[6] M. Cotlar, *Some generalizations of the Hardy-Littlewood maximal theorem*, Rev. Mat. Cuyana 1 (1955), 85–104.
[7] J.-L. Journée, *Calderón–Zygmund Operators, Pseudo-differential Operators and the Cauchy Integral of Calderón*, Lecture Notes in Math. 994, Springer, 1983.
[8] M. Kaneko and S. Yano, *Weighted norm inequalities for singular integrals*, J. Math. Soc. Japan 27 (1975), 570–588.
[9] D. S. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. 255 (1979), 343–362.
[10] —, —, *A note on singular integrals with weights*, Proc. Amer. Math. Soc. 81 (1981), 391–397.
[11] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.
[12] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, ibid. 87 (1958), 159–172.

FACULTAD DE CIENCIAS EXACTAS Y NATURALES
UNIVERSIDAD DE BUENOS AIRES
and
INSTITUTO ARGENTINO DE MATHEMÁTICA (CONICET)

Received February 17, 1987
Revised version August 14, 1987 (2279)

STUDIA MATHEMATICA, T. XCII. (1989)

Existence of bases and the dual splitting relation for Fréchet spaces

by

JÖRG KRONE (Wuppertal)

Abstract. The present article contains a topological condition for nuclear Fréchet spaces E and F which is sufficient for the existence of a basis in the range of every continuous linear operator from E to F. Surprisingly our condition is in some sense dual to the ones describing the splitting of short exact sequences of Fréchet spaces.

The problem of Pełczyński [20] whether complemented subspaces of nuclear Fréchet spaces with basis also have a basis is still unsolved. Positive solutions were obtained by Mityagin and Henkin [16, 17, 18] in the case of finite type power series spaces, by Vogt and Dubinsky [9, 26] in the case of tame power series spaces and by Dubinsky [8] and Fachinger [10] in some more general cases.

The present paper modifies the method of Mityagin and Henkin (see 1.1) so that it can give a necessary and sufficient topological condition for the existence of a basis in complemented subspaces of regular nuclear Köthe spaces (see 1.2). This method leads to a sufficient condition on E and F for the range of every operator $T: E \to F$ to have a basis. Here E and F are nuclear Fréchet spaces or Köthe Schwartz spaces or their dual spaces. Some examples of nuclear Köthe spaces without this property are given in Krone [13, 14]. It is an interesting fact that the above-mentioned sufficient condition can be interpreted as the dual splitting relation (see Apiola [2], Ketonen–Nyberg [12], Krone–Vogt [15], Nyberg [19], Vogt [24, 25]). Hence it is called DS and in the case that E and F are Köthe spaces we have a simpler version called DS* (see 1.3).

The condition DS is easy to check in the standard cases. If both spaces are power series spaces we obtain exactly the above-mentioned well-known results (see 2.1). If one space is a shift-stable power series space the characteristic properties of the DS partner spaces are the conditions $\underline{\mathrm{DN}}$, $\overline{\mathrm{DN}}$, Ω and $\tilde{\Omega}$ which are introduced in Vogt [22, 23] and Wagner [27]. In 2.3 we evaluate the DS condition for Dragilev spaces.

The examples considered in the second chapter underline and clarify the dual character of the splitting and the DS conditions, but it is not clear at all whether there is a deeper theoretical explanation for this surprising result.

I want to thank D. Vogt for many helpful suggestions.

Notation (see also Dubinsky [8]). E and F are always locally convex spaces. For a subset M of E the *gauge functional* is defined by $p_M(x) := \inf\{c \geq 0\colon x \in cM\}$, $\inf\{\} := +\infty$. M is a *Hilbert ball* if M is an absolutely convex, closed set and if p_M is a Hilbert norm on $E_M := \bigcup_{c\geq 0} cM$, i.e. if the completion $(E_M, p_M)\hat{}$ is a Hilbert space.

For fundamental systems of absolutely convex and closed neighbourhoods of zero $(U_a)_{a\in A}$ in E and $(V_b)_{b\in B}$ in F, the $\|\ \|_a := p_{U_a}$ and $\|\ \|_b := p_{V_b}$ are the corresponding fundamental systems of seminorms. If E is a Fréchet space ((F)-space) we can choose $A = N$ and $U_n \supset U_{n+1}$.

For a continuous linear mapping $T \in L(E, F)$ we define

$$\|T\|_{N,M} := \inf\{c \geq 0\colon\ T(M) \subset cN\}, \qquad \|T\|_{b,a} := \|T\|_{V_b, U_a}.$$

The range of T is always endowed with the subspace topology of F.

$A = (a_{j,k})_{j,k\in N}$ with $0 \leq a_{j,k} \leq a_{j,k+1}$ and $\sup_{k\in N} a_{j,k} > 0$ for all $j, k \in N$ is a *Köthe matrix.*

A *Köthe space* (*Köthe sequence space*) *of type* $p \in N$ or $p = +\infty$ is defined by

$$\lambda^p(A) := \{(x_j)_{j\in N} \subset K\colon\ \|x\|_k^p := \Big(\sum_{j=1}^{\infty} |x_j a_{j,k}|^p\Big)^{1/p} < \infty \text{ for all } k \in N\},$$

$$\lambda^\infty(A) := \{(x_j)_{j\in N} \subset K\colon\ \|x\|_k^\infty := \sup_{j\in N} |x_j a_{j,k}| < \infty \text{ for all } k \in N\}.$$

A Köthe space of type 1, $\lambda^1(A)$, is simply called a Köthe space $\lambda(A)$.

