **Convolution.** Recall that for a function $f$ defined on $N$ and for $x$ and $y$ in $N$, $l_x f(y) = f(x^{-1} y)$ and $\hat{f}(x) = f(x^{-1})$. Thus, if $f \in \mathcal{S}(N/H, \chi_2)$, $f \in \mathcal{S}(N/H, \chi_{-1})$. For $D \in \mathcal{S}(N/H, \chi_{-1})$ and $f \in \mathcal{S}(N/H, \chi_2)$ the function $D * f$ is defined on $N$ by

$$
D * f(x) = \langle D, l_x f \rangle.
$$

Note that if $\lambda = 0$ and $D = g \in \mathcal{S}(N)$, this definition agrees with the usual one, i.e.

$$
g * f(x) = \langle g, l_x f \rangle = \int g(\exp(Y)) f(\exp(-Y) x) dY.
$$
Since convolution (on the left) by $D$ commutes with right translation, $D * f$ will be a smooth function on $N$ for each $D \in \mathcal{D}(N/H, \chi_d)$ and $f \in \mathcal{S}(N/H, \chi_d)$. Thus to show that $D * f$ is again in $\mathcal{S}(N/H, \chi_d)$ it suffices to show that $D * f$ is rapidly decreasing. More precisely, we have

**Lemma 3.1.** Let $D \in \mathcal{D}(N/H, \chi_d)$ and suppose that for each nonnegative integer $p$ there exist a constant $C_p$ and nonnegative integers $p', q'$ such that

$$\|D * f\|_{p,0} \leq C_p \|f\|_{p',q'},$$

for each $f \in \mathcal{S}(N/H, \chi_d)$. Then for each $q$ there exist a constant $C_q$ independent of $f$ and nonnegative integers $p''$, $q''$ such that

$$\|D * f\|_{p''} \leq C_q \|f\|_{p'',q''}.$$

**Proof.** We give the proof for $s = 2$:

$$\|D * f\|_{p''} = \left(\sum |\hat{f}_t(D * f)|_{L^2(\mathbb{R}^d)}^2\right)^{1/2},$$

the sum being over an orthonormal basis for $\mathcal{S}(\mathbb{R}^d)$. Thus

$$\|D * f\|_{p''} \leq \left(\sum |\hat{f}_t(D * f)|_{L^2(\mathbb{R}^d)}^2\right)^{1/2},$$

$$\|D * f\|_{p',q'} = \left(\sum |\hat{f}_t(D * f)|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \leq C_q \|f\|_{p',q'},$$

where

$$C_q = \sup \{\|L \|L\|/\|L\|\|L\|: L \in \mathcal{S}(\mathbb{R}^d), L \in \mathcal{S}(\mathbb{R}^d)\}.$$

Let $\mathfrak{p}$ be an ideal in $\mathfrak{n}$ that contains $\mathfrak{h}$, and let $P = \exp(\mathfrak{p})$. Let $R: \mathcal{S}(N/H, \chi_d) \rightarrow \mathcal{S}(P/H, \chi_d)$ be the restriction mapping, and denote its adjoint by $R^*$. The following lemma shows that if $D \in \mathcal{D}(P/H, \chi_d)$ then $R^* D$ is an operator on $\mathcal{S}(P/H, \chi_d)$.

**Lemma 3.2.** Let $D \in \mathcal{D}(P/H, \chi_d)$ and suppose that for each nonnegative integer $p$ there exist a constant $C_p$ and nonnegative integers $p', q'$ such that for each $f \in \mathcal{S}(P/H, \chi_d)$,

$$\|D * f\|_{p,0} \leq C_p \|f\|_{p',q'},$$

Then there exist a constant $C_p$ and nonnegative integers $p''$, $q''$ such that for each $f \in \mathcal{S}(N/H, \chi_d)$,

$$\|R^* D * f\|_{p,0} \leq C_p \|f\|_{p',q''},$$

**Proof.** We give the proof for $s = 2$:

$$\int_{N/H} |D * f(x)|^2 (1 + \|\log(x)\|^2)^q dx$$

$$= \int_{P/H} |D * f(y)|^2 (1 + \|\log(y)\|^2)^q dy$$

$$\leq C \int_{P/H} \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

(for some $r = r(p)$, and $C$ independent of $f$ and $D$)

$$\leq C \int_{P/H} \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

$$\leq C C_p \int_{P/H} \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

$$= C C_p \sum_{p''} \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

the sum is over an orthonormal basis of $\mathcal{S}(\mathbb{R}^d) (p/0)$

$$= C C_p \sum_{p''} \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

(by Lemma 1.2)

$$\leq C C_p \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

$$= C C_p \|D * f(y)\|^2 (1 + \|\log(y)\|^2)^{\frac{q}{2}} dy$$

$\sup_{\mathfrak{p}''} \mathfrak{p}''$.

**Lemma 3.3.** Suppose that $D \in \mathcal{D}(N/H, \chi_d)$ such that $(P\mathfrak{c} D)^{-1}$ is a smooth function on $\mathfrak{n}^+ + \lambda$, all of whose derivatives have polynomial bounds. Then the function $Y \rightarrow D * f(\exp(X + Y))$ is $\mathcal{S}(\mathfrak{g})$. More specifically, suppose that for each integer $j \geq 0$ there is an integer $l$ and a constant $C_j(D)$ such that for $\eta \in S^j(\mathfrak{n}^+)$,

$$|\partial_{\eta} D(v + \lambda)| \leq C_j(D) \|\eta\| (1 + \|\lambda\|^2)^j, \quad v \in \mathfrak{n}^+.$$

Then, given a nonnegative integer $p$, an $X \in \mathfrak{h}$, and an $f \in \mathcal{S}(N/H, \chi_d)$, there is a constant $C_p(X)$ and positive integers $l$, $p$, $q'$, independent of $f$, such that

