[22] D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. 37 (1983), 269–301.
[23] —, *Fréchetrüume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.
[24] —, *Some results on continuous linear maps between Fréchet spaces*, in: K.-D. Bierstedt and B. Fuchssteiner (eds.), Functional Analysis: Surveys and Recent Results III, North-Holland Math. Stud. 90, 1984, 349–381.
[25] —, *On the functors* $\mathrm{Ext}^1(E, F)$ *for Fréchet spaces*, Studia Math. 85 (1987), 163–197.
[26] —, *Tame spaces and power series spaces*, Math. Z. 196 (1987), 523–536.
[27] M. J. Wagner, *Quotientenräume von stabilen Potenzreihenräumen endlichen Typs*, Manuscripta Math. 31 (1980), 97–109.

FACHBEREICH MATHEMATIK
BERGISCHE UNIVERSITÄT
GESAMTHOCHSCHULE WUPPERTAL
Gaußstr. 20, D-5600 Wuppertal 1, F.R.G.

Received March 2, 1987 (2283)

STUDIA MATHEMATICA, T. XCII. (1989)

C^0-Scalar operators on cyclic spaces

by

RALPH deLAUBENFELS (Athens, O.)

Abstract. We consider (possibly unbounded) linear operators A on a Banach space X for which there exists x_0 such that $X = \overline{\operatorname{span}}\{A^n x_0 \mid n = 0, 1, 2, \ldots\}$. We show that A is C^0-scalar with real spectrum if and only if there exists a certain type of Banach lattice Y of complex-valued functions on the real line and a homeomorphism U from X onto Y such that $(UAU^{-1}f)(t) = tf(t)$, with the domain of UAU^{-1} equal to $\{f \text{ in } Y \mid t \to tf(t) \text{ is in } Y\}$.

When the quasi-analytic growth condition $\sum_{n=0}^{\infty}(\sup_{k\leqslant n}\|A^k x_0\|^{-1/k}) = \infty$ is satisfied, and the domain of A equals span $\{A^n x_0 \mid n = 0, 1, \ldots\}$, we show that $\bar{A}$ is C^0-scalar with real spectrum if and only if there exists an equivalent norm with respect to which X is a Banach lattice, with

$$(X^*)^+ = \{\varphi \text{ in } X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ is positive-definite}\}.$$

Similar results for C^0-scalar operators with nonnegative spectrum are given.

Introduction. Scalar-type spectral operators generalize, to an arbitrary Banach space, selfadjoint operators on a Hilbert space. A (possibly unbounded) linear operator A is *scalar* if there exists a projection-valued measure E such that

$$Ax = \lim_{n\to\infty} \int_{-n}^{n} t\, d(E(t)x),$$

with the domain of A equal to the set of all x for which that limit exists. The spectral theorem asserts that a selfadjoint operator is scalar.

Another form of the spectral theorem asserts that A is selfadjoint if and only if it is unitarily equivalent to a multiplication operator on an L^2-space. In particular, if the Hilbert space H is *cyclic*, i.e. equals the closure of the span of $\{A^n x_0 \mid n = 0, 1, 2, \ldots\}$, for some x_0, then there exists a Borel measure m and unitary U from H onto $L^2(\boldsymbol{R}, m)$ such that

$$(UAU^{-1}f)(t) = tf(t).$$

We prove a similar result for scalar operators on a cyclic Banach space. More generally, we consider C^0-*scalar* operators (see Definition 1). These are operators that have a functional calculus defined for continuous functions vanishing at infinity. Scalar operators have a functional calculus defined for bounded Borel measurable functions. $(Af)(t) \equiv tf(t)$, on $C[0, 1]$, is an example of a C^0-scalar operator that is not scalar.

We show that the operator A on the cyclic Banach space X is C^0-scalar if and only if it is homeomorphically equivalent to multiplication by $g(t) = t$ on a certain type of Banach lattice of complex-valued functions on the real line (see Definition 4), Y. That is, there exists a homeomorphism U from X onto Y such that

$$(UAU^{-1} f)(t) = tf(t)$$

(see Theorem 5).