It is nuclear if and only if for all k there is an m so that $\sum_{j\in J_m} a_{j,k}/a_{j,m} < \infty$ for $J_m := \{j \in N\colon\ a_{j,m} > 0\}$.

It is a Schwartz space if and only if for all k there is an m with

$$\lim_{\substack{j\to\infty\\ j\in J_m}} a_{j,k}/a_{j,m} = 0.$$

It is called *regular* if $a_{j,k+1}/a_{j,k} \leq a_{i,k+1}/a_{i,k}$ for all $i, j \in J_k$, $k \in N$, with $j \leq i$.

If it has a continuous norm we may assume that $a_{j,k} > 0$ for all $j, k \in N$.

If A has the form $a_{j,k} = \exp(r_k \alpha_j)$, where α is a sequence with $0 \leq \alpha_j \leq \alpha_{j+1} \to +\infty$ and $r_k < r_{k+1} \to r$, $r \in R$ or $r = +\infty$, then $\lambda(A)$ is called a *power series space* $\lambda_r(\alpha)$. If $a_{j,k} = \exp(f(r_k\alpha_j))$ with an odd rapidly increasing function f, we call $\lambda(A)$ a *Dragilev space* $L_f(\alpha, r)$ (see Dragilev [7] and Ahonen [1]).

1.1. First we want to show the existence of a basis in the range of some operators in a quite general setting.

Lemma 1.1. *Let E and F be complete locally convex spaces and let $T \in L(E, F)$. If there are Hilbert balls K and U so that E_K is dense in E, K is compact, U is a neighbourhood of zero in F and $\{L \in L(E, F)\colon\ L(K) \subset T(K)$ and $T^{-1}(U) \subset L^{-1}(U)\}$ is equicontinuous, then the range of T has a basis.*

Proof. If we endow E_k with the topology of p_K and if $F_U := (F, p_U)\hat{}$, then $T\colon E_K \to F_U$ is a compact mapping between Hilbert spaces. Because of the spectral theorem for compact operators we can write T in the form

$$Tx = \sum_{j=1}^{\infty} \alpha_j \langle x, e_j\rangle_K f_j$$

for all $x \in E_K$, where $\{e_j\}$ and $\{f_j\}$ are othonormal systems in E_K and F_U and where $\alpha_j \geq \alpha_{j+1} \to 0$. We may assume that $\alpha_j > 0$ since otherwise the image of T is finite-dimensional.

Since $T(e_j) = \alpha_j f_j$, we have $f_j \in \operatorname{range}(T)$.

From $\sum_{j=1}^{n} a_j f_j \to 0$ as $n \to +\infty$ in range(T) it follows that the series is also convergent in F_U, hence $a_j = 0$ for all j and the set $\{f_j\}$ is linearly independent in the range of T.

For $x \in E$, $n \in N$ we define $T_n(x) := \sum_{j=1}^{n} \langle Tx, f_j\rangle_U f_j$ and we get

$$T_n(T^{-1}(U)) \subset \{\sum_{j=1}^{n} \langle z, f_j\rangle_U f_j\colon\ z \in U\} \subset U,$$

$$T_n(x) = T\Big(\sum_{j=1}^{n} x_j e_j\Big) \quad \text{if } x = \sum_{j=1}^{\infty} x_j e_j \text{ in } E_K,$$

$$T_n(K) = \{\sum_{j=1}^{n} \alpha_j x_j f_j\colon\ \sum_{j=1}^{\infty} |x_j|^2 \leq 1\} \subset T(K).$$

Hence $\{T_n\}_{n\in N}$ is equicontinuous. Since $T_n(x) \to T(x)$ for $x \in E_K$, which is dense in E, we obtain the convergence on the whole of E by a standard argument, e.g. used in the Banach–Steinhaus theorem.

For $y_j(x) := \langle Tx, f_j\rangle_U$ we have $y_j \in E'$ and $\sum_{j=1}^{\infty} a_j y_j = 0$ in E'_σ implies

$$\alpha_i a_i = \sum_{j=1}^{\infty} a_j \langle Te_i, f_j\rangle_U = 0.$$

Hence we obtain

Corollary 1.1. *Under the assumptions of Lemma 1.1 there is an index set $I \subset N$ and linearly independent sets $\{y_i\}_{i\in I}$ in E'_σ and $\{f_i\}_{i\in I}$ in F so that $Tx = \sum_{i\in I} y_i(x) f_i$ for all $x \in E$. We even get $f_i \in \operatorname{range}(T)$, $\{y_i\}_{i\in I}$ equicontinuous* *and* $y_i = f_i' \circ T$ *for an equicontinuous biorthogonal* ($f_i'(f_j) = \delta_{i,j}$) *system* $\{f_i'\}$ *in* F'.