$(i)$

$$\|D * f(\exp(X + Y))\| \leq C_p(X) C_{p''}(D) \|f\|_{p'',q''},$$

$(ii)$

$$C_p(X) \leq C (1 + \|X\|^2)^l.$$
Proof. Let \( F = (P_{*W}D) \) and set \( g \in \mathcal{S}(N) \) such that \( P_{*W}g = f \). Then
\[
D * f(x) = \langle \mathcal{D}, l_{\mathcal{D}}(P_{*W}g) \rangle = \langle P_{*W}D, l_{\mathcal{D}}g \rangle = \langle F, (l_{\mathcal{D}}g) \rangle.
\]
If \( Y \in \mathbb{N} \) and \( h \in \mathcal{S}(N) \) then \( (l_{\exp_{Y}}h) = e^{2\pi i \langle \lambda, \varphi \rangle} \langle h \rangle \). Thus, setting
\[
\Delta = \sum_{n=0}^{h-1} \frac{1}{4\pi^2} \int \varphi_x^2 \int_{\mathbb{R}^n} d
\]
one has \( (1 - \alpha)^n(l_{\exp_{Y}}h) = (1 + |\gamma|^2)^n(l_{\exp_{Y}}h) \). For positive integer \( k \), define \( \varphi_k \) on \( h^k + \lambda \) by \( \varphi_k(x + \lambda) = (1 + |x|^2)^{\lambda} \). Then, for \( X \in \mathbb{N} \) and \( Y \in \mathbb{N} \),
\[
(1 + |Y|^2)^n D * f(\exp(X + Y))
= \int \varphi_k^{-1}(\xi + \lambda)(1 - \alpha)^k F(\xi + \lambda)(l_{\exp_{Y}}(\xi + \lambda) \varphi_k \varphi_k(\xi + \lambda) d\xi
= \left( \int_{\mathbb{R}^n} \varphi_k^{-1}(\xi + \lambda)(1 - \alpha)^k F(\xi + \lambda)(l_{\exp_{Y}}(\xi + \lambda) \varphi_k \varphi_k(\xi + \lambda) d\xi
= \left( \int_{\mathbb{R}^n} \varphi_k^{-1}(\xi + \lambda)(1 - \alpha)^k F(\xi + \lambda)(l_{\exp_{Y}}(\xi + \lambda) \varphi_k \varphi_k(\xi + \lambda) d\xi
\]
where \( \Delta = \sum_{n=0}^{h-1} \frac{1}{4\pi^2} \int \varphi_x^2 \int_{\mathbb{R}^n} d\).
for all \( \eta \in \mathcal{S}(n^*) \). Then for each integer \( p \geq 0 \) there exist integers \( j, p', q' \), and a constant, \( C_p \) such that for each \( f \in \mathcal{S}(N) \),

\[
\|D^s f\|_{p,q'} \leq C_p C_J(D)\|f\|_{p',q'}.
\]

**Proof.**

\[
\sup_{\{\theta_n\}} \|D^s f\|^2 \leq C_{C_p} C_J(D)\|f\|_{p',q'}^2 + C_{C_p} C_J(D)\|f\|_{p',q'}^2,
\]

where the symbol \( \sup_{\{\theta_n\}} \) converges to \( \theta \) in \( \mathbb{PB}_\mathbb{F}(n^*) \) if for each \( j \) and all \( i \) sufficiently large, \( \rho_{n,i}(\theta_n - \theta) \to 0 \). Given \( \theta \in \mathbb{PB}_\mathbb{F}(n^*) \), we let \( D_\theta \) denote the element of \( \mathcal{S}_j^*(N) \) such that \( (D_\theta)^{-1} = \theta \). Theorem 3.3 may be rephrased as

**Theorem 3.3'.** The mapping \( \mathbb{PB}_\mathbb{F}(n^*) \times \mathcal{S}_j(N) \to \mathcal{S}_j(N) \) given by \( (\theta, f) \to D_\theta \) is jointly continuous.

Let \( \mathcal{S}_j(N/H, \chi) \) denote the space of all bi-invariant distributions \( D \) in \( \mathcal{S}_j(N/H, \chi) \) such that \( D \in \mathcal{S}_j(N/H, \chi) \) for each \( f \in \mathcal{S}_j(N/H, \chi) \), where \( D \in \mathcal{S}_j(N/H, \chi) \) is topologized so that \( D \to 0 \) in \( \mathcal{S}_j(N/H, \chi) \) if \( D \to f \) in \( \mathcal{S}_j(N/H, \chi) \).

**Theorem 3.9.** The mapping \( \theta \to D_\theta \) is a homeomorphism of \( \mathbb{PB}_\mathbb{F}(n^*) \) onto \( \mathcal{S}_j(N/H, \chi) \).

**Remark.** The fact that this mapping is also an algebra homomorphism is proved in Corollary 4.4.

**Proof.** It remains only to show that the Fourier transform of each bi-invariant Schwartz multiplier is in \( \mathbb{PB}_\mathbb{F}(n^*) \). We must first show that for \( D \in \mathcal{S}_j(N/Z, \chi) \), \( D \in \mathbb{PB}_\mathbb{F}(3^j + 1) \). For this, note that if \( D \in \mathcal{S}_j(N/H, \chi) \), then \( D \) is in \( \mathcal{S}_j(N/Z, \chi) \) for each \( f \in \mathcal{S}_j(N/H, \chi) \), where \( P_f : \mathcal{S}_j(N/H, \chi) \to \mathcal{S}_j(N/Z, \chi) \) is defined by

\[
P_f(x) = \int (x \exp(Y)e^{-2\pi \langle a, Y \rangle} dY,
\]

where the symbol \( \sup_{\{\theta_n\}} \) converges to \( \theta \) in \( \mathbb{PB}_\mathbb{F}(n^*) \) if for each \( j \) and all \( i \) sufficiently large, \( \rho_{n,i}(\theta_n - \theta) \to 0 \). Given \( \theta \in \mathbb{PB}_\mathbb{F}(n^*) \), we let \( D_\theta \) denote the element of \( \mathcal{S}_j^*(N) \) such that \( (D_\theta)^{-1} = \theta \). Theorem 3.3 may be rephrased as

**Theorem 3.3'.** The mapping \( \mathbb{PB}_\mathbb{F}(n^*) \times \mathcal{S}_j(N) \to \mathcal{S}_j(N) \) given by \( (\theta, f) \to D_\theta \) is jointly continuous.

