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Behaviour of L^r-Dini singular integrals in weighted L^1 spaces

by

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Abstract. Singular integral operators K with kernels satisfying an L^r-Dini condition of the type introduced by D. S. Kurtz and R. L. Wheeden are considered here. Weighted norm inequalities and weak type estimates for the maximal singular integral operator K^* are obtained in Theorem 1. The convergence in L^1_w of the truncated singular integrals $K_\varepsilon f$ to Kf is proved in Theorem 2 under the assumption that f and Kf belong to L^1_w and $w^{r'} \in A_1$.

1. Notation. Let $\boldsymbol{R}^n$ stand for the n-dimensional euclidean space. As usual, let $(x\cdot y) = \sum_{i=1}^n x_i y_i$ and $|x| = (x\cdot x)^{1/2}$. The ball $B(x, r)$ is the set $\{y: |x-y| < r\}$. If $B = B(x, r)$ and $\lambda > 0$, λB denotes the ball $B(x, \lambda r)$. The cube $Q(x, r)$ is the set $\{y: |x_i - y_i| < r,\ i = 1, \ldots, n\}$ and if $Q = Q(x, r)$ and $\lambda > 0$, λQ will denote the cube $Q(x, \lambda r)$. All the sets and functions that will be considered here are Lebesgue measurable. The Lebesgue measure of a set E is denoted by $|E|$. If $w(x) \geqslant 0$ is a locally integrable function, the measure of E with respect to the weight $w(x)$ is denoted by $w(E) = \int_E w(x)\,dx$. The element of the surface area of the unit sphere $\Sigma = \{x: |x| = 1\}$ is denoted by $d\sigma_x$. The Hardy–Littlewood maximal function $Mf(x)$ is defined as

$$Mf(x) = \sup_{x\in B}\Big(|B|^{-1} \int_B |f(x)|\,dy\Big),$$

where B is a ball. For $r \geqslant 1$, we define $M_r f(x) = M(|f|^r)(x)^{1/r}$. The class A_p of B. Muckenhoupt [11], $1 < p < \infty$, consists of all nonnegative and locally integrable functions $w(x)$ (called *weights*) such that

$$\Big(|B|^{-1} \int_B w(x)\,dx\Big)\Big(|B|^{-1} \int_B w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant c_w$$

for a constant c_w and any ball B. For $p = 1$, $w \in A_1$ means that $Mw(x) \leqslant c_w w(x)$ a.e. The L^p spaces with respect to the measure $w(x)\,dx$ are denoted by L^p_w and the norm in these spaces by $\|f\|_{L^p_w}$.

2. Introduction. In 1979, D. S. Kurtz and R. L. Wheeden introduced in [9] a class of singular integral kernels $k(x) = \Omega(x)/|x|^n$, $x \in \boldsymbol{R}^n$, with the following assumptions on $\Omega(x)$:

(i) $\Omega(x)$ is a positively homogeneous function of degree zero.

(ii) $\Omega \in L^r(\Sigma)$, $1 \leqslant r \leqslant \infty$, and if

$$\omega_r(\delta) = \sup_{|\varrho| < \delta} \Big(\int_{\Sigma} |\Omega(\varrho x) - \Omega(x)|^r \, d\sigma_x\Big)^{1/r},$$

where ϱ denotes a rotation and $|\varrho| = \sup_{x \in \Sigma} |\varrho x - x|$, then

$$\int_0^1 \omega_r(\delta) \frac{d\delta}{\delta} < \infty.$$

(iii) Moreover, $\int_\Sigma \Omega(x)\, d\sigma_x = 0$.

We shall refer to kernels $k(x) = \Omega(x)/|x|^n$ with Ω satisfying (i)–(iii) as *L^r-Dini singular integral kernels.*

For the case $r = 1$, these kernels were introduced in 1967 by A. P. Calderón, M. Weiss and A. Zygmund [3]. They showed that the singular integral operator Kf associated to $k(x)$ and defined as

$$Kf(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} k(x-y) f(y)\, dy$$

is well defined for a.e. x and that it is of strong type (p, p), $1 < p < \infty$, and of weak type (1, 1) with respect to the Lebesgue measure. In the already mentioned paper [9], D. S. Kurtz and R. L. Wheeden considered the action of these singular integral operators on L^p_w spaces, $1 \leqslant p < \infty$, for suitable classes of weights assuming the stronger condition that $k(x)$ be an L^r-Dini singular integral kernel with $r > 1$. We now state the results of [9] that will be needed in this paper:

THEOREM A. *Let $k(x)$ be an L^r-Dini singular integral kernel, $1 < r \leqslant \infty$, and let K be its associated singular integral operator. If*

(i) *$r' < p < \infty$ and $w \in A_{p/r'}$, or*

(ii) *$1 < p < r$ and $w^{-1/(p-1)} \in A_{p'/r'}$,*

then there is a constant c independent of f such that

$$\|Kf\|_{L^p_w} \leqslant c\|f\|_{L^p_w}.$$

When $r < \infty$, we may take $p = r'$ in (i) *and $p = r$ in* (ii).

If (iii) *$w^{r'} \in A_1$, then*

$$w(\{x: |Kf(x)| > s\}) \leqslant cs^{-1}\|f\|_{L^1_w}, \quad s > 0,$$

where c is independent of f and s.