1.2. Now we are able to give a complete topological characterization of those complemented subspaces of regular nuclear Köthe spaces with continuous norm which have a basis.

THEOREM 1.2. *If E is a complemented subspace of a nuclear regular Köthe space with continuous norm, then E has a basis if and only if there is a compact Hilbert ball K in E with E_K dense and a Hilbert ball U which is a neighbourhood of zero so that $\{L \in L(E, E): L(K) \subset K$ and $L(U) \subset U\}$ is equicontinuous.*

Proof. The condition is sufficient since we may apply Lemma 1.1 with $E = F$ and $T = \mathrm{id}$.

If E has a basis it is again a nuclear regular Köthe space with continuous norm (cf. Bessaga [3], Dragilev [6] and Dubinsky [8]). Let $E = \lambda(A)$. For every k we find an S so that

$$a_{i,k}/a_{j,k} \leqslant S \max(a_{i,i}/a_{j,j},\ a_{i,1}/a_{j,1}) \quad \text{for all } i, j \in N.$$

With

$$U := \{x \in E: \sum_{i=1}^{\infty} |x_i a_{i1}|^2 \leqslant 1\}, \qquad K := \{x \in E: \sum_{i=1}^{\infty} |x_i a_{ii}|^2 \leqslant 1\}$$

we get our condition.

1.3. Since it is not easy to decide whether the complemented subspaces of a given space always satisfy the condition in 1.2, we want to introduce a sufficient condition which has nice properties concerning subspaces and quotients. In general this condition is not necessary (cf. Krone [14]) but it is not clear whether it is necessary under some mild and reasonable assumptions on regularity or stability.

We want to introduce that condition by the "dualization" of the splitting relation. For Köthe spaces $\lambda(A)$ and $\lambda(B)$, where $\lambda(A)$ is not a Banach space, every exact sequence of Fréchet spaces $0 \to \lambda(B) \to G \to \lambda(A) \to 0$ splits if and only if

(S*) $\quad \forall \mu\ \exists p, k\ \forall m, v\ \exists n, S\ \forall i, j: \quad a_{i,m}/b_{j,k} \leqslant S \max(a_{i,n}/b_{j,v},\ a_{i,p}/b_{j,\mu})$.

Now we want to give a dual condition by exchanging $\exists$ and $\forall$, $b_{i,k}$ and $1/b_{i,k}$, $1/a_{i,m}$ and $a_{i,m}$ and so on. This condition will be called DS* and the basis free version DS.

DEFINITION 1.3. $(E, F) \in (\mathrm{DS})$ iff there is a bounded set B in E so that

$$\exists \mu\ \forall p, k\ \exists m, v, S: \quad \|T\|_{k,m} \leqslant S \max(\|T\|_{v,B},\ \|T\|_{\mu,p})$$

for every $T \in L(E, F)$.

$(\lambda(A), \lambda(B)) \in (\mathrm{DS}^*)$ iff $\exists \mu\ \forall p, k\ \exists m, v\ \forall n\ \exists s\ \forall i, j$

$$b_{j,k}/a_{i,m} \leqslant S \max(b_{j,v}/a_{i,n},\ b_{j,\mu}/a_{i,p}).$$

First we want to state two obvious facts:

Remark. 1. The conditions DS and DS* are independent of the choice of the fundamental system of seminorms.

2. $(E, F) \in (\mathrm{DS})$ implies $(E_0, F_0) \in (\mathrm{DS})$ for every quotient space E_0 of E and every subspace F_0 of F.

Now we have to show that the condition DS* is exactly the condition DS for Köthe spaces.

PROPOSITION 1.3. *For all $p, q \in N \cup \{+\infty\}$ the following are equivalent:*

(1) $\quad (\lambda(A), \lambda(B)) \in (\mathrm{DS}^*)$.

(2) $\quad (\lambda^p(A), \lambda^q(B)) \in (\mathrm{DS})$.

(3) $\quad ((\lambda^q(B))_b', (\lambda^p(A))_b') \in (\mathrm{DS})$.

Proof. If $\lambda^p(A)$ or $\lambda^q(B)$ is a Banach space the three conditions are satisfied. Hence we may assume that they are not Banach spaces.

Step 1: (2) *is equivalent to*

(2′) $\quad \exists \mu,\ x \in \lambda^+(A)\ \forall n_0, k\ \exists m, v, s$:

$$b_{j,k}/a_{i,m} \leqslant S \max(x_i b_{j,v},\ b_{j,\mu}/a_{i,n_0})$$

for all i, j and with $\lambda^+(A) := \{x = (x_i)_{i \in N} \in \lambda^\infty(A): x_i > 0$ *for all* $i\}$.

Proof. Let $(e_i)_{i \in N}$ and $(f_i)_{i \in N}$ be the canonical bases in $\lambda(A)$ and $\lambda(B)$. The dual bases are denoted by *. If we evaluate (2) at $L := e_i^* \otimes f_j$ we obtain (2′) with $x_i := \sup_{y \in K} |e_i^*(y)|$. $x := (x_i)_{i \in N}$ is in $\lambda^\infty(A)$ since K is bounded, and $x_i > 0$ if $\lambda^q(B)$ is not a Banach space.