In a previous paper ([4]), we showed that when the sequence $\{\varphi(A^n x)\}_{n=0}^{\infty}$ was positive-definite (see Definition 6), for sufficiently many φ in X^* and x in X, then $\bar{A}$ was C^0-scalar. In this paper, we show that the converse is also true, on a cyclic space, i.e. when A is C^0-scalar, then a large supply of these positive-definite sequences does exist, enough to form the positive cone of a Banach lattice (Theorem 8). Theorem 11 gives necessary and sufficient conditions, in terms of these sequences, for $\bar{A}$ to be C^0-scalar, when the domain of A equals span $\{A^n x_0 \mid n = 0, 1, 2, \ldots\}$, for some x_0.

Similar results for C^0-scalar operators with nonnegative spectrum are in Theorems 9 and 14.

Corollaries 12 and 15 characterize bounded C^0-scalar operators with real or nonnegative spectrum, respectively, on cyclic spaces.

A characterization of strongly closed operator algebras generated by a σ-complete Boolean algebra of projections on a cyclic Banach space appears in [1]. This is also treated in [1]. This characterizes scalar operators on cyclic spaces, by using their projection-valued measure. In this paper, dealing with C^0-scalar operators, we may have no projections to work with.

Some recent results on unbounded scalar operators appear in [4], [7], and [8]. Basic material on scalar operators may be found in [5] and [6]. Generalized scalar operators, which include C^n-scalar operators, are covered in [3] and [9]. A reference for the classical analysis of the moment problem, used in Theorems 9 and 10, is [12]; some of this is stated in Definition 6. A reference for Banach lattices is [11]. We will also be using basic facts about semigroups of operators and their generators; a reference for this is [10]. We will write "e^{tA}" for the semigroup generated by A. Reference [4] contains some introductory material for all these topics.

All operators are linear, on a Banach space X. By "group" ("semigroup") we shall mean "strongly continuous group" ("...semigroup"). X^* is the dual of X, $B(X, Y)$ is all bounded linear operators from X into Y, $B(X)$ is $B(X, X)$. When A is an operator on X, $D(A)$ will be the domain of A. $\bar{A}$ will be the closure of the operator A. When X is a Banach lattice, X^+ will be the positive cone.

DEFINITION 1. If $T \in B(X)$, and D is a bounded subset of the complex plane, then T is *C^0-scalar on D* if there exists a continuous algebra homomorphism Λ from $C(D)$ into $B(X)$ such that $\Lambda(f_0) = I$, $\Lambda(f_1) = T$, where $f_0(z) \equiv 1$, $f_1(z) \equiv z$.

The (possibly unbounded) operator A is *C^0-scalar on* $[0, \infty)$ if A generates a uniformly bounded semigroup $\{e^{-tA}\}_{t \geq 0}$, and $(1+A)^{-1}$ is C^0-scalar on $[0, 1]$.

The operator A is *C^0-scalar on $\boldsymbol{R}$* if iA generates a uniformly bounded group $\{e^{itA}\}_{t \in \boldsymbol{R}}$, and $(1-iA)(1+iA)^{-1}$ is C^0-scalar on the unit circle.

The operator A is C^0-scalar on $\boldsymbol{R}$ if and only if there exists a continuous algebra homomorphism Λ from $C_l(\boldsymbol{R}) \equiv \{\text{continuous } f\colon \boldsymbol{R} \to \boldsymbol{C} \mid \lim_{x \to \pm\infty} f(x) \text{ exists}\}$ into $B(X)$ such that $\Lambda(f_0) = I$, $\Lambda(g) = (1-iA)(1+iA)^{-1}$, where $f_0(t) \equiv 1$, $g(t) \equiv (1-it)(1+it)^{-1}$.

The analogous assertion holds for being C^0-scalar on $[0, \infty)$.