Let \( \mathcal{S}_j(N/H, \chi) \) denote the space of all bi-invariant distributions \( D \) in \( \mathcal{S}_j(N/H, \chi) \) such that \( D \in \mathcal{S}_j(N/H, \chi) \) for each \( f \in \mathcal{S}_j(N/H, \chi) \), where \( D \in \mathcal{S}_j(N/H, \chi) \) is topologized so that \( D \to 0 \) in \( \mathcal{S}_j(N/H, \chi) \) if \( D \to f \) in \( \mathcal{S}_j(N/H, \chi) \).

**Theorem 3.9.** The mapping \( \theta \to D_\theta \) is a homeomorphism of \( \mathbb{PB}_\mathbb{F}(n^*) \) onto \( \mathcal{S}_j(N/H, \chi) \).

**Remark.** The fact that this mapping is also an algebra homomorphism is proved in Corollary 4.4.

**Proof.** It remains only to show that the Fourier transform of each bi-invariant Schwartz multiplier is in \( \mathbb{PB}_\mathbb{F}(n^*) \). We must first show that for \( D \in \mathcal{S}_j(N/Z, \chi) \), \( D \in \mathbb{PB}_\mathbb{F}(3^j + 1) \). For this, note that if \( D \in \mathcal{S}_j(N/H, \chi) \), then \( D \) is in \( \mathcal{S}_j(N/Z, \chi) \) for each \( f \in \mathcal{S}_j(N/H, \chi) \), where \( P_f : \mathcal{S}_j(N/H, \chi) \to \mathcal{S}_j(N/Z, \chi) \) is defined by

\[
P_f(x) = \int (x \exp(Y)e^{-2\pi \langle a, Y \rangle} dY,
\]
and $\chi_{+A}$ is the character defined on $Z = \text{exp}(g) = \text{exp}(\text{ad} h) \chi_{+A}(\text{exp}(Y + W)) = e^{2\pi i (\nu + \lambda, Y + W)}$. To see that $D_\nu$ is well defined, let $(\phi_\nu) \in \mathcal{S}(N/H, \chi_{+A})$ be an approximate identity. Then for $f \in \mathcal{S}(N/H, \chi_{+A})$, $D_\nu f = \lim_{\nu \to 0} (D_\nu \phi_\nu) * f$. Thus, if $F \in \mathcal{S}(N, \chi_{+A})$ and $F = P_\nu f = P_\nu g$, then

$$\langle D_\nu, (P_\nu f) * \rangle = P_\nu (D_\nu f)(0) = \lim_{\nu \to 0} \langle D_\nu \phi_\nu, P_\nu f(0) \rangle = \lim_{\nu \to 0} \langle D_\nu \phi_\nu, P_\nu g(0) \rangle.$$

To see that $D_\nu$ is continuous on $\mathcal{S}(N, \chi_{+A})$ it suffices to note that if $F_\nu \to F$ in $\mathcal{S}(N, \chi_{+A})$, then one can construct a sequence $f_\nu \to f$ in $\mathcal{S}(N/H, \chi_{+A})$ such that $F_\nu = P_\nu f_\nu$ and $F = P_\nu f$. It follows that $\langle D_\nu, F_\nu \rangle = P_\nu (D_\nu f_\nu)(0) = P_\nu (D_\nu f)(0) = \langle D_\nu, F \rangle$. This also shows that $D_\nu F_\nu \to D_\nu F$ in $\mathcal{S}(N, \chi_{+A})$.

We now show that if $D \in \mathcal{M}\mathcal{S}(N, \chi_{+A})$, $D \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. The proof is by induction on dim$(n)$. If dim$(n) = 2$, the result is trivial. Thus, assume dim$(n) \geq 3$.

Suppose first that dim$(\nu) = 1$. Pick $X, Y, Z, \eta_0$, and $n_1$ as in the beginning of the proof of Theorem 3.7. We denote by $(t, W)$ the group element $\exp(\text{ad} t W)$, where $t \in R$, $W \in \mathfrak{n}_1$. For $x \in \mathfrak{n}_2$, denote by $f^x$ the function defined on $N$ by $f^x(y) = f(\nu^{-1} x y)$. Note that $f^x(t, W) = f(t, W + s t x)$. Thus, using a partition of unity in the $t$-direction, one can show that if $f \in \mathcal{S}(N)$ such that $f(0, W) = 0$ for all $W \in \mathfrak{n}_1$, then $\langle D_\nu, f \rangle = 0$. Hence, $D = D_1 + D_2$, where $D_1 \in \mathcal{M}\mathcal{S}(N_1)$ ($N_1 = \exp(\eta_0)$) and $D_2 \in \mathcal{M}\mathcal{S}(N)$ with $\eta_0 = 0$. However, since $D_2 \in \mathcal{S}(N)$ for each $f \in \mathcal{S}(N)$, $D_2 = 0$. If $\chi$ is the center of $n_1$, the induction hypothesis yields that $D_1 \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. The Ad*-invariance of $D$ shows that $D \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. Suppose therefore that $\chi$ is not the center of $n_1$. Let $\tilde{\eta}_1$ be a subspace of $\mathfrak{n}_1$ such that $\tilde{\eta}_1 \subset \mathfrak{h}_1$ is the center of $\tilde{\eta}_1$. The mapping defined on $\tilde{\eta}_1$ by $Y \to D \nu f(x \exp(Y)) \in \mathcal{S}(\tilde{\eta}_1)$. Thus, by Fourier inversion,

$$D \nu f(x) = \int D \nu f(x \exp(y)) dy = \int D_\nu f(x \exp(y)) dy,$$

where $D_\nu \in \mathcal{S}(N_1, Z_1, \chi_{+A})$. By the induction assumption, $D_\nu \in \mathcal{P}\mathcal{B}_N(\lambda^{+} + \tilde{\eta}_1 \nu + \lambda)$. From the fact that $D$ is given by the function defined on $\lambda^{+} + \tilde{\eta}_1 \nu + \lambda - \tilde{\eta}_1 \nu + \lambda$, it follows that $D \nu f(x \exp(y)) \in \mathcal{S}(\tilde{\eta}_1 \nu^{+} + \lambda)$. Thus,

$$D \nu f(x) \in \mathcal{S}(\tilde{\eta}_1 \nu^{+} + \lambda) \subset \mathcal{S}(\tilde{\eta}_1 \nu^{+} + \lambda).$$