For the case $r = 1$ there are counterexamples to this theorem (see [10]). The case $r = \infty$ had been considered previously by M. Kaneko and S. Yano in [8]. They proved even more; in fact, they considered the maximal singular integral operator K^* defined as

$$K^* f(x) = \sup_{\varepsilon > 0} |K_\varepsilon f(x)|$$

and proved that K^* is of strong type (p, p), $1 < p < \infty$, with respect to weights $w \in A_p$ and of weak type (1, 1) if $w \in A_1$. For $r = 1$, A. P. Calderón and A. Zygmund [5] gave a simple and powerful method for proving the weak type (1, 1) of K^* with respect to the Lebesgue measure.

The main purpose of this paper is to prove the convergence of the truncated singular integral $K_\varepsilon f$ to its limit Kf in L^1_w whenever f and Kf belong to L^1_w. In addition, we obtain norm estimates for the maximal singular integral operator K^*, proving that Theorem A holds with K^* in place of K. In particular, this implies the convergence of $K_\varepsilon f$ to Kf in the norm of L^p_w for p and w suitably related. Using a different technique, R. J. Bagby and D. S. Kurtz obtained part (i) of Theorem A for K^* (see [1], p. 81).

3. The maximal singular integral operator. The following lemma, due to D. S. Kurtz and R. L. Wheeden, is the basis for the study of the L^r-Dini singular integral kernels.

LEMMA 1. *Let $k(x)$ be an L^r-Dini singular integral kernel. There exists a constant $\alpha_0 > 0$ such that if $|y| < \alpha_0 R$, then*

$$\Big(\int_{R < |x| < 2R} |k(x-y) - k(x)|^r \, dx\Big)^{1/r} \leqslant cR^{-n/r'} \Big\{|y|/R + \int_{|y|/(2R) < \delta < |y|/R} \omega_r(\delta)\, d\delta/\delta\Big\}.$$

Proof. See Lemma 5, p. 359 of [9].

LEMMA 2. *Let $k(x)$ be an L^r-Dini singular integral kernel. Assume $1 < r \leqslant \infty$. Then*

$$\int_{|x| \geqslant 4|y|} |k(x-y) - k(x)|\,|f(x)|\, dx \leqslant cM_{r'} f(\xi),$$

where $|\xi| \leqslant 8|y|$. The constant c is independent of f, y and ξ.

Proof. By Hölder's inequality,

$$\int_{|x| \geqslant 4|y|} |k(x-y) - k(y)|\,|f(x)|\, dx \tag{3.1}$$

$$\leqslant \sum_{j=2}^{\infty} \Big(\int_{2^j|y| \leqslant |x| < 2^{j+1}|y|} |k(x-y) - k(x)|^r \, dx\Big)^{1/r} \Big(\int_{|x| < 2^{j+1}|y|} |f(x)|^{r'} \, dx\Big)^{1/r'}.$$

By Lemma 1, (3.1) is bounded by

$$c \sum_{j=2}^{\infty} \Big\{2^{-j} + \int_{2^{-j-1} \leqslant |x| < 2^{-j}} \omega_r(\delta)\, d\delta/\delta\Big\} \big((2^{j+1}|y|)^{-n} \int_{|x| \leqslant 2^{j+1}|y|} |f(x)|^{r'} \, dx\big)^{1/r'}.$$

Then, if $|\xi| \leqslant 8|y|$ we have $|\xi| \leqslant 2^{j+1}|y|$ for $j \geqslant 2$. Thus,

$$\int_{|x| \geqslant 4|y|} |k(x-y)-k(x)|\,|f(x)|\,dx \leqslant c\,\{1+\int_0^1 \omega_r(\delta)\,d\delta/\delta\}\,M_{r'} f(\xi).$$

COROLLARY. *If* $1 < r \leqslant \infty$ *and* $w^{r'} \in A_1$ *then*

(i) $$\int_{|x| \geqslant 4|y|} |k(x-y)-k(x)|\,w(x)\,dx \leqslant cw(y) \qquad y\text{-a.e.},$$

(ii) $$\int_{|x| \geqslant 4|y|} |k(x-y)-k(x)|\,w(x+z)\,dx \leqslant cw(z) \qquad z\text{-a.e.}$$

LEMMA 3 (Cotlar's type lemma, see [6] and [7]). *Let* $k(x)$ *be an* L^r*-Dini singular integral kernel,* $1 < r < \infty$. *Suppose that* $f \in L^p_w$, $w \in A_{p/r'}$ *and* $p \geqslant r'$. *Then*

$$K^* f(x) \leqslant MKf(x) + cM_{r'} f(x)$$

with a constant c *independent of* x *and* f.

Proof. Let $B = B(x, \varepsilon/4)$ and $B' = B(x, \varepsilon)$. Set $f_1 = f\chi_{B'}$ and $f_2 = f - f_1$. Then

$$K_\varepsilon f(x) = Kf_2(x) = Kf(z) - Kf_1(z) + Kf_2(x) - Kf_2(z).$$

Integrating with respect to z on B and dividing by $|B|$, we obtain

(3.2) $$|K_\varepsilon f(x)| \leqslant |B|^{-1}\int_B |Kf(z)|\,dz + |B|^{-1}\int_B |Kf_1(z)|\,dz + |B|^{-1}\int_B |Kf_2(x) - Kf_2(z)|\,dz.$$

Since $z \in B$, we have $|x-z| < \varepsilon/4$. Therefore,

$$|Kf_2(x) - Kf_2(z)| \leqslant \int_{|x-y|>\varepsilon} |k(x-y)-k(z-y)|\,|f(y)|\,dy \leqslant \int_{|y|>4|x-z|} \big|k(y)-k\big(y-(x-z)\big)\big|\,|f(x-y)|\,dy.$$

By Lemma 2, the last integral is bounded by a constant times $M_{r'} f(x)$.