Note that this map $f \to f(A) \equiv \Lambda f$ extends to a continuous algebra homomorphism from the set of all bounded Borel measurable functions, with the supremum norm, into $B(X, X^{**})$, by

$$[f(A)x]\varphi \equiv \int f(t)\, dE_{\varphi,x}(t),$$

where $E_{\varphi,x}$ is as in Lemma 3 below.

DEFINITION 2. $D(x_0, A) \equiv \text{span}\{A^n x_0 \mid n = 0, 1, 2, \ldots\}$. X is *cyclic* if $X = \overline{D(x_0, A)}$, for some x_0.

LEMMA 3. *Suppose A is C^0-scalar on $\boldsymbol{R}$ and, for φ in X^* and x in X, $E_{\varphi,x}$ is the unique complex-valued Borel measure such that*

$$\varphi(f(A)x) = \int_{\boldsymbol{R}} f(t)\, dE_{\varphi,x}(t), \quad \text{for all } f \in C_l(\boldsymbol{R}).$$

Then

(a) $dE_{\varphi,Ax}(t) = t\, dE_{\varphi,x}(t)$, *for all x in $D(A)$ and φ in X^*.*

(b) $E_{\varphi,x} \ll E_{\varphi,x_0}$, *for all φ in X^* and x in* $\overline{D(x_0, A)}$ (see Definition 2).

(c) $\dfrac{dE_{\psi,x}}{dE_{\psi,x_0}} = \dfrac{dE_{\varphi,x}}{dE_{\varphi,x_0}}$, *for all φ, ψ in X^* and x in* $\overline{D(x_0, A)}$.

Proof. (a) It is clear from the uniqueness of $E_{\varphi,x}$ that, for f in $C_l(\boldsymbol{R})$, $dE_{\varphi,f(A)x}(t) = f(t)\, dE_{\varphi,x}(t)$, for all φ in X^* and x in X. Thus, for any x in $D(A)$, since $x = (i+A)^{-1}((i+A)x)$, we have

$$dE_{\varphi,x}(t) = (i+t)^{-1}\, dE_{\varphi,(i+A)x}(t),$$

which gives (a).

(b) Suppose $E_{\varphi,x_0}(B) = 0$. Define θ in X^* by $\theta x \equiv E_{\varphi,x}(B)$. By (a), $\theta \equiv 0$ on the dense set $D(x_0, A)$, hence, since θ is bounded, $\theta \equiv 0$ on X, as desired.

(c) For fixed ψ, φ, consider θ in X^* defined by

$$\theta x \equiv \frac{dE_{\psi,x}}{dE_{\psi,x_0}} - \frac{dE_{\varphi,x}}{dE_{\varphi,x_0}}.$$

By (a), $\theta \equiv 0$ on the dense set $D(x_0, A)$, hence $\theta = 0$ on X, as desired.

DEFINITION 4. We will say that a Banach lattice Y of complex-valued Borel measurable functions on $\boldsymbol{R}$ is a *C^0-lattice on $\boldsymbol{R}$* if the following hold:

(1) $Y^+ = \{f \text{ in } Y \mid f(t) \geqslant 0 \text{ for all } t \text{ in } \boldsymbol{R}\}$.

(2) $fg \in Y$ whenever $g \in Y$, $f \in C_l(\boldsymbol{R})$ (see Definition 1).

THEOREM 5. *Suppose X is cyclic. Then A is C^0-scalar on $\boldsymbol{R}$ if and only if there exists Y, a C^0-lattice on $\boldsymbol{R}$, and a homeomorphism U from X onto Y such that*

$$U(D(A)) = \{f \in Y \mid t \to tf(t) \text{ is in } Y\}, \quad \text{with}$$

$$(UAU^{-1}f)(t) = tf(t).$$

Proof. Suppose $(Bf)(t) \equiv tf(t)$, on Y, a C^0-lattice on $\boldsymbol{R}$, with the domain given in the statement of the theorem. For $\lambda > 0$, $(\lambda \pm iB)^{-1}$ exists given by