Suppose now that dim$(\nu) \geq 2$. Let $\Theta \in \mathcal{S}(N/Z, \lambda_{+})$ be the character defined on $Z = \exp(g) = \exp(h) \chi_{+}(\exp(Y + W)) = e^{2\pi i (\nu + \lambda, Y + W)}$. To see that $D_\nu$ is well defined, let $(\phi_\nu) \in \mathcal{S}(N/H, \chi_{+A})$ be an approximate identity. Then for $f \in \mathcal{S}(N/H, \chi_{+A})$, $D_\nu f = \lim_{\nu \to 0} (D_\nu \phi_\nu) * f$. Thus, if $F \in \mathcal{S}(N, \chi_{+A})$ and $F = P_\nu f = P_\nu g$, then

$$\langle D_\nu, (P_\nu f) * \rangle = P_\nu (D_\nu f)(0) = \lim_{\nu \to 0} \langle D_\nu \phi_\nu, P_\nu f(0) \rangle = \lim_{\nu \to 0} \langle D_\nu \phi_\nu, P_\nu g(0) \rangle.$$

By the previous argument, $D_\nu \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. Repeating the argument used in the paragraph above, one concludes that $D_\nu \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. Thus, dim$(n) = 2$, the result is trivial. Thus, assume dim$(n) \geq 3$.

Suppose first that dim$(\nu) = 1$. Pick $X, Y, Z, \eta_0$, and $n_1$ as in the beginning of the proof of Theorem 3.7. We denote by $(t, W)$ the group element $\exp(\text{ad} t W)$, where $t \in R$, $W \in \mathfrak{n}_1$. For $x \in \mathfrak{n}_2$, denote by $f^x$ the function defined on $N$ by $f^x(y) = f(\nu^{-1} x y)$. Note that $f^x(t, W) = f(t, W + s t x)$. Thus, using a partition of unity in the $t$-direction, one can show that if $f \in \mathcal{S}(N)$ such that $f(0, W) = 0$ for all $W \in \mathfrak{n}_1$, then $\langle D_\nu, f \rangle = 0$. Hence, $D = D_1 + D_2$, where $D_1 \in \mathcal{M}\mathcal{S}(N_1)$ ($N_1 = \exp(\eta_0)$) and $D_2 \in \mathcal{M}\mathcal{S}(N)$ with $\eta_0 = 0$. However, since $D_2 \in \mathcal{S}(N)$ for each $f \in \mathcal{S}(N)$, $D_2 = 0$. If $\chi$ is the center of $n_1$, the induction hypothesis yields that $D_1 \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. The Ad*-invariance of $D$ shows that $D \in \mathcal{P}\mathcal{B}_N(\nu^{+} + \lambda)$. Suppose therefore that $\chi$ is not the center of $n_1$. Let $\tilde{\eta}_1$ be a subspace of $\mathfrak{n}_1$ such that $\tilde{\eta}_1 \subset \mathfrak{h}_1$ is the center of $\tilde{\eta}_1$. The mapping defined on $\tilde{\eta}_1$ by $Y \to D \nu f(x \exp(Y)) \in \mathcal{S}(\tilde{\eta}_1)$. Thus, by Fourier inversion,

$$D \nu f(x) = \int D_\nu f(x \exp(y)) dy = \int D_\nu f(x \exp(y)) dy,$$

where $D_\nu \in \mathcal{M}\mathcal{S}(N_1, Z_1, \chi_{+A})$. By the induction assumption, $D_\nu \in \mathcal{P}\mathcal{B}_N(\lambda^{+} + \tilde{\eta}_1 \nu + \lambda)$ for each $x \in \mathfrak{n}_1$. Thus,

$$D \nu f(x) \in \mathcal{S}(\tilde{\eta}_1 \nu^{+} + \lambda) \subset \mathcal{S}(\tilde{\eta}_1 \nu^{+} + \lambda).$$

4. Representations and bi-invariant Schwartz multipliers. Given $\xi \in \mathfrak{n}_2$, a polarization of $\xi$ is a subalgebra $\mathfrak{m}$ of $\mathfrak{n}$ of maximum dimension with $[\mathfrak{m}, \mathfrak{m}] \subset \ker(\xi)$. (Note that a polarization will always contain $\mathfrak{z}$, the center of $\mathfrak{n}$.) Given such a polarization, a unitary character $\Psi_\xi$ is defined on $M = \exp(\mathfrak{m})$ by

$$\Psi_\xi(\exp(t \mathfrak{m})) = e^{-2\pi i \xi(t \mathfrak{m})}.$$

If $\mathfrak{m}$ is a subalgebra of $\mathfrak{n}$ that contains $\mathfrak{z}$, and $f \in \mathcal{S}(N)$, where $N' = \exp(\mathfrak{m})$, we define $Q_\xi(f)$ on $N'$ by

$$Q_\xi(f)(n') = \int f(n' \exp(y)) \Psi_\xi(\exp(y)) dy.$$
Theorem 4.2. Let \( m \) be a polarization of \( \xi \in \mathbb{N}^* \), and let \( D \in S^*(N) \) such that \( D \in PB^\mathbb{N}_m(n^*) \). Then for each \( f \in \mathcal{S}(N) \),
\[
Q_\xi(D \ast f) = \hat{D}(\xi) Q_\xi(f).
\]

Proof. The proof is by induction on \( \dim(n) \). If \( n \) is Abelian, \( m = n \), and for \( f \in \mathcal{S}(N) \), \( Q_\xi(f) = \check{f}(\xi) \). Thus, \( Q_\xi(D \ast f) = (D \ast f)(\xi) = \hat{D}(\xi) \check{f}(\xi) = \hat{D}(\xi) Q_\xi(f) \).