(2′) implies (2) with $K := \{y \in \lambda^p(A): \sum_i |y_i/x_i|^p \leqslant 1\}$ since for $p, q \in N$ and $W := \{y: \sum_i |y_i|^p \leqslant 1\}$ we obtain the estimates

$$\begin{aligned}\|T\|_{k,m}^q &\leqslant \sup_{y \in W} \sum_{i,j} |b_{j,k} T_{j,i} y_i/a_{i,m}|^q \\ &\leqslant \sup_{y \in W} S^q \max\Big(\sum_{i,j} |b_{j,v} T_{j,i} y_i x_i|^q,\ \sum_{i,j} |b_{j,\mu} T_{j,i} y_i/a_{i,n_0}|^q\Big) \\ &\leqslant S^q \max\Big(\sup_{y \in W} \sum_{i,j} |b_{j,v} T_{j,i} y_i x_i|^q,\ \sup_{y \in W} \sum_{i,j} |b_{j,\mu} T_{j,i} y_i/a_{j,n_0}|^q\Big) \\ &\leqslant S^q \max(\|T\|_{v,K}^q,\ \|T\|_{\mu,n_0}^q).\end{aligned}$$

For $p = +\infty$ or $q = +\infty$ there are analogous estimates.

Step 2: (3) *is equivalent to*

(3′) $\quad \exists \mu,\ x \in \lambda^+(A)\ \forall y \in \lambda^+(B),\ z \in \lambda^+(A)\ \exists v \in \lambda^+(B),\ w \in \lambda^+(A)$:

$$z_j/v_i \leqslant \max(x_j/y_i,\ w_j b_{i,\mu}).$$

Proof. Since the sets $(K_x^0)_{x\in\lambda^+(A)}$ defined as above are a fundamental system of neighbourhoods of zero in $(\lambda^p(A))_b'$ (see Bierstedt–Meise–Summers [4]) the proof follows the same lines as in the first step.

Step 3: (1) *is equivalent to* (2′).

Proof. Obviously (2′) implies (1).

If $\lambda^p(B)$ is not a Banach space (1) implies analogously to Krone–Vogt [15]

$$\exists\,\mu\ \forall n_0, k\ \exists m, v\ \forall n, R>0\ \exists s\colon\quad \frac{b_{j,k}}{a_{i,m}} \leqslant \max\left(s\frac{b_{j,v}}{a_{i,n}}, \frac{1}{R}\frac{b_{j,\mu}}{a_{i,n_0}}\right).$$

Hence there are sequences $(S_{n,k})_{n,k\in N}$ and $(m_k)_{k\in N}$ with

$$\frac{b_{j,k}}{a_{i,m_k}} \leqslant \max\left(S_{n,k}\frac{b_{j,m_k}}{a_{i,n}}, \frac{b_{j,\mu}}{a^{i,k}}\right) \quad \text{for all } n, k, i, j\in N.$$

Let $S_{n,k}\geqslant 1$ and $S_m := \max_{n,k\leqslant m} S_{n,k}$. For $x_i := \inf_n S_n/a_{i,n}$ and $x := (x_i)$ we have $x\in\lambda^\infty(A)$ and

$$b_{j,k}/a_{i,m_k} \leqslant \max(S_k x_i b_{j,m_k}, b_{j,\mu}/a_{i,k}) \quad \text{for all } i, j, k.$$

Step 4: (2′) *is equivalent to* (3′).

Proof. (2′) implies

$$\exists\,\mu, x\in\lambda^+(A)\ \forall y\in\lambda^+(B), z\in\lambda^+(A), n_0', k\ \exists S\colon$$

$$b_{j,k} z_i \leqslant S\max(x_i/y_j, b_{j,\mu}/a_{i,n_0}) \quad \text{for all } i \text{ and } j.$$

If $\lambda^p(A)$ is not a Banach space we get

$$\exists\,\mu, x\in\lambda^+(A)\ \forall y\in\lambda^+(B), z\in\lambda^+(A), n_0, k\ \exists S\colon$$

$$b_{j,k} z_i \leqslant \max(x_i/y_j, S b_{j,\mu}/a_{i,n_0}).$$

This implies with the same S_k

$$\exists\,\mu, x\in\lambda^+(A)\ \forall y\in\lambda^+(B), z\in\lambda^+(A)\ \exists (S_k)_{k\in N}\colon$$

$$b_{j,k} z_i \leqslant \max(x_i/y_j, S_k S_{n_0} b_{j,\mu}/a_{i,n_0}) \quad \text{for all } i, j, k, n_0.$$

For $v_j := \inf_k S_k/b_{j,k}$ and $w_i := \inf_n S_n/a_{i,n}$ we get (3′).