$$(\lambda \pm iB)^{-1} f(t) = (\lambda \pm it)^{-1} f(t).$$

Thus $|(\lambda \pm iB)^{-1} f(t)| \leqslant |\lambda|^{-1} |f(t)|$. Since Y is a Banach lattice, this implies that

$$\|(\lambda \pm iB)^{-1} f\| \leqslant |\lambda|^{-1} \|f\|,$$

for all f in Y, so that $\|(\lambda \pm iB)^{-1}\| \leqslant |\lambda|^{-1}$, for all $\lambda > 0$. By the Hille–Yosida theorem (see [10]), this implies that iB generates a one-parameter group, given by $(e^{isB} f)(t) = e^{ist} f(t)$.

Define an algebra homomorphism $f \to f(B)$ from $C_l(\boldsymbol{R})$ into $B(Y)$ (see Definition 1) by $f(B)g \equiv fg$. Since Y is a Banach lattice,

$$\|fg\| \leqslant \big\| \|f\|_\infty g \big\| = \|f\|_\infty \|g\|,$$

so that $\|f(B)\| \leqslant \|f\|_\infty$, i.e. the algebra homomorphism is continuous, as desired.

Conversely, suppose $X = \overline{D(x_0, A)}$ (see Definition 2) and A is C^0-scalar on $\boldsymbol{R}$. By Lemma 3(b),(c), for any x in X, there exists a Borel measurable function Ux on $\boldsymbol{R}$ such that

$$dE_{\varphi,x}/dE_{\varphi,x_0} = Ux, \quad \text{for all } \varphi \text{ in } X^*.$$

Let $Y \equiv \{Ux \mid x \in X\}$. Define an ordering on Y by $Y^+ \equiv \{f \in Y \mid f(t) \geqslant 0$ for all $t\}$. Define a norm on Y by

$$\|Ux\| \equiv \sup\{\|z\| \mid z \in X,\ |Uz(t)| \leqslant |Ux(t)|, \text{ for all } t\}.$$

Since $|f|(t) = |f(t)|$, for all $f \in Y$ and $t \in \boldsymbol{R}$, this is clearly a Riesz norm on Y.

To see that U is a homeomorphism, suppose $|Uz(t)| \leqslant |Ux(t)|$, for all t. Let

$$g(t) = \begin{cases} 0 & \text{if } Ux(t) = 0, \\ Uz(t)/Ux(t) & \text{otherwise}. \end{cases}$$

Then $z = g(A)x$ (see Definition 1). Since the map $g \to g(A)$ is continuous, and $\|g\|_\infty \leqslant 1$, there exists $M < \infty$ such that

$$\|Ux\| \leqslant M\|x\|, \quad \text{for all } x \in X.$$

Clearly $\|x\| \leqslant \|Ux\|$. Thus U is a homeomorphism from X onto Y.

Since Y has a Riesz norm, and contains a dense lattice {polynomially bounded continuous functions}, Y is a Banach lattice.

Since $fUx = U(f(A)x)$, for any f in $C_l(\boldsymbol{R})$, Y satisfies condition (2) of Definition 4. Thus Y is a C^0-lattice on $\boldsymbol{R}$.

By Lemma 3(a), $(U(Ax))(t) = t(Ux)(t)$, for all x in $D(A)$. It follows that the one-parameter group generated by iA satisfies

$$(Ue^{isA}U^{-1}f)(t) = e^{ist} f(t),$$

so that $U(D(A))$ is as stated.

DEFINITION 6. A sequence $\{a_k\}_{k=0}^{\infty}$ of real numbers is *positive-definite* if

$$\sum_{k,j} \alpha_k \bar{\alpha}_j a_{k+j} \geqslant 0,$$

for any finite sequence $\{\alpha_k\}$ of complex numbers. It satisfies a *Stieltjes moment condition* if, in addition,

$$\sum_{k,j} \alpha_k \bar{\alpha}_j a_{k+j+1} \geqslant 0.$$

There exists a nondecreasing function g such that

$$a_n = \int_{\boldsymbol{R}} t^n \, dg(t), \quad n = 0, 1, 2, \ldots$$

if and only if $\{a_n\}_{n=0}^{\infty}$ is positive-definite. The function g is supported on $[0, \infty)$ if and only if $\{a_n\}$ satisfies a Stieltjes moment condition.