Assume that \( \dim(j) = 1 < \dim(n) \). Let \( n_1 \) be a Kirillov subalgebra of \( n \), i.e., there exist elements \( X_0, Y_0, Z_0 \in \mathfrak{g} \) such that \( \mathfrak{z} \) is spanned by \( Z_0 \) which is equal to \([X_0, Y_0]\), \( n_1 \) is the centralizer of \( Y_0 \), and \( n = RX_0 @ n_1 \). Let \( X_0^* \) be the element of \( n^* \) that is dual to \( X_0 \) (with respect to \( n_1 \)). Then \( Ad^*([\exp(Y_0)](\eta)) = \eta + i(\langle \eta, Z_0 \rangle X_0^*) \). Thus, if \( \langle \eta, Z_0 \rangle \neq 0 \), \( D(\eta + i X_0^*) = \hat{D}(\eta) \) for all \( \eta \in n^* \).

Let \( n_1^* = (RX_0)^*, n_2^* = RX_0^*, \) and \( n_0 = RX_0 \). Then, for \( m \in M \) and \( n \in N \),
\[
Q_\xi(D \ast f)(m) = \int \hat{D}(\eta) \int \int \int (\exp(Y) \exp(X) e^{-2\pi i (\langle X, Y \rangle + \langle \xi, Y \rangle)} dYdXd\eta,
\]
\[
= \int \hat{D}(\eta_0 + \eta_1) \int \int \int (\exp(Y) \exp(X_0 + X_1)) \times e^{-2\pi i (\langle (\eta_0, X_0) + (\eta_1, X_1) \rangle + \langle X_0, Y \rangle)} dYdX_0 dX_1 d\eta_0 d\eta_1,
\]
\[
= \int \hat{D}(\eta_1) \int \int \int (\exp(Y) \exp(X_1)) \times e^{-2\pi i (\langle (\eta_1, X_1) + (\xi, X_1) \rangle + \langle X_1, Y \rangle)} dYdX_1 d\eta_1,
\]
\[
= Q_\xi(D \ast (l_{-1} f)) (m) = Q_\xi(f)(m),
\]
where \( D \) is the element of \( S^*(N_1) \) whose Fourier transform agrees with \( D \) on \( n_1^* \), and \( (l_{-1} f) \) is the restriction of \( f \) to \( N_1 \). Since \( D \) is constant on \( n_1^* \) cosets, one may assume that \( \xi \in n_1^* \). Also note that since \( M \subset N_1 \), \( Q_\xi(l_{-1} f)(m) = Q_\xi(f)(m) \). Thus, by the induction assumption, \( Q_\xi(D \ast f) = \hat{D}(\xi) Q_\xi(f) \).

Suppose now that \( \dim(j) \geq 2 \). Given \( \xi \in \mathbb{N}^* \), pick a basis \( \{X_0, X_1, \ldots, X_n\} \) of \( n \) so that \( \{X_1, X_0\} \subset \mathcal{A} \) and \( \langle \xi, X_0 \rangle = 0 \). Let \( n_1 = span \{X_1, \ldots, X_n\}, n_0 = RX_0 \), and let \( n_1 \) be a subspace of \( m \) so that \( m = n_1 @ n_0 \). Let \( X_0^*, \ldots, X_n^* \) be the dual basis in \( n^* \), with \( X_0^* \) contained in \( n^* \), and \( X_1^*, \ldots, X_n^* \). Finally, assume that for fixed \( n_1 \in n_1^* \), \( n_0 \rightarrow D(n_1 + n_0) \in \mathcal{S}(n_0) \).

For \( U_1 \in n_1, U_0 \in n_0 \),
\[
(4.3) \quad Q_\xi(D \ast f)(\exp(U_1 + U_0)) = \int \hat{D}(\eta_1 + \eta_0) \int \int \int \int (\exp(U_1 + U_0) \exp(V_1 + V_0)) \times \exp(T_1 + T_0) e^{-2\pi i (\langle \eta_1, V_1 \rangle + \langle \eta_0, Y_0 \rangle + \langle X, Y \rangle)} dT_0 dT_1 dV_0 dV_1 d\eta_0 d\eta_1.
\]

Finally, the assumption that \( \eta_0 \rightarrow D(n_1 + n_0) \in \mathcal{S}(n_0) \) was required for the Fourier inversion used in the third equality in (4.3). For more general \( D \), approximate \( D \) in \( PB^\mathbb{N}_m(n^*) \) by \( [\delta_{\eta_0}] \subset PB^\mathbb{N}_m(n^*) \) with support of \( \delta_{\eta_0} \) contained in the slabs \( \{\eta: \|\eta, X_0\| < n\} \). Then, by Theorem 3.8,
\[
Q_\xi(D \ast f) = \lim_{n \to \infty} Q_\xi(D_{\eta_0} \ast f) = \lim_{n \to \infty} \theta(\xi) Q_\xi(f) = \hat{D}(\xi) Q_\xi(f).
\]

Let \( \xi \in \mathbb{N}^* \), and let \( \tau_n \) be the irreducible unitary representation of \( N \) corresponding to the Ad^*-orbit of \( \xi \). \( \tau_n \) can be realized as left translation on \( L^2(N/M, \mathcal{P}_n) \), the space of all measurable functions \( f \) defined on \( N \) with \( F(nm) = \mathcal{P}_n(m) F(n) \) for \( n \in N, m \in M \), and having
\[
\int |F(n)|^2 dm < \infty,
\]
where \( m = \log(M) \) is a polarization of \( \xi \) and \( \mathcal{P}_\xi \) is the character defined in (4.1).