(3′) implies the condition

(*) $$\exists\,\mu, x\in\lambda^+(A)\ \forall y\in\lambda^+(B), z\in\lambda^+(A), n_0, k\ \exists S\colon$$

$$b_{j,k} z_i \leqslant S\max(x_i/y_j, b_{j,\mu}/a_{i,n_0}).$$

If (2′) is not satisfied there are k, n_0 and increasing sequences (i_m), (j_m) so that for every m

$$b_{j_m,k}/a_{i_m,m} > m\max(b_{j_m,m} x_{i_m}, b_{j_m,\mu}/a_{i_m,n_0}).$$

For every i there is an $l_i \geqslant i$ with $a_{i,l_i} > 0$ and $b_{i,l_i} > 0$. The following z and y yield a contradiction to (*):

$$z_i := \begin{cases} 1/a_{i,m} & \text{for } i = i_m,\\ 1/a_{i,l_i} & \text{otherwise,}\end{cases} \qquad y_j := \begin{cases} 1/b_{j,m} & \text{for } j = j_m,\\ 1/b_{j,l_j} & \text{otherwise.}\end{cases}$$

Finally we want to prove the main result of this chapter: the condition DS is sufficient for the basis property.

THEOREM 1.3. *Let E and F be complete locally convex spaces so that E has a fundamental system for the bounded sets consisting of compact Hilbert balls, and F has a fundamental system for the neighbourhoods of zero consisting of Hilbert balls. Then $(E, F)\in$(DS) implies that the range of every operator $T\in L(E, F)$ has a basis.*

Moreover, we can write T in the form $T(x) = \sum_{i\in I} y_i(x) f_i$ for all $x\in E$ and linearly independent sequences $(f_i)_{i\in I}$ in the range of T and $(y_i)_{i\in I}$ in E'_σ and an index set I contained in N.

Remark. The assumptions on E and F are satisfied for nuclear Fréchet spaces, for Köthe–Schwartz spaces $\lambda^2(A)$ and their dual spaces.

Proof of the theorem. There is a compact Hilbert ball K in E meeting the condition DS.

If E_K is not dense in E then DS implies that F is a Hilbert space, hence the proof is elementary.

If $\|\ \|_\mu$ is not a norm on F then DS implies that E is a Hilbert space, hence E is of finite even dimension.

In the case that E_K is dense in E and $\|\ \|_\mu$ is a norm on F, we may apply Lemma 1.1. Then we have to show that the set $\{L\in L(E, F)\colon L(K)\subset T(K)$ and $T^{-1}(V_\mu)\subset L^{-1}(V_\mu)\}$ is always equicontinuous. But this is a consequence of the condition DS since

$$\|L\|_{k,m} \leqslant S\max(\|L\|_{v,K}, \|L\|_{\mu,p}) \leqslant S\max(\|T\|_{v,K}, \|T\|_{\mu,p}).$$

2.1. In the second chapter we apply Theorem 1.3 in some specific cases, hence we have to evaluate the condition (DS) or (DS*). First we consider power series spaces. Here we obtain the well-known results for the basis property due to Mityagin [16], [17] and Mityagin–Henkin [18] for finite type and to Dubinsky–Vogt [7] and Vogt [26] for infinite type. To demonstrate the dual character of the conditions DS* and S* we also state the corresponding results for the splitting relation (see Ketonen–Nyberg [12], Nyberg [19] and Hebbecker [11]).

Let M denote the set of all finite limit points of $\{\beta_j/\alpha_i\colon i, j\in N\}$ and let $s\in\boldsymbol{R}$ or $s = +\infty$.

THEOREM 2.1. *$(\lambda_r(\alpha), \lambda_s(\beta)) \in (\mathrm{DS}^*)$ resp. $(\lambda_r(\beta), \lambda_s(\alpha)) \in (\mathrm{S}^*)$ if and only if*

	DS*	S*
$r = +\infty$	*M bounded*	*always*
$r = 0$	*always*	*M bounded*

Proof. First we consider $r = +\infty$. For $s_k = k$ or $s_k = -1/k$ the condition DS* implies: $\exists \mu, m \ \forall n \ \exists S$:

$$s_{\mu+1}\beta_j - m\alpha_i \leqslant S + \max(s_m\beta_j - n\alpha_i,\ s_\mu\beta_j - \alpha_i) \quad \text{for all } i, j.$$

Hence we obtain for all $x \in M$, $n \in N$

$$s_{\mu+1}x - m \leqslant \max(s_m x - n,\ s_\mu x - 1).$$

For n sufficiently large this implies $s_{\mu+1}x - m \leqslant s_\mu x - 1$ or $x \leqslant (m-1)/(s_{\mu+1} - s_\mu)$. Hence we have an upper bound for M.