DEFINITION 7. Suppose X is a Banach lattice. Then $F\colon \boldsymbol{R} \to B(X)$ is *positive-definite* if

$$\sum_{k,j} \alpha_k \bar{\alpha}_j F(s_k - s_j)$$

is positive, for any finite sequence of complex numbers $\{\alpha_k\}$.

The function $G: [0, \infty) \to B(X)$ is *completely monotone* if, for φ in $(X^*)^+$ and x in X^+,

$$(-1)^n \frac{d^n}{ds^n} \varphi(G(s)x)$$

exists, and is nonnegative, for all $s \geqslant 0$ and all nonnegative integers n.

In [4], we showed that, if iA generates a positive-definite group, then A is C^0-scalar on $\boldsymbol{R}$, and if $-A$ generates a completely monotone semigroup, then A is C^0-scalar on $[0, \infty)$.

THEOREM 8. *Suppose* $X = \overline{D(x_0, A)}$ (see Definition 2) *and* A *is* C^0*-scalar on* $\boldsymbol{R}$. *Then there exists an equivalent norm with respect to which* X *is a Banach lattice, with*

$$(X^*)^+ = \{\varphi \text{ in } X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ is positive-definite}\}.$$

With respect to this ordering, iA *generates a positive-definite group.*

Proof. Let U and Y be as in Theorem 5. Then $|||x||| \equiv \|Ux\|$ is an equivalent norm on X, and if we define an ordering on X by $x_1 \leqslant x_2$ if and only if $Ux_1 \leqslant Ux_2$, then $(X, |||\ |||)$ is a Banach lattice.

Since A is C^0-scalar, for any φ in X^*, there exists a unique complex-valued measure E_φ such that

$$\varphi(f(A)x_0) = \int_{\boldsymbol{R}} f(t)\, dE_\varphi(t),$$

for all f in $C_l(\boldsymbol{R})$. This means that

$$\varphi(x) = \int_{\boldsymbol{R}} (Ux)(t)\, dE_\varphi(t),$$

for all x in X, so that φ is in $(X^*)^+$ if and only if E_φ is a positive measure.

By Lemma 3(a), we have

$$\varphi(A^n x_0) = \int_{\boldsymbol{R}} t^n\, dE_\varphi(t).$$

Thus (see Definition 6), $\varphi \in (X^*)^+$ if and only if $\{\varphi(A^n x_0)\}_{n=0}^{\infty}$ is positive-definite, as desired.

Since A is C^0-scalar on $\boldsymbol{R}$, iA generates a one-parameter group $\{e^{isA}\}_{s\in\boldsymbol{R}}$. Since $(Ue^{isA}U^{-1}f)(t) = e^{ist}f(t)$, we have

$$\varphi(e^{isA}x) = \int e^{ist}(Ux)(t)\, dE_\varphi(t),$$

so that iA generates a positive-definite group.

The same proof gives the following:

THEOREM 9. *Suppose* $X = \overline{D(x_0, A)}$ *and* A *is* C^0*-scalar on* $[0, \infty)$. *Then there exists an equivalent norm with respect to which* X *is a Banach lattice, with*

$$(X^*)^+ = \{\varphi \text{ in } X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ satisfies a Stieltjes moment condition}\}.$$

With respect to this ordering, $-A$ *generates a completely monotone semigroup.*

To obtain converses to Theorems 8 and 9, we will need growth conditions on $\|A^n x_0\|$. These are easily satisfied when A is bounded.