Corollary 4.3. Let \( \theta \in PB^\mathbb{N}_m(n^*) \) and \( f \in \mathcal{S}(N) \). For \( \xi \in \mathbb{N}^* \),
\[
\tau_n(D_{\ast} f) = \theta(\xi) \pi_\xi(f).
\]

Proof. First note that \( Q_\xi(S^*(N)) \) is a dense subspace of \( L^2(N/M, \mathcal{P}_\xi) \). Given \( f, g \in \mathcal{S}(N), \theta \in PB^\mathbb{N}_m(n^*) \),
\[
\pi_\xi(D_{\ast} f) Q_\xi(g) = Q_\xi(D_{\ast} f \ast g) = \theta(\xi) Q_\xi(f \ast g) = \theta(\xi) \pi_\xi(f) Q_\xi(g).
\]
Corollary 4.4. The mapping $\theta \to D_\theta$ is an algebra homomorphism from $\mathrm{PB}_\mathbb{F}(n^*)$ to $\mathcal{S}(N)$.

Proof. Given $0, \varphi \in \mathrm{PB}_\mathbb{F}(n^*)$, $f \in \mathcal{S}(N)$, and $\zeta \in n^*$,

$$
\pi_\zeta(D_{\varphi} \ast f) = \theta(\zeta) \varphi (\zeta) \pi_\xi(f) = \theta(\zeta)\pi_\xi(D_\varphi \ast f) = \pi_\xi(D_\varphi \ast D_\varphi \ast f).
$$

Thus, $D_{\varphi} \ast f = D_\varphi \ast D_\varphi \ast f$. Since $f$ was arbitrary in $\mathcal{S}(N)$, $D_{\varphi} E = E D_{\varphi}$.

Linearity is obvious.

By the Plancherel Theorem for $N$, there is a measure on $n^* / \text{Ad}^*$, $\Omega$, such that for $f \in \mathcal{S}(N)$,

$$
f(\xi) = \int_{n^* / \text{Ad}^*} \text{Tr}(\pi_\xi(f)) d\Omega(\xi).
$$

It follows that

$$
\|f\|_{L^2}^2 = \int_{n^* / \text{Ad}^*} \text{Tr}(\pi_\xi(f) \pi_\xi(f^*)) d\Omega(\xi).
$$

Thus, we have

Corollary 4.5. If $\theta \in \mathrm{PB}_\mathbb{F}(n^*)$ is bounded, then $\|D_{\theta} \ast f\| \leq \|\theta\|_{\infty} \|f\|_{L^2}$, i.e. $D_{\theta}$ extends to a bounded operator on $L^2(N)$.

5. Applications. The first application concerns eigenfunction expansion for certain left-invariant differential operators on $N$. For this, we require that $N$ be stratified, i.e. $n$ has a direct sum decomposition, $n = n_1 \oplus \ldots \oplus n_k$, such that $[n_i, n_j] \subset n_{i+j}$, $1 \leq i, j \leq k$, and such that $n_1$ generates $n$. A one parameter family of dilations $\delta_{t_1, \ldots, t_k}$ is defined on $n$ by setting $\delta_t X = tX$ for $X \in n_i$, and extending linearly. Let $Q = k(k+1)/2$. For a function $f$ defined on $n^*$ set

$$
\hat{f}(\exp(X)) = t^{-Q} f(\exp(\delta_{-t} X)),
$$

and for a function $\theta$ defined on $n^*$, set $\theta(\xi) = \theta(\delta_t^* \xi)$, where $\langle \delta_t^* \xi, X \rangle = \langle \xi, \delta_t X \rangle$.

A left-invariant differential operator $L$ (we drop the distinction between $L$ and $\theta L$) is said to be a Rockland operator (of degree $\gamma$) if $L(f \circ \delta_t) = t^\gamma Lf \circ \delta_t$, and if for every nontrivial irreducible unitary representation $\pi$ of $N$, $\pi(L)$ is injective on the space of $C^\infty$-vectors. In virtue of Helffer and Nourrigat [HN], a positive Rockland operator $L$ is hypoelliptic, and thus is essentially selfadjoint on $C^\infty_c(N)$ in $L^2(N)$ by a theorem of Nelson and Stein [NS]. Thus, the closure of $-L$ is the infinitesimal generator of a convolution semigroup $\{P_t\}_{t > 0}$. Folland and Stein [FS] have shown that $\{P_t\} \subset \mathcal{S}(N)$.

Let $\mathcal{A}$ denote the closed subalgebra of $L^1(N)$ spanned by $\{P_t\}_{t > 0}$. It follows from the homogeneity of $L$ that $\mathcal{A}$ is closed under the mapping $f \mapsto f$. Consequently, the Gelfand space of $\mathcal{A}$ can be identified with $R^*$ in such a way that given a spectral resolution of $L$,

$$
L f = \int_0^\infty \lambda dE(\lambda) f, \quad f \in C^\infty_c(N),
$$

and $k \in \mathcal{A}$, one has

$$
k \ast f = \int_0^\infty \lambda dE(\lambda) f, \quad f \in L^2(N),
$$

where $\hat{\lambda}$ is the Gelfand transform of $k$. Also, if $L$ is homogeneous of degree $\gamma$, then $(k \ast \hat{\lambda}) = \hat{\lambda}(t^\gamma \lambda)$. It easily follows that if $\int k(x) dx = 1$, then for $f \in L^p(N)$,

$$
I^p \lim_{t \to \infty} k_t \ast f = f.
$$

Thus, if $k \in \mathcal{A}$ and $\int k(x) dx = 1$,

$$
f = I^p \lim_{t \to \infty} \int_0^\infty \lambda dE(\lambda) f, \quad f \in L^p(N).
$$

In [H], Hulanicki gave conditions on a function $K$ defined on $R^+$ that are sufficient to imply that $K = k$ for some $k \in \mathcal{A}$.