Now suppose K is an upper bound of M. There is a sequence $(j_i)_{i\in N}$ with $\beta_{j_i - 1} \leqslant 2K\alpha_i \leqslant \beta_{j_i}$. For any k, $n \in N$ there is an i_0 with $\beta_{j_i} \geqslant 2kn\alpha_i$ for all $i \geqslant i_0$. Hence we get for $j \geqslant j_i$

$$-\frac{1}{k}\beta_j \leqslant n\alpha_{i_0} - \frac{1}{2k}\beta_j - n\alpha_i$$

resp.

$$k\beta_j \leqslant n\alpha_{i_0} + (k+1)\beta_j - n\alpha_i.$$

On the other hand, we obtain for $j < j_i$

$$-\frac{1}{k}\beta_j - (2K+k)\alpha_i \leqslant -\beta_j - k\alpha_i$$

resp.

$$k\beta_j - (2Kk+k)\alpha_i \leqslant \beta_j - k\alpha_i.$$

Hence $\forall n, k \ \exists C \ \forall i, j$:

$$s_k\beta_j - (2Kk+k)\alpha_i \leqslant \max(C + s_{2k}\beta_j - n\alpha_i,\ s_1\beta_j - k\alpha_i).$$

This implies $(\lambda_\infty(\alpha), \lambda_s(\beta)) \in (\mathrm{DS}^*)$.

Now we have to show that $(\lambda_0(\alpha), \lambda_s(\beta)) \in (\mathrm{DS}^*)$.

First, if $\beta_j \leqslant \frac{1}{2k^2}\alpha_i$ then

$$-\frac{1}{k}\beta_j + \frac{1}{2k}\alpha_i \leqslant \frac{1}{k}\alpha_i - \frac{1}{2k^2}\alpha_i \leqslant -\beta_j + \frac{1}{k}\alpha_i,$$

$$k\beta_j + \frac{1}{2k}\alpha_i \leqslant \beta_j + \frac{1}{k}\alpha_i,$$

and if $\beta_j \geqslant \frac{1}{2k^2}\alpha_i$ then

$$-\frac{1}{k}\beta_j + \frac{1}{4k^3}\alpha_i \leqslant -\frac{1}{k}\beta_j + \frac{1}{2k}\beta_j \leqslant -\frac{1}{2k}\beta_j + \frac{1}{n}\alpha_i,$$

$$k\beta_j + \frac{1}{2k}\alpha_i \leqslant 2k\beta_j + \frac{1}{n}\alpha_i.$$

This completes the proof.

2.2. In the case that one space is a shift stable power series space (there is a C with $\alpha_{j+1} \leqslant C\alpha_j$ for all j), we can characterize the DS relation in terms of the well-known conditions of DN and Ω type. Namely, we use the following properties of a Fréchet space E introduced by Vogt [22, 23] and Wagner [27]:

$\underline{\mathrm{DN}}$: $\exists p \ \forall k \ \exists n, C, d > 0 \ \forall x \in E$: $\quad \|x\|_k^{1+d} \leqslant C\|x\|_p^d \|x\|_n$,

$\overline{\mathrm{DN}}$: $\exists k \ \forall n \ \exists N \ \forall d \ \exists C \ \forall x \in E$: $\quad \|x\|_n^{1+d} \leqslant C\|x\|_k^d \|x\|_N$,

$\bar{\Omega}$: $\forall p, d \ \exists k \ \forall n \ \exists C \ \forall y \in E'$: $\quad \|y\|_{-k}^{1+d} \leqslant C\|y\|_{-n}\|y\|_{-p}^d$,

$\tilde{\Omega}$: $\forall p \ \exists d, k \ \forall n \ \exists C \ \forall y \in E'$: $\quad \|y\|_{-k}^{1+d} \leqslant C\|y\|_{-n}\|y\|_{-p}^d$.

If E is a Fréchet–Schwartz space, $\bar{\Omega}$ is equivalent to (see Vogt [22])

$\bar{\Omega}'$: There is a bounded set B in E so that

$$\forall p, d \ \exists k, C \ \forall y \in E': \quad \|y\|_{-k}^{1+d} \leqslant C\|y\|_{-B}\|y\|_{-p}^d,$$

and $\tilde{\Omega}$ is equivalent to (see Dineen–Meise–Vogt [5])

$\tilde{\Omega}'$: There is a bounded set B in E so that

$$\forall p \ \exists d, k, C \ \forall y \in E': \quad \|y\|_{-k}^{1+d} \leqslant C\|y\|_{-B}\|y\|_{-p}^d.$$

THEOREM 2.2. *Let α be shift stable* ($\sup_j \alpha_{j+1}/\alpha_j < +\infty$) *and E a Fréchet space. Then*

$$(\lambda_0(\alpha), E) \in (\mathrm{DS}) \Leftrightarrow E \text{ has } \underline{\mathrm{DN}},$$

$$(E, \lambda_0(\alpha)) \in (\mathrm{DS}) \Leftrightarrow E \text{ has } \bar{\Omega}',$$

$$(\lambda_\infty(\alpha), E) \in (\mathrm{DS}) \Leftrightarrow E \text{ has } \overline{\mathrm{DN}},$$

$$(E, \lambda_\infty(\alpha)) \in (\mathrm{DS}) \Leftrightarrow E \text{ has } \tilde{\Omega}'.$$

The proof uses the same arguments as similar evaluations of conditions in this form e.g. in Vogt [23]–[25].

2.3. For Dragilev spaces the DS* condition implies some new results for the basis problem (see also Dubinsky [8] and Fachinger [10]). The corresponding evaluations of the splitting relation S* are contained in Hebbecker [11].