Since K is of strong type (r', r') with respect to the Lebesgue measure, it follows that

$$|B|^{-1}\int_B |Kf_1(z)|\,dz \leqslant c\big(|B'|^{-1}\int_{B'} |f(y)|^{r'}\,dy\big)^{1/r'} \leqslant cM_{r'} f(x).$$

Then from (3.2) we obtain $|K_\varepsilon f(x)| \leqslant MKf(x) + cM_{r'} f(x)$, which implies the lemma.

THEOREM 1. *Let* $k(x)$ *be an* L^r*-Dini singular integral kernel,* $1 < r \leqslant \infty$, *and let* K^* *be the maximal singular integral operator associated to* $k(x)$. *If*

(i) $r' < p < \infty$ *and* $w \in A_{p/r'}$, *or*

(ii) $1 < p < r$ *and* $w^{-1/(p-1)} \in A_{p'/r'}$,

then there is a constant c *independent of* f *such that*

$$\|K^* f\|_{L^p_w} \leqslant c\|f\|_{L^p_w}.$$

When $r < \infty$, *we may take* $p = r'$ *in* (i) *and* $p = r$ *in* (ii).

If (iii) $w^{r'} \in A_1$, *then*

$$w(\{x\colon K^* f(x) > s\}) \leqslant cs^{-1}\|f\|_{L^1_w}, \qquad s > 0,$$

where the constant c *is independent of* f *and* s.

The case $r = \infty$ was considered by M. Kaneko and S. Yano in [8].

Proof. To prove part (i), i.e. $r' < p < \infty$, we observe that if $w \in A_{p/r'}$ then

(3.3) $$\|M_{r'} f\|_{L^p_w} = \big(\int (M|f|^{r'})^{p/r'} w(x)\,dx\big)^{1/p} \leqslant c\big(\int |f|^p w(x)\,dx\big)^{1/p}.$$

On the other hand, since $A_{p/r'} \subset A_p$, by Theorem A we have

(3.4) $$\|MKf\|_{L^p_w} \leqslant c\|Kf\|_{L^p_w} \leqslant c\|f\|_{L^p_w}.$$

Therefore, by Lemma 3, (3.3) and (3.4) we obtain (i). To prove (ii) and (iii) we need the already proved part (i) and the following lemma:

LEMMA 4. *Let* $k(x)$ *be an* L^r*-Dini singular integral kernel,* $1 < r \leqslant \infty$, *and suppose that* $w^{-1/(p-1)} \in A_{p'/r'}$, $1 < p < r$, *or* $w^{r'} \in A_1$ *if* $p = 1$. *Then the maximal singular integral operator* K^* *is of weak type* (p, p), *i.e.*

$$w(\{x\colon K^* f(x) > s\}) \leqslant c(\|f\|_{L^p_w}/s)^p,$$

where the constant c *is independent of* f *and* s.

Proof. The proof of this lemma follows the same lines as the classical proof of A. P. Calderón and A. Zygmund for $p = 1$ and $w = 1$ ([4] and [5]). Therefore, an outline of the proof is given here, pointing out the modifications to their proof.

Let $t > 0$. Since $f \in L^p_w$ and since $w(x)\,dx$ is a doubling measure, we can apply the Calderón–Zygmund decomposition lemma obtaining a sequence of dyadic cubes $\{Q_i\}$ such that

(a) the cubes $\{Q_i\}$ are pairwise disjoint,

(3.5) (b) $t^p < w(Q_i)^{-1}\int_{Q_i} |f(x)|^p w(x)\,dx \leqslant ct^p$,

(c) $|f(x)| \leqslant t$ if $x \notin \bigcup Q_i$, x-a.e.

Since $w^{-1/(p-1)} \in A_{p'/r'} \subset A_{p'}$, we have $w \in A_p$. From $w^{r'} \in A_1$, we have $w \in A_1$, Therefore, for any cube Q,

$$|Q|^{-1}\int_Q |f(x)|\,dx \leqslant c\big(w(Q)^{-1}\int_Q |f(x)|^p w(x)\,dx\big)^{1/p}. \tag{3.6}$$

In particular, the average $m_{Q_i}f = |Q_i|^{-1}\int_{Q_i} f(x)\,dx$ satisfies $|m_{Q_i}f| \leqslant ct$, where c is independent of f and t.

Let $b_i(x) = \big(f(x)-m_{Q_i}f\big)\chi_{Q_i}(x)$ and $b(x) = \sum_i b_i(x)$. We denote by $\bar{Q}_i$ the cube Q_i expanded λ times, with λ chosen so that the distance from Q_i to $\complement\bar{Q}_i$ be twice the diameter of Q_i. For $x \notin \bigcup \bar{Q}_i$, let

$$h(x) = \sum_i \int |k(x-y)-k(x-y_i)|\,a_i(y)\,dy,$$

where y_i is the center of Q_i and $a_i(y) = \big(|b_i(y)|+t\big)\chi_{Q_i}(y)$. It follows from the definitions that

$$\int a_i(y)^p w(y)\,dy \leqslant c\int_{Q_i} |f(y)|^p w(y)\,dy. \tag{3.7}$$

Next, we shall estimate the L^p_w-norm of $h(x)$ on the complement of $\bigcup\bar{Q}_i$. Let $s(x) \geqslant 0$ be supported outside $\bigcup\bar{Q}_i$. Then

$$\int h(x)s(x)\,dx = \sum\int a_i(y)\int |k(x-y)-k(x-y_i)|\,s(x)\,dx\,dy.$$

By Lemma 2, if $1 < p < r$, this expression is bounded by

$$\sum_i \int a_i(y)\,M_{r'}s(y)\,dy$$

$$\leqslant \Big(\int\big(\sum_i a_i(y)\big)^p w(y)\,dy\Big)^{1/p}\Big(\int M_{r'}s(y)^{p'} w(y)^{-1/(p-1)}\,dy\Big)^{1/p'}.$$