Suppose now that $N$ contains a discrete cocompact subgroup $\Gamma$, then $L$ is naturally defined as a differential operator on $\Gamma \backslash N$. Since $\{P_t\} \subset \mathcal{S}(N)$ ($L^1(N)$ is sufficient), it follows that the spectrum of $L$, $\sigma(L)$, is a discrete subset of $R^*$ with finite multiplicities. One looks for summability kernels $K$, defined on $R^*$, so that for $F \in L^1(\Gamma \backslash N)$, $1 \leq p \leq \infty$,

$$
F = \lim_{t \to \infty} \sum_{\lambda \in \sigma(L)} K(\lambda) F_\lambda,
$$

where $F_\lambda$ is the projection of $F$ onto the eigensubspace corresponding to $\lambda$.

By the Kirillov theory, the irreducible unitary representations of $N$, $\tilde{N}$, can be identified with the $\text{Ad}^*$-orbits in $n^*$. Let $\pi_\lambda$ denote the representation corresponding to the orbit of $\xi \in n^*$. There is a discrete subspace $(\Gamma \backslash N)^* \subset \tilde{N}$ such that

$$
I^2(\Gamma \backslash N) \cong \bigoplus_{\pi_\lambda} \mathcal{H}_\lambda,
$$

where right translation on $\mathcal{H}_\lambda$ is a finite multiple of $\pi_\lambda$ (cf. [R]). Thus, $\sigma(L)$ is the union of $\sigma(\pi_\lambda(L))$, $\pi_\lambda \in (\Gamma \backslash N)^*$, not counting multiplicities. Therefore, there is a natural identification of $\sigma(L)$ with a subset of $n^* \times R^*$, and so one looks for summability kernels on this set.

Let $\theta \in \mathrm{PB}_\mathbb{F}(n^*)$ with $\theta(0) = 1$. It follows from Theorem 3.8' that there is a function $s : R^* \to R^*$ such that

$$
\lim_{t \to 0} \|D_{\theta(t)} \ast k_t - k_t\| = 0.
$$
Combining this with Corollary 4.3 gives

**Theorem 5.1.** If \( \theta \in \text{PB}_\text{c}^*(\mathfrak{n}^*) \) with \( \theta(0) = 1 \), and if \( k \in \mathcal{A} \cap \mathcal{U} \mathcal{V} \), then there is an \( s : R^* \to R^* \) such that for \( F \in \text{IP}(\Gamma \setminus N) \), \( 1 \leq p \leq \infty \),

\[
F = I_p - \lim_{t \to 0} \sum \theta(\delta_{x_0}^s \xi) \mathcal{K}(\lambda) F_x,
\]

where the sum is over \( \sigma(L) \) in \( \mathfrak{n}^* \times R^* \).

A second application concerns local solvability. A left-invariant differential operator \( L \) on \( N \) is said to be locally solvable if there is an open set \( U \subset N \) such that \( C_c^\infty(U) \subset L(C_c^\infty(U)) \), i.e. if for each \( f \in C_c^\infty(U) \) there is a \( u \in C_c^\infty(U) \) such that \( Lu = f \).

Let \( \sigma(\xi) \) denote the \( \text{Ad}^* \)-orbit in \( \mathfrak{n}^* \) that contains \( \xi \), and having fixed a norm on \( \mathfrak{n}^* \), set \( ||\sigma(\xi)|| = \inf \{ ||\xi'\sigma(\xi)|| : \xi' \in \sigma(\xi) \} \). There is a linear subspace \( V \subset \mathfrak{n}^* \) and a Zariski open subset \( V_0 \subset V \) such that the elements in \( V_0 \) parametrize an open dense set of orbits in \( \mathfrak{n}^* \). Representations corresponding to elements of \( V_0 \) are said to be in general position.

The Plancherel measure \( \Omega \) is supported on \( V_0 \), and in fact is absolutely continuous with respect to the Lebesgue measure on \( V \) with density given by a rational function.

**Theorem 5.2.** Let \( L \) be a left-invariant differential operator on \( N \).

(i) Suppose that for each \( \xi \in \sigma(L) \), \( \pi_\xi(L) \) has a bounded right inverse \( A_\xi \), that \( \xi \to A_\xi \) is measurable, and that the norm of \( A_\xi \) is bounded by \( \text{polynomial in } ||\sigma(\xi)|| \).

(ii) Suppose that \( \mathcal{N} \) contains a discrete cocompact subgroup \( \Gamma \), for each \( \xi \in \sigma(L) \), \( \pi_\xi(L) \) has a bounded right inverse \( A_\xi \), and that the norm of \( A_\xi \) is bounded by a polynomial in \( ||\sigma(\xi)|| \).

If either (i) or (ii) holds then \( L \) is locally solvable.

**Remarks.** The theorem with condition (i) is (essentially) a theorem due to Corwin [C], and our proof is an adaptation of his proof. The theorem with condition (ii) was proved by Corwin and Greenleaf [CG] with the additional assumption that all the representations in general position were induced from a common normal subgroup.

**Proof.** (i) Let \( Z \) be a bi-invariant differential operator on \( N \) such that \( \pi_\xi(Z) = 0 \) for each representation \( \pi_\xi \) not in general position. Since \( Z \) has a fundamental solution (cf. [R]), it suffices to show that for some \( U \) and each \( f \in C_c^\infty(U) \) there is a \( g \in C_c^\infty(U) \) satisfying \( Lg = Zf \).