Let again M denote the set of all finite limit points of $\{\beta_j/\alpha_i\colon\ i, j \in N\}$ and $N := \{1/(1-x)\colon\ x < 1 \text{ and } x \in M\}$.

THEOREM 2.3. *$(L_f(\alpha, r), L_f(\beta, r)) \in (\mathrm{DS}^*)$ resp. $(L_f(\beta, r), L_f(\alpha, r)) \in (\mathrm{S}^*)$ if and only if*

	DS*	S*
$r = +\infty$	*M bounded*	*always*
$r = 1$	*always*	*N bounded*
$r = 0$	*always*	*M bounded*
$r = -1$	*N bounded*	*always*

Proof. The condition DS* is for Dragilev spaces equivalent to

$$\exists \mu\ \forall p, k\ \exists m, v\ \forall n\ \exists R\ \forall i, j > R\colon$$

$$f(r_p \alpha_j) - f(r_m \alpha_j) < f(r_\mu \beta_i) - f(r_k \beta_i),$$

$$f(r_k \beta_i) - f(r_v \beta_i) < f(r_m \alpha_j) - f(r_n \alpha_j).$$

First we consider the cases with $r > 0$. Since f is rapidly increasing that condition is equivalent to

$$\forall k\ \exists m, v > k\ \forall n > m\ \exists K\ \forall i, j > K\colon$$

$$f(r_m \alpha_j) > f(r_k \beta_i) \quad \text{or} \quad f(r_n \alpha_j) < f(r_v \beta_i),$$

i.e. $r_m/r_k > \beta_i/\alpha_j$ or $r_n/r_v < \beta_i/\alpha_j$. Here we assume $0 < r_k < r_{k+1}$.

For $r = 1$ we have $\forall k\ \exists m, v\ \forall n\colon r_n/r_v < r_m/r_k$, hence the condition is always satisfied.

For $r = +\infty$ the condition has the same form as in the proof of Theorem 2.1 for power series spaces of infinite type, hence we obtain the same result.

For $r \leqslant 0$ we obtain as an equivalent formulation of DS*:

$$\exists \mu\ \forall p, k\ \exists m, J\ \forall i, j > J\colon \quad f(r_p \alpha_j) < f(r_\mu \beta_i) \text{ or } f(r_k \beta_i) < f(r_m \alpha_j),$$

i.e. $r_p/r_\mu > \beta_i/\alpha_j$ or $r_m/r_k < \beta_i/\alpha_j$. This is equivalent to $\exists \mu\ \forall p, k\ \exists m\colon r_p/r_\mu > x$ or $r_m/r_k < x$ for all $x \in M$.

For $r = 0$ this is satisfied for every $x \in \boldsymbol{R}$.

For $r = -1$ this is equivalent to $\exists \delta > 0\ \forall x \in M\colon x < 1 - \delta$ or $x \geqslant 1$, hence $1/(1-x) < 1/\delta$ for all $x \in M$ with $x < 1$.

Finally, we want to point out that our results imply the following facts for every nuclear power series or Dragilev space E: Either every exact sequence of Fréchet spaces $0 \to E \to F \to E \to 0$ splits or the range of every continuous linear mapping $T\colon E \to E$ has a basis. If moreover E is shift stable then E satisfies exactly one of the two relations $(E, E) \in (\mathrm{DS}^*)$ and $(E, E) \in (\mathrm{S}^*)$. On the other hand, if E is an unstable Köthe space with DN or $\bar{\Omega}$, Terzioglu [21] showed that (E, E) satisfies both DS* and S*.

References

[1] H. Ahonen, *On nuclear Köthe spaces defined by Dragilev functions*, Dissertation, Helsinki 1981.

[2] H. Apiola, *Characterization of subspaces and quotients of nuclear $L_f(\alpha, \infty)$-spaces*, Compositio Math. 50 (1983), 65–81.

[3] C. Bessaga, *Some remarks on Dragilev's theorem*, Studia Math. 31 (1968), 307–318.

[4] K.-D. Bierstedt, R. Meise and W. H. Summers, *Köthe sets and Köthe sequence spaces*, in: J. A. Barroso (ed.), Functional Analysis, Holomorphy and Approximation Theory, North-Holland Math. Stud. 71, 1982, 27–91.

[5] S. Dineen, R. Meise and D. Vogt, *Characterization of nuclear Fréchet spaces in which every bounded set is polar*, Bull. Soc. Math. France 112 (1984), 41–68.

[6] M. M. Dragilev, *Standard form of basis for the space of analytic functions*, Uspekhi Mat. Nauk 15 (2) (1960), 181–188 (in Russian).

[7] —, *On regular bases in nuclear spaces*, Amer. Math. Soc. Transl. (2) 93 (1970), 61–82.

[8] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer, 1979.

[9] E. Dubinsky and D. Vogt, *Bases in complemented subspaces of power series spaces*, Bull. Polish Acad. Sci. Math. 34 (1986), 65–67.

[10] U. Fachinger, Diplomarbeit, Wuppertal 1985.

[11] J. Hebbecker, *Auswertung der Splittingbedingungen* (S_1^*) *und* (S_2^*) *für Potenzreihenräume und L_f-Räume*, Diplomarbeit, Wuppertal 1984.