Since $w^{-1/(p-1)} \in A_{p'/r'}$, $p' > r'$, and from (3.7) it follows that

$$\int h(x)s(x)\,dx \leqslant c\Big(\int_{\bigcup Q_i} |f(x)|^p w(x)\,dx\Big)^{1/p}\Big(\int s(x)^{p'} w(x)^{-1/(p-1)}\,dx\Big)^{1/p'}.$$

Hence

$$\int_{\complement[\bigcup\bar{Q}_i]} h(x)^p w(x)\,dx \leqslant c\int_{\bigcup Q_i} |f(x)|^p w(x)\,dx, \tag{3.8}$$

where c is independent of f and $t > 0$. This estimate holds even if $p = 1$. In fact, integrating $h(x)$ on $\complement[\bigcup\bar{Q}_i]$ with respect to $w(x)\,dx$, we get

$$\int_{\complement[\bigcup\bar{Q}_i]} h(x)w(x)\,dx \leqslant \sum_i \int a_i(y)\int_{\complement\bar{Q}_i} |k(x-y)-k(x-y_i)|\,w(x)\,dx\,dy.$$

By part (i) of the Corollary of Lemma 2 and (3.7), we obtain

$$\int_{\complement[\bigcup\bar{Q}_i]} h(x)w(x)\,dx \leqslant c\int_{\bigcup Q_i} |f(x)|\,w(x)\,dx. \tag{3.9}$$

Now, following [5], we find that for $x \notin \bigcup\bar{Q}_i$, $K^*b(x) \leqslant c\big(h(x)+t\big)$.

Therefore, if we take $s = 4ct$, by (3.8) or (3.9) and (3.5) (b) we obtain

$$w\big(\{x\colon K^*b(x) > s/2\}\big) \leqslant cs^{-p}\int_{\complement[\bigcup\bar{Q}_i]} h(x)^p\,dx + cw\big([\bigcup\bar{Q}_i]\big) \tag{3.10}$$

$$\leqslant cs^{-p}\int_{\bigcup Q_i} |f(x)|^p w(x)\,dx.$$

Let $g = f-b$. It follows from (3.6) that $|g(x)| \leqslant ct = c's$. Therefore, $g \in L^{p_1}_w$ for any $p_1 > p$. On the other hand, since $w^{-1/(p-1)} \in A_{p'/r'} \subset A_{p'}$, we have $w \in A_p \subset A_{p_1}$. Let us choose $p_1 > pr'$; then $w \in A_{p_1/r'}$ and, by Theorem 1, part (i) already proved, we have

$$w\big(\{x\colon K^*g(x) > s/2\}\big) \leqslant c\big(\|g\|_{L^{p_1}_w}/s\big)^{p_1} \leqslant c\big(\|g\|_{L^p_w}/s\big)^p$$

$$\leqslant c\big(\|f\|_{L^p_w}/s\big)^p.$$

This estimate, together with (3.10), finishes the proof of the lemma.

Now, we can continue with the proof of Theorem 1. We observe that for $1 < p \leqslant r$, if $w^{-1/(p-1)} \in A_{p'/r'}$, then there exists p_0, $1 < p_0 < p \leqslant r$, such that $w^{-1/(p_0-1)} \in A_{p_0'/r'}$. On the other hand, as was shown in the proof of Lemma 4, there is $p_1 > p$ such that $w \in A_{p_1/r'}$. Thus, by Theorem 1(i) and Lemma 4, K^* is of strong type (p_1, p_1) and of weak type (p_0, p_0) with respect to $w(x)\,dx$. Applying the Marcinkiewicz interpolation theorem, we obtain the strong type (p, p) of K^* with respect to the measure $w(x)\,dx$.

This proof includes the limit case $p = r$. For the case $p = r'$ the result follows by the argument used in [8], of interpolation with changes of measures, which remains valid for sublinear operators (see [12]). Finally, (iii) is Lemma 4 for $p = 1$.

4. Convergence in L^1_w of the truncated singular integral. If $k(x)$ is a singular integral kernel, the *truncated kernel* $k_\varepsilon(x)$ is defined as $k_\varepsilon(x) = k(x)$ if $|x| > \varepsilon$ and $k_\varepsilon(x) = 0$ otherwise. The *truncated singular integral* $K_\varepsilon f$ is given by

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x-y)f(y)\,dy = (k_\varepsilon * f)(x).$$

In order to simplify the notation, sometimes we shall write $\tilde{h}(x)$ instead of the singular integral $Kh(x)$.

LEMMA 5. *Let $k(x)$ be an L^r-Dini singular integral kernel, $r > 1$, and let $\varphi(x) \geqslant 0$ be a C^∞ function with support in $|x| \leqslant 1$ and $\int\varphi(x)\,dx = 1$. If $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, then*

$$\delta_\varepsilon(x) = \tilde{\varphi}_\varepsilon(x) - k_\varepsilon(x) \tag{4.1}$$

satisfies

$$\|\delta_\varepsilon * f\|_{L^1_w} \leqslant c\|f\|_{L^1_w}$$

provided that $w^{r'} \in A_1$.