DEFINITION 10. The vector $x \in \bigcap_{n=0}^{\infty} D(A^n)$ is a *quasi-analytic vector* if

$$\sum_{n=0}^{\infty} \sup_{k \leqslant n} \|A^k x\|^{-1/k} = \infty.$$

If $\{\varphi(A^n x_0)\}_{n=0}^{\infty}$ is positive-definite (see Definition 6), and x_0 is a quasi-analytic vector, then the solution of the moment problem

$$\varphi(A^n x_0) = \int_{\boldsymbol{R}} t^n\, dg(t), \qquad n = 0, 1, 2, \ldots,$$

is unique (see [12]).

If $D(A) = D(x_0, A)$ (see Definition 2), where x_0 is a quasi-analytic vector, and A has an extension that generates a one-parameter group, then $\bar{A}$ generates a one-parameter group (see [2]).

THEOREM 11. *Suppose* $D(A) = D(x_0, A)$ (see Definition 2) *and* x_0 *is quasi-analytic. Then* $\bar{A}$ *is* C^0*-scalar on* $\boldsymbol{R}$ *if and only if there exists an equivalent norm with respect to which* X *is a Banach lattice, with*

$$(X^*)^+ = \{\varphi \in X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ is positive-definite}\}.$$

With respect to this ordering, $i\bar{A}$ *generates a positive-definite group.*

Proof. Suppose X is a Banach lattice, with the given $(X^*)^+$. For $\varphi \in (X^*)^+$, since x_0 is a quasi-analytic vector (see comments after Definition 10), there exists a unique positive measure E_φ such that

$$\varphi(A^n x_0) = \int_{\boldsymbol{R}} t^n\, dE_\varphi(t), \qquad n = 0, 1, 2, \ldots$$

Let $C_c(\boldsymbol{R}) \equiv \{$continuous complex-valued functions of compact support$\}$. For f in $C_c(\boldsymbol{R})$, define $\theta(f) \in X^{**}$ by

$$[\theta(f)](\varphi) \equiv \int f(t)\, dE_\varphi(t), \qquad \text{for } \varphi \in (X^*)^+.$$

Note that the uniqueness of E_φ guarantees that $\theta(f)$ extends to a linear functional on X^*.

Let $D \equiv \{\theta(f) \mid f \in C_c(\boldsymbol{R})\}$, $Y \equiv \bar{D}$, in X^{**}. Define an operator S on Y by $D(S) \equiv D$, $S(\theta(f)) \equiv \theta(f_1 f)$, where $f_1(t) \equiv t$. We wish to show that $i\bar{S}$

generates a positive-definite group. By [4], Theorem 3.20, it is sufficient to show the following.

(1) $D^+ = \{\theta(f) \mid f \in C_c(\boldsymbol{R}), f(t) \geqslant 0, \text{ for all } t\}$ contains only quasi-analytic vectors (see Definition 10).

(2) $\{\psi(S^n x)\}_{n=0}^{\infty}$ is positive-definite, for all ψ in $(Y^*)^+$ and x in D^+.

To prove (1), suppose $\theta(f) \in D^+$, and $M \equiv \sup\{|t| \mid f(t) > 0\}$. If $\varphi \in (X^*)^+$, then

$$|\varphi(S^n \theta(f))| = \left|\int t^n f(t)\, dE_\varphi(t)\right| \leqslant M^n \int f(t)\, dE_\varphi(t) = M^n \varphi(\theta(f)).$$

It follows that $\|S^n \theta(f)\| \leqslant 4M^n \|\theta(f)\|$. Thus $\theta(f)$ is a quasi-analytic vector, as desired.

To prove (2), suppose $\psi \in (Y^*)^+$. The map $f \to \psi(\theta(f))$ is a positive linear functional on $C_c(\boldsymbol{R})$, thus there exists a unique positive measure m_ψ such that

$$\psi(\theta(f)) = \int_{\boldsymbol{R}} f(t)\, dm_\psi(t),$$

for all f in $C_c(\boldsymbol{R})$. Hence, if $\theta(f) \in D^+$, then

$$\psi(S^n \theta(f)) = \int_{\boldsymbol{R}} t^n f(t)\, dm_\psi(t),$$

so that $\{\psi(S^n \theta(f))\}_{n=0}^{\infty}$ is positive-definite, proving (2).