[12] T. Ketonen and K. Nyberg, *Twisted sums of nuclear Fréchet spaces*, Ann. Acad. Sci. Fenn. Ser. AI Math. 7 (1982), 323–335.

[13] J. Krone, *Bases in the range of operators between Köthe spaces*, DOGA Math. Ser. 10/1 (1986), 162–166.

[14] —, *Basisprobleme in Frécheträumen*, Dissertation, Wuppertal 1986.

[15] J. Krone and D. Vogt, *The splitting relation for Köthe spaces*, Math. Z. 190 (1985), 387–400.

[16] B. S. Mityagin, *Sur l'équivalence des bases inconditionnelles dans les échelles de Hilbert*, C. R. Acad. Sci. Paris 269 (1969), 426–428.

[17] —, *Equivalence of bases in Hilbert scales*, Studia Math. 37 (1971), 111–137 (in Russian).

[18] B. S. Mityagin and G. Henkin, *Linear problems in complex analysis*, Uspekhi Mat. Nauk 26 (4) (1971), 93–152 (in Russian).

[19] K. Nyberg, *Tameness of pairs of nuclear power series spaces and related topics*, Trans. Amer. Math. Soc. 283 (1984), 646–660.

[20] A. Pełczyński, *Problem 37*, Studia Math. 38 (1970), 476.

[21] T. Terzioglu, *Unstable Köthe spaces and the functor* Ext, DOGA Math. Ser. 10/1 (1986), 227–231.

[22] D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. 37 (1983), 269–301.
[23] –, *Fréchetträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.
[24] –, *Some results on continuous linear maps between Fréchet spaces*, in: K.-D. Bierstedt and B. Fuchssteiner (eds.), Functional Analysis: Surveys and Recent Results III, North-Holland Math. Stud. 90, 1984, 349–381.
[25] –, *On the functors* $\mathrm{Ext}^1(E, F)$ *for Fréchet spaces*, Studia Math. 85 (1987), 163–197.
[26] –, *Tame spaces and power series spaces*, Math. Z. 196 (1987), 523–536.
[27] M. J. Wagner, *Quotientenräume von stabilen Potenzreihenräumen endlichen Typs*, Manuscripta Math. 31 (1980), 97–109.

FACHBEREICH MATHEMATIK
BERGISCHE UNIVERSITÄT
GESAMTHOCHSCHULE WUPPERTAL
Gaußstr. 20, D-5600 Wuppertal 1, F.R.G.

Received March 2, 1987 (2283)

STUDIA MATHEMATICA, T. XCII. (1989)

C^0-Scalar operators on cyclic spaces

by

RALPH DELAUBENFELS (Athens, O.)

Abstract. We consider (possibly unbounded) linear operators A on a Banach space X for which there exists x_0 such that $X = \overline{\mathrm{span}}\{A^n x_0 \mid n = 0, 1, 2, \ldots\}$. We show that A is C^0-scalar with real spectrum if and only if there exists a certain type of Banach lattice Y of complex-valued functions on the real line and a homeomorphism U from X onto Y such that $(UAU^{-1}f)(t) = tf(t)$, with the domain of UAU^{-1} equal to $\{f \text{ in } Y \mid t \to tf(t) \text{ is in } Y\}$.

When the quasi-analytic growth condition $\sum_{n=0}^{\infty}(\sup_{k\le n}\|A^k x_0\|^{-1/k}) = \infty$ is satisfied, and the domain of A equals span $\{A^n x_0 \mid n = 0, 1, \ldots\}$, we show that $\bar{A}$ is C^0-scalar with real spectrum if and only if there exists an equivalent norm with respect to which X is a Banach lattice, with

$$(X^*)^+ = \{\varphi \text{ in } X^* \mid \{\varphi(A^n x_0)\}_{n=0}^{\infty} \text{ is positive-definite}\}.$$

Similar results for C^0-scalar operators with nonnegative spectrum are given.

Introduction. Scalar-type spectral operators generalize, to an arbitrary Banach space, selfadjoint operators on a Hilbert space. A (possibly unbounded) linear operator A is *scalar* if there exists a projection-valued measure E such that

$$Ax = \lim_{n\to\infty} \int_{-n}^{n} t\, d(E(t)x),$$

with the domain of A equal to the set of all x for which that limit exists. The spectral theorem asserts that a selfadjoint operator is scalar.

Another form of the spectral theorem asserts that A is selfadjoint if and only if it is unitarily equivalent to a multiplication operator on an L^2-space. In particular, if the Hilbert space H is *cyclic*, i.e. equals the closure of the span of $\{A^n x_0 \mid n = 0, 1, 2, \ldots\}$, for some x_0, then there exists a Borel measure m and unitary U from H onto $L^2(\boldsymbol{R}, m)$ such that

$$(UAU^{-1}f)(t) = tf(t).$$

We prove a similar result for scalar operators on a cyclic Banach space. More generally, we consider *C^0-scalar* operators (see Definition 1). These are operators that have a functional calculus defined for continuous functions vanishing at infinity. Scalar operators have a functional calculus defined for