Proof. First, we show that

(4.2) $$|\tilde{\varphi}_\varepsilon(x)| \leqslant c\varepsilon^{-n} \quad \text{if } |x| \leqslant 4\varepsilon.$$

In fact, we have

$$\tilde{\varphi}_\varepsilon(x) = \text{p.v.}\int k(y)\,\varphi_\varepsilon(x-y)\,dy = \int_{|y|\leqslant 5\varepsilon} k(y)\,[\varphi_\varepsilon(x-y)-\varphi_\varepsilon(x)]\,dy.$$

Taking absolute values and recalling that $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, it follows that

$$|\tilde{\varphi}_\varepsilon(x)| \leqslant \|\nabla\varphi\|_\infty \varepsilon^{-n-1} \int_{|y|\leqslant 5\varepsilon} |k(y)|\,|y|\,dy = c\varepsilon^{-n}.$$

Now, let $\delta_\varepsilon^{(1)}(x) = \delta_\varepsilon(x)$ if $|x| \leqslant 4\varepsilon$ and $\delta_\varepsilon^{(1)}(x) = 0$ otherwise. We define $\delta_\varepsilon^{(2)}(x) = \delta_\varepsilon(x) - \delta_\varepsilon^{(1)}(x)$. We shall prove that

$$\|\delta_\varepsilon^{(j)} * f\|_{L^1_w} \leqslant c\|f\|_{L^1_w} \quad \text{for } j = 1, 2,$$

which obviously implies the lemma.

Let us consider $j = 1$. From (4.2) and the definition of $k_\varepsilon(x)$ we get

$$|\delta_\varepsilon^{(1)}(x)| \leqslant c\varepsilon^{-n}(1+|\Omega(x)|).$$

Then

$$\|\delta_\varepsilon^{(1)} * f\|_{L^1_w} \leqslant c\varepsilon^{-n}\int\Big(\int_{|x-y|\leqslant 4\varepsilon} (1+|\Omega(x-y)|)\,|f(y)|\,dy\Big)\,w(x)\,dx$$

$$\leqslant c\int\Big(\varepsilon^{-n}\int_{|x|\leqslant 4\varepsilon}(1+|\Omega(x)|)^r\,dx\Big)^{1/r}$$

$$\times\Big(\varepsilon^{-n}\int_{|x-y|\leqslant 4\varepsilon} w(x)^{r'}\,dx\Big)^{1/r'}|f(y)|\,dy.$$

From the assumption that $w^{r'} \in A_1$ it follows that

$$\|\delta_\varepsilon^{(1)} * f\|_{L^1_w} \leqslant c\Big(\int_\Sigma (1+|\Omega(x)|)^r\,d\sigma_x\Big)^{1/r}\int|f(y)|\,w(y)\,dy.$$

Let us consider the case $j = 2$. If $|x| \geqslant 4\varepsilon$, we have

$$\delta_\varepsilon^{(2)}(x) = \tilde{\varphi}_\varepsilon(x) - k(x) = \int_{|y|\leqslant\varepsilon}[k(x-y)-k(x)]\,\varphi_\varepsilon(y)\,dy.$$

Then

(4.3) $$|\delta_\varepsilon^{(2)}(x)| \leqslant \int_{|y|\leqslant\varepsilon} |k(x-y)-k(x)|\,\varphi_\varepsilon(y)\,dy.$$

Now, we have

$$\|\delta_\varepsilon^{(2)} * f\|_{L^1_w} \leqslant \int\Big(\int|\delta_\varepsilon^{(2)}(x)|\,|f(z-x)|\,dx\Big)\,w(z)\,dz.$$

By Fubini's theorem and a change of variables, we see that the right-hand side of the inequality above is equal to

$$\int\Big(\int|\delta_\varepsilon^{(2)}(x)|\,w(x+z)\,dx\Big)\,|f(z)|\,dz.$$

Thus, using (4.3), we obtain

$$\|\delta_\varepsilon^{(2)} * f\|_{L^1_w}$$

$$\leqslant \int\Big(\int_{|x|\geqslant 4\varepsilon}\Big(\int_{|y|\leqslant\varepsilon}|k(x-y)-k(x)|\,\varphi_\varepsilon(y)\,dy\Big)\,w(x+z)\,dx\Big)\,|f(z)|\,dz$$

$$\leqslant \int\Big(\int\Big(\int_{|x|\geqslant 4|y|}|k(x-y)-k(x)|\,w(x+z)\,dx\Big)\,\varphi_\varepsilon(y)\,dy\Big)\,|f(z)|\,dz.$$

Therefore, by part (ii) of the Corollary of Lemma 2, we obtain

$$\|\delta_\varepsilon^{(2)} * f\|_{L^1_w} \leqslant c\int\varphi_\varepsilon(y)\,dy\int|f(z)|\,w(z)\,dz,$$

which completes the proof of the lemma.

Lemma 6. *Let* $k(x)$ *be an* L^r*-Dini singular integral kernel and* $r > 1$. *Assume that* $w^{r'} \in A_1$, $f \in L^1_w$ *and* $g \in C_0(\mathbf{R}^n)$. *Then*

$$(f * g)^{\tilde{}}(x) = (f * \tilde{g})(x) \quad \text{a.e.}$$

Proof. First assume $r < \infty$. We shall show (writing $L = L^{r'}_{w^{r'}}$ for simplicity)

(4.4) $$\|f * \tilde{g}\|_L \leqslant c_g\|f\|_{L^1_w},$$

(4.5) $$\|(f * g)^{\tilde{}}\|_L \leqslant c_g\|f\|_{L^1_w},$$

where the constant c_g does not dependent on f.