Thus, $i\bar{S}$ generates a positive-definite group.

We now claim that $\bar{S}$ is an extension of A. To prove this, we need to show that the graph of A is contained in the closure of the graph of S.

So suppose p is a polynomial. For any n, let p_n be a continuous function supported on $[-(n+1), n+1]$, with $p_n(t) = p(t)$, for $|t| \leqslant n$, and $|p_n(t)| \leqslant |p(t)|$, for $|t| \leqslant n+1$.

There exists even N such that $\lim_{|t| \to \infty} p(t)/t^N = 0$. For any k, let $M_k \equiv \sup\{|p(t)/t^N| \mid |t| \geqslant k\}$.

Suppose $\varphi \in (X^*)^+$. Then

$$\begin{aligned} |[\theta(p_k)](\varphi) - \varphi(p(A)x_0)| &= \left|\int_{\boldsymbol{R}} (p_k - p)(t)\, dE_\varphi(t)\right| \\ &\leqslant 2 \int_{|t| \geqslant k} |p(t)|\, dE_\varphi(t) \leqslant 2M_k \int_{|t| \geqslant k} t^N\, dE_\varphi(t) \\ &\leqslant 2M_k \int_{\boldsymbol{R}} t^N\, dE_\varphi(t) = 2M_k\, \varphi(A^N x_0). \end{aligned}$$

Thus $\|\theta(p_k) - p(A)x_0\| \leqslant 8M_k \|A^N x_0\|$; since $M_k \to 0$ as $k \to \infty$, $\theta(p_k)$ converges to $p(A)x_0$.

The same argument shows that $S\theta(p_k)$ converges to $Ap(A)x_0$ as $k \to \infty$. Thus the graph of A is contained in the closure of the graph of S, as desired.

Since iA has an extension, $i\bar{S}$, that generates a positive-definite group, and x_0 is quasi-analytic, $i\bar{A}$ generates a positive-definite group (see comments after Definition 10). By Proposition 3.7 in [4], $\bar{A}$ is C^0-scalar on $\boldsymbol{R}$.

COROLLARY 12. *If A is bounded and X is cyclic, then A is C^0-scalar on $[-\|A\|, \|A\|]$ if and only if there exists an equivalent norm with respect to which X is a Banach lattice, with*

$$(X^*)^+ = \{\varphi \in X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ is positive-definite}\}.$$

$\{e^{itA}\}_{t \in \boldsymbol{R}}$ is positive-definite with respect to this ordering.

DEFINITION 13. The vector x in $\bigcap_{n=0}^{\infty} D(A^n)$ satisfies a *Stieltjes growth condition* if

$$\sum_{n=0}^{\infty} \sup_{k \leqslant n} \|A^k x\|^{-1/(2k)} = \infty.$$

If $D(A) = D(x_0, A)$ (see Definition 2), where x_0 satisfies a Stieltjes growth condition, and A has an extension that generates a uniformly bounded semigroup, then $\bar{A}$ generates a uniformly bounded semigroup (see [2]).

If x_0 satisfies a Stieltjes growth condition, and $\{\varphi(A^n x_0)\}_{n=0}^{\infty}$ satisfies a Stieltjes moment condition (see Definition 6), then the solution of the moment problem

$$\varphi(A^n x_0) = \int_0^{\infty} t^n\, dg(t), \quad n = 0, 1, 2, \ldots$$

is unique (see [12]).

Essentially the same proof as in Theorem 11 gives the following.

THEOREM 14. *Suppose $D(A) = D(x_0, A)$ and x_0 satisfies a Stieltjes growth condition. Then $\bar{A}$ is C^0-scalar on $[0, \infty)$ if and only if there exists an equivalent norm with respect to which X is a Banach lattice, with*

$$(X^*)^+ = \{\varphi \in X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ satisfies a Stieltjes moment condition}\}.$$

With respect to this ordering, $-\bar{A}$ generates a completely monotone semigroup.