Given \( f \), by [DM] there exist \( g_1, \ldots, k \in C_c^\infty(U) \), \( 1 = 1, \ldots, k \), such that

\[
f = \sum_{i=1}^{k} g_i \ast h_i.
\]

Let \( j \) be a positive integer sufficiently large so that

\[
\max_{\xi} \|\A_{\xi}\|/(1 + |\sigma(\xi)|)^j \, d\xi < \infty,
\]

and let \( \phi \) be a smooth function defined on the complex numbers with values in \([0, 1]\) such that \( \phi(z) = 0 \) if \( |z| \leq 1/2 \), and \( \phi(z) = 1 \) if \( |z| \geq 1 \). Define \( \theta \) on \( V \) by

\[
\theta(\xi) = \phi(\pi_\xi(Z)(1 + |\sigma(\xi)|)^j/(1 + |\sigma(\xi)|)^j)^j + 1.
\]

(Note that since \( Z \) is bi-invariant, \( \xi \to \pi_\xi(Z) \) is an \( \text{Ad}^* \)-invariant polynomial on \( \mathfrak{n}^* \).) Then \( \theta \) and \( f \) have unique extensions to elements of \( \text{PB}_\text{c}^*(\mathfrak{n}^*) \). There exist elements \( u_i \in L^2(N) \) such that for \( v \in L^2(N) \),

\[
\langle u_i, v \rangle = \int_{\mathcal{N}} A_G(Q_{\xi}(D_{1,0} \ast Z_h)_{V_0}) \, Q_{\xi} v \rangle \, dv(\xi),
\]

where \( Q_{\xi} \) is defined before Theorem 4.2. It follows that \( Lu_i = Zh \). Thus, if we let

\[
u = \sum_{i=1}^{k} (D_{1,0} \ast g_i) \ast u_i,
\]

then

\[
Lu = \sum_{i=1}^{k} (D_{1,0} \ast g_i) \ast Lu_i = \sum_{i=1}^{k} (D_{1,0} \ast g_i) \ast (D_{1,0} \ast Z_h) = \sum_{i=1}^{k} g_i \ast Zh = Zf.
\]

The proof of (ii) is similar. For \( f \in \mathcal{F}(\mathcal{N}) \) we define \( \tau f \in L^2(\Gamma \setminus N) \) by

\[
\tau f = \sum_{\gamma \in \mathcal{F}} f(\gamma)
\]

Define \( D_\gamma : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{N}) \) by \( \langle D_\gamma(D_{1,0}), f \rangle = \langle D_{1,0} \ast f \rangle \) for \( D \in \mathcal{F}(\mathcal{N}) \) and \( f \in \mathcal{F}(\mathcal{N}) \). Then, for \( f, g \in \mathcal{F}(\mathcal{N}) \),

\[
\langle D_\gamma(f), g \rangle = \langle f, D_\gamma(g) \rangle = \langle \sum_{\gamma} f(\gamma)D_{1,0} \ast g(\gamma^{-1}) \rangle dn
\]

\[
= \sum_{\gamma \in \mathcal{F}} \langle \gamma f, D_{1,0} \ast g \rangle \quad (\text{where } \gamma f(n) = f(\gamma n))
\]

\[
= \sum_{\gamma \in \mathcal{F}} \langle \gamma(D_{1,0} \ast f), g \rangle = \langle \gamma(D_{1,0} \ast f), g \rangle.
\]

Thus,

\[
D_\gamma(f) = \tau (D_{1,0} \ast f).
\]

It easily follows from (5.2) and (5.3) that for \( f, g \in \mathcal{F}(\mathcal{N}) \),

\[
(D_{1,0} \ast f) \ast \tau g = f \ast (D_{1,0} \ast g).
\]
Let $P_\xi$ denote the orthogonal projection from $L^2(\Gamma N)$ to $\mathcal{H}_\xi$. In [1] we proved that

$$D_\xi(\sigma f) = \sum_{\nu \in (\Gamma N)^0} \theta(\xi) P_\xi(\sigma f).$$

In particular then, $P_\xi(D_\xi(\sigma f)) = \theta(\xi) P_\xi(\sigma f)$. Let $f$ be a bi-invariant differential operator on $N$ such that $\pi_\xi(Z) = 0$ for each representation $\pi$ not in general position. As noted in [CG], to prove the theorem, it suffices to show that for some $U$ and each $f \in C_c^\infty(U)$ there is a $g \in C_c^\infty(U)$ satisfying $Lg = Zf$. Let $U = U^{-1}$ be a neighborhood of the identity in $N$ such that $U^3 \cap \Gamma = \{e\}$, and let $f \in C_c^\infty(U)$. By [DM] there exist $g_i, h_i \in C_c^\infty(U)$, $i = 1, \ldots, k$, such that

$$f = \sum_{i=1}^k g_i \ast h_i.$$

Let $j$ be a positive integer sufficiently large so that

$$\sum_{\xi \in (\Gamma N)^0, \nu \neq 0} \max \{ 1, |A_\xi| \} (1 + |\sigma(\xi)|)^j < \infty.$$

Define $\varphi$ and $\theta$ as before, and set

$$\nu_i = \sum_{\xi \in (\Gamma N)^0, \nu \neq 0} A_\xi P_\xi(D_{1/i}(\tau(Z\xi))).$$

By splitting the sum into two parts, one containing the terms for which $|\sigma(\xi)| < (1 + |\sigma(\xi)|)^{-j}$, and using (5.5), one can show that $u_i \in L^2(\Gamma N)$. Using the same argument as the one used in [CG], one can show that $Lu_i = \tau(D_{1/i} * Z h_i)$. Thus, if we let

$$u = \sum_{i=1}^k (D_\nu \ast g_i) \ast u_i,$$

then, using (5.4) and the fact that $D_\nu \ast D_{1/i} = I$,

$$Lu = \sum_{i=1}^k (D_\nu \ast g_i) \ast Lu_i = \sum_{i=1}^k D_\nu \ast g_i \ast \tau(D_{1/i} \ast Z h_i) = Z(\sum_{i=1}^k g_i \ast \tau(h_i)).$$

Since $\text{supp}(g_i \ast \tau(h_i)) \cap U = \text{supp}(g_i \ast h_i) \cap U$, $Lu = Zf$ on $U$.

References