By Minkowski's integral inequality and Theorem A, we have

$$\|f * \tilde{g}\|_L \leqslant \int\Big(\int|\tilde{g}(x-y)|^{r'}\,w(x)^{r'}\,dx\Big)^{1/r'}|f(y)|\,dy$$

$$\leqslant c\int\Big(\int|g(x-y)|^{r'}\,w(x)^{r'}\,dx\Big)^{1/r'}|f(y)|\,dy.$$

If g vanishes outside a ball B, since g is bounded and $w^{r'} \in A_1$, we get

$$\|f * \tilde{g}\|_L \leqslant c|B|\,\|g\|_\infty\int\Big(|B|^{-1}\int_B w(x+y)^{r'}\,dx\Big)^{1/r'}|f(y)|\,dy$$

$$\leqslant c_g\int|f(y)|\,w(y)\,dy,$$

which proves (4.4), Again, by Theorem A and Minkowski's integral inequality, we have

$$\|(f*g)^{\sim}\|_L \leqslant c\|f*g\|_L$$
$$\leqslant c\int\left(\int|g(x-y)|^{r'}w(x)^{r'}dx\right)^{1/r'}|f(y)|\,dy.$$

Arguing as before, we get (4.5).

Let f_k be a sequence of $C_0(\boldsymbol{R}^n)$ functions converging to f in L^1_w. By Lemma 3 of [2] we have $(f_k*g)^{\sim}(x)=(f_k*\tilde{g})(x)$ a.e. Then by (4.4) and (4.5) we obtain

$$\|(f*g)^{\sim}-f*\tilde{g}\|_L \leqslant \|(f*g)^{\sim}-(f_k*g)^{\sim}\|_L+\|f_k*\tilde{g}-f*\tilde{g}\|_L$$
$$\leqslant 2c_g\|f-f_k\|_{L^1_w},$$

showing that $\|(f*g)^{\sim}-f*\tilde{g}\|_L=0$. This proves the lemma for $r<\infty$.

The case $r=\infty$ reduces to the case $1<r<\infty$. In fact, since $w\in A_1$, by the reverse Hölder inequality, there exists $1<r'<\infty$ such that $w^{r'}\in A_1$. On the other hand, the L^∞-Dini condition on $k(x)$ implies the L^r-Dini condition for $r=r'/(r'-1)$. Therefore, we can assume that $k(x)$ and w satisfy the hypothesis for this finite value of r. This ends the proof of the lemma.

Let m be a positive integer and $k=(k_1,\ldots,k_n)$ an n-tuple of integers, i.e. $k\in\boldsymbol{Z}^n$. Let y_k^m stand for the point $2^{-m}k\in\boldsymbol{R}^n$ and $Q_k^m=\{y\colon 2^{-m}k_i\leqslant y_i<2^{-m}(k_i+1),\ 1\leqslant i\leqslant n\}$. We observe that $y_k^m\in Q_k^m$ and that the length of the sides of Q_k^m is 2^{-m}. In addition, for any given m, the family $\{Q_k^m\colon k\in\boldsymbol{Z}^n\}$ is a partition of $\boldsymbol{R}^n$. Let us assume that g is a bounded function with bounded support, $w\in A_1$ and $f\in L^1_w$. Then, for any given positive integer m, we define

$$C_m(f)(x)=\sum_{k\in\boldsymbol{Z}^n}f(x-y_k^m)\int_{Q_k^m}g(y)\,dy. \tag{4.6}$$

The set I of indices k with the property that Q_k^m has a nonempty intersection with the support of g is finite. If $\operatorname{supp}g$ is contained in the unit ball, the points $\{y_k^m\}_{k\in I}$ satisfy $|y_k^m|\leqslant 1+2^{-m}\sqrt{n}$.

In the sequel we shall assume m large enough so that $|y_k^m|\leqslant 2$.

LEMMA 7. *Let $w\in A_1$, $f\in L^1_w$ and let g be a bounded function with bounded support. Then, for every $R>0$,*

$$\lim_{m\to\infty}\int_{|x|\leqslant R}|C_m(f)(x)-(f*g)(x)|\,dx=0.$$

Proof. If $f\in L^1_w$ and $w\in A_1$, it is well known that f is locally integrable with respect to the Lebesgue measure. If the support of g is assumed to be contained in $|x|\leqslant 1$, then the function $f_0(x)=f(x)$ if $|x|\leqslant R+2$ and $f_0(x)=0$ otherwise satisfies

$$C_m(f_0)(x)=C_m(f)(x),\qquad (f_0*g)(x)=(f*g)(x),$$

for every $|x|\leqslant R$. Since $f_0(x)$ belongs to L^1, our lemma reduces to formula (16), p. 325 of [2].

LEMMA 8. *Let $k(x)$ be an L^r-Dini singular integral kernel with $r>1$ and $\tilde{f}=Kf$ the singular integral associated to $k(x)$. If $w^{r'}\in A_1$ and f and $\tilde{f}$ belong to L^1_w, then*

$$(f*g)^{\sim}(x)=(\tilde{f}*g)(x)\quad a.e. \tag{4.7}$$

provided that g is a bounded function with bounded support.