COROLLARY 15. *If A is bounded and X is cyclic, then A is C^0-scalar on $[0, \|A\|]$ if and only if there exists an equivalent norm with respect to which X is a Banach lattice, with*

$$(X^*)^+ = \{\varphi \in X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ satisfies a Stieltjes moment condition}\}.$$

$\{e^{-tA}\}_{t \geqslant 0}$ is completely monotone with respect to this ordering.

References

[1] W. G. Bade, *On Boolean algebras of projections and algebras of operators*, Trans. Amer. Math. Soc. 80 (1955), 345–360.
[2] P. R. Chernoff, *Quasi-analytic vectors and quasi-analytic functions*, Bull. Amer. Math. Soc. 81 (4) (1975), 637–646.
[3] I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York 1968.
[4] R. deLaubenfels, *Unbounded scalar operators on Banach lattices*, Honam Math. J. 8 (1986), 1–19.
[5] H. R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, 1978.
[6] N. Dunford and J. T. Schwartz, *Linear Operators, Part III, Spectral Operators*, Wiley-Interscience 1971.
[7] S. Kantorovitz, *Characterization of unbounded spectral operators with spectrum in a half-line*, Comment. Math. Helv. 56 (2) (1981), 163–178.
[8] —, *Spectrality criteria for unbounded operators with real spectrum*, Math. Ann. 256 (1) (1981), 19–28.
[9] —, *Spectral Theory of Banach Space Operators*, Lecture Notes in Math. 1012, Springer, 1983.
[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer, 1983.
[11] H. Schaefer, *Banach Lattices and Positive Operators*, Springer, 1974.
[12] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Math. Surveys 1, Amer. Math. Soc., 1943.

DEPARTMENT OF MATHEMATICS
OHIO UNIVERSITY
Athens, Ohio 45701, U.S.A.

Received March 27, 1987
Revised version July 24, 1987 (2293)

STUDIA MATHEMATICA, T. XCII. (1989)

Weighted inequalities on product domains

by

SHUICHI SATO (Sendai)

Abstract. We prove weighted integral inequalities between the Lusin area functions and the nontangential maximal functions of biharmonic functions on product domains. Furthermore, we study the duality of weighted BMO spaces and weighted H^1 spaces in the two-parameter theory.

§1. Introduction. Let $A(u)$ and $N(u)$ be the Lusin area function and the nontangential maximal function, respectively, of a biharmonic function u on the product space $R_+^{n_1+1} \times R_+^{n_2+1}$, where $R_+^{n_i+1} = R^{n_i} \times (0, \infty)$ $(i = 1, 2)$. In this note we consider a weight function w which satisfies the two-parameter analogue of the Muckenhoupt A_∞ condition and we prove the weighted L^p-"norm" inequalities:

$$\|N(u)\|_{L_w^p} \leqslant c\,\|A(u)\|_{L_w^p} \quad (0 < p < \infty)$$

for biharmonic functions satisfying a reasonable condition (see Theorem 3 in §3). This is an extension to the weighted L^p-spaces of a result of Gundy–Stein [13]. For the proof of Theorem 3, results of Wheeden [23] (see also [11] and [14]) about harmonic majorization and H^p spaces of conjugate harmonic functions are extended to the case of biharmonic functions on the product domains (see Theorems 1 and 2 in §3). These results together with the weighted inequalities for the Lusin functions and the nontangential maximal functions (of the one-parameter theory) proved in [14] are applied to obtain the desired result if we argue as in [13].

As for the converse, we have obtained only a partial result, which we can derive from a weighted analogue of a result of Merryfield [16] (see also [12], [13] and [15] for the unweighted case). We will state these results in §3 without proofs.

Finally, we also study the duality of weighted BMO spaces and weighted H^1 spaces on the product domains (see Theorem 5 in §3).