Proof. Without loss of generality we may suppose that $\operatorname{supp}g$ is contained in the unit ball. By Theorem A, Kf is defined for f in L^1_w and since $f*g\in L^1_w$, $(f*g)^{\sim}(x)$ is well defined a.e. Analogously, $\tilde{f}$ in L^1_w implies that $(\tilde{f}*g)(x)$ is defined a.e. Let $T>0$ and $\varepsilon>0$. We choose $R>T+10$ satisfying

$$\int_{|y|>R-2}|f(y)|\,w(y)\,dy<\varepsilon^2. \tag{4.8}$$

Applying Lemma 7 to f and $\tilde{f}$ we see that for m large enough, $|y_k^m|\leqslant 2$,

$$\int_{|y|\leqslant R}|C_m(f)(y)-(f*g)(y)|\,dy<\varepsilon^2, \tag{4.9}$$

$$\int_{|y|\leqslant R}|C_m(\tilde{f})(y)-(\tilde{f}*g)(y)|\,dy<\varepsilon^2. \tag{4.10}$$

Denote by B_T the ball $\{x\colon |x|<T\}$ and let $\chi_R(x)$ be the characteristic function of the ball $\{x\colon |x|<R\}$. Then we have

$$\begin{aligned}(4.11)\quad &\left|\{x\in B_T\colon |(f*g)^{\sim}(x)-(\tilde{f}*g)(x)|>\varepsilon\}\right|\\ &\leqslant\left|\{x\in B_T\colon |(f*g)^{\sim}(x)-(\chi_R(f*g))^{\sim}(x)|>\varepsilon/4\}\right|\\ &\quad+\left|\{x\in B_T\colon |(\chi_R(f*g))^{\sim}(x)-(\chi_R C_m(f))^{\sim}(x)|>\varepsilon/4\}\right|\\ &\quad+\left|\{x\in B_T\colon (\chi_R C_m(f))^{\sim}(x)-C_m(\tilde{f})(x)|>\varepsilon/4\}\right|\\ &\quad+\left|\{x\in B_T\colon |C_m(\tilde{f})(x)-(\tilde{f}*g)(x)|>\varepsilon/4\}\right|\\ &=\alpha+\beta+\gamma+\delta.\end{aligned}$$

Let us estimate α. Since $w^{r'}\in A_1$, by Theorem A(iii), K is of weak type $(1,1)$. Therefore,

$$\begin{aligned}(4.12)\quad \alpha &\leqslant c\left(|B_T|/w(B_T)\right)w\left(\{x\colon |(f*g)^{\sim}(x)-(\chi_R(f*g))^{\sim}(x)|>\varepsilon/4\}\right)\\ &\leqslant c\varepsilon^{-1}\left(|B_T|/w(B_T)\right)\|(1-\chi_R)(f*g)\|_{L^1_w}.\end{aligned}$$

For this last norm, we have

$$\|(1-\chi_R)(f*g)\|_{L^1_w}\leqslant\int_{|x|\geqslant R}\left\{\int|f(y)|\,|g(x-y)|\,dy\right\}w(x)\,dx.$$

Since $\operatorname{supp} g$ is contained in the unit ball, the relevant values of x and y in the integrand satisfy $|x| \geqslant R$ and $|x-y| \leqslant 1$. Therefore, $R-|y| \leqslant |x|-|y| \leqslant 1$, which implies $R-1 \leqslant |y|$. Then by the condition A_1 and (4.8) we get

$$\|(1-\chi_R)(f*g)\|_{L^1_w} \leqslant \int_{|y|\geqslant R-1} |f(y)| \Big\{ \int_{|x-y|\leqslant 1} |g(x-y)|\, w(x)\, dx \Big\} dy$$
$$\leqslant |B(0,1)|\, \|g\|_\infty \int_{|y|\geqslant R-1} |f(y)| \Big\{ |B(0,1)|^{-1} \int_{|x-y|\leqslant 1} w(x)\, dx \Big\} dy$$
$$\leqslant c\|g\|_\infty \int_{|y|\geqslant R-1} |f(y)|\, w(y)\, dy \leqslant c\|g\|_\infty\, \varepsilon^2,$$

which together with (4.12) gives

$$\alpha \leqslant c\big(|B_T|/w(B_T)\big)\varepsilon.$$

Estimation of β. By the weak type (1, 1) of the operator K with respect to the Lebesgue measure and (4.9) we obtain

$$\beta \leqslant \big|\{x:\ |(\chi_R C_m(f))^\sim(x) - (\chi_R(f*g))^\sim(x)| > \varepsilon/4\}\big|$$
$$\leqslant 4c\varepsilon^{-1} \int_{|x|\leqslant R} |C_m(f)(x) - (f*g)(x)|\, dx \leqslant 4c\varepsilon.$$

Estimation of γ. By (4.6), we have

$$(\chi_R C_m(f))^\sim(x) - C_m(\tilde f)(x)$$
$$= \sum_k \text{p.v.} \int k(x-y-y_k^m)[\chi_R(y+y_k^m)-1] f(y)\, dy \int_{Q_k^m} g(z)\, dz.$$

Thus,

$$|(\chi_R C_m(f))^\sim(x) - C_m(\tilde f)(x)|$$
$$\leqslant \sum_k \int |k(x-y-y_k^m) - k(x-y)|\, |\chi_R(y+y_k^m)-1|\, |f(y)|\, dy \int_{Q_k^m} |g(z)|\, dz$$
$$+ \Big|\int k(x-y) f(y) \sum_k [\chi_R(y+y_k^m)-1] \int_{Q_k^m} g(z)\, dz\, dy\Big|$$
$$= I(x) + J(x).$$

If $|y| \leqslant R-2$, since we assume $|y_k^m| \leqslant 2$, we have $|y+y_k^m| \leqslant R$ and therefore $\chi_R(y+y_k^m)-1 = 0$. On the other hand, if $x \in B_T$, since $T < R-10$, we have $|x-y| \geqslant |y|-|x| \geqslant (R-2)-(R-10) = 8$. Thus,

$$(4.13)\qquad \int_{B_T} I(x)\, w(x)\, dx$$
$$\leqslant \sum_k \int_{|y|\geqslant R-2} \Big[\int_{|x-y|\geqslant 8} |k(x-y-y_k^m) - k(x-y)|$$
$$\times w(x)\, dx\Big] |f(y)|\, dy \int_{Q_k^m} |g(z)|\, dz.$$

Now, by part (ii) of the Corollary of Lemma 2, we have

$$\int_{|x-y|\geqslant 8} |k(x-y-y_k^m) - k(x-y)|\, w(x)\, dx$$
$$= \int_{|x|\geqslant 8} |k(x-y_k^m) - k(x)|\, w(x+y)\, dx$$
$$\leqslant \int_{|x|\geqslant 4|y_k^m|} |k(x-y_k^m) - k(x)|\, w(x+y)\, dx \leqslant cw(y) \qquad y\text{-a.e.}$$

Using this in (4.13) and recalling (4.8) we obtain

$$\int_{B_T} I(x)\, w(x)\, dx \leqslant c \int_{|y|\geqslant R-2} |f(y)|\, w(y)\, dy \cdot \|g\|_1 \leqslant c\|g\|_1\, \varepsilon^2.$$

By Chebyshev's inequality and taking into account that $w \in A_1$, we obtain

$$|\{x \in B_T:\ I(x) > \varepsilon/8\}| \leqslant c\big(|B_T|/w(B_T)\big) w\big(\{x \in B_T:\ I(x) > \varepsilon/8\}\big)$$
$$\leqslant 8c\big(|B_T|/w(B_T)\big)\varepsilon^{-1} \int_{B_T} I(x)\, w(x)\, dx \leqslant c\big(|B_T|/w(B_T)\big)\varepsilon.$$

The weak type (1, 1) of the operator K for $w^{r'} \in A_1$ (Theorem A) allows us to write

$$|\{x \in B_T:\ J(x) > \varepsilon/8\}| \leqslant c\big(|B_T|/w(B_T)\big) w\big(\{x:\ J(x) > \varepsilon/8\}\big)$$
$$\leqslant c\varepsilon^{-1}\big(|B_T|/w(B_T)\big) \int_{|y|>R-2} |f(y)|\, w(y)\, dy \cdot \|g\|_1 \leqslant c\big(|B_T|/w(B_T)\big)\varepsilon.$$

Collecting our estimates for I and J we obtain

$$\gamma \leqslant |\{x \in B_T: I(x) > \varepsilon/8\}| + |\{x \in B_T:\ J(x) > \varepsilon/8\}| \leqslant c\big(|B_T|/w(B_T)\big)\varepsilon.$$

Estimation of δ. By Chebyshev's inequality and (4.10), we get

$$\big|\{x \in B_T:\ |C_m(\tilde f)(x) - (\tilde f * g)(x)| > \varepsilon/4\}\big|$$
$$\leqslant 4\varepsilon^{-1} \int_{|x|\leqslant R} |C_m(\tilde f)(x) - (\tilde f * g)(x)|\, dx \leqslant 4\varepsilon.$$

The estimates just obtained for α, β, γ and δ show that (4.11) is smaller than a constant c_T times ε. Letting ε go to zero for any given T, we obtain (4.7).

Lemma 9. *Let $w \in A_1$ and $f \in L^1_w$. Let $\varphi(x)$ be a bounded function with support contained in $|x| \leqslant 1$ and $\int \varphi(x)\, dx = 1$. Then*

(i) $$\|f * \varphi_\varepsilon\|_{L^1_w} \leqslant c\|\varphi\|_\infty \|f\|_{L^1_w},$$

(ii) $$\lim_{\varepsilon\to 0} \|f * \varphi_\varepsilon - f\|_{L^1_w} = 0.$$

Proof. (i) follows from

$$\int |f*\varphi_\varepsilon(x)|\, w(x)\,dx \leqslant \int |f(y)| \int |\varphi_\varepsilon(x-y)|\, w(x)\,dx$$
$$\leqslant \int |f(y)|\, \|\varphi\|_\infty \Big(\varepsilon^{-n} \int_{|x-y|<\varepsilon} w(x)\,dx\Big)\,dy \leqslant c\|\varphi\|_\infty \int |f(y)|\, w(y)\,dy.$$

For (ii), let $\eta > 0$ and let g be a continuous function with compact support such that $\|f-g\|_{L^1_w} < \eta$. Then

$$\|f*\varphi_\varepsilon - f\|_{L^1_w} \leqslant \|(f-g)*\varphi_\varepsilon\|_{L^1_w} + \|g*\varphi_\varepsilon - g\|_{L^1_w} + \|g-f\|_{L^1_w}.$$

The uniform convergence of $g*\varphi_\varepsilon$ to g and part (i) of this lemma show that

$$\limsup_{\varepsilon\to 0} \|f*\varphi_\varepsilon - f\|_{L^1_w} \leqslant (c\|\varphi\|_\infty + 1)\,\eta,$$

proving part (ii) of the lemma.

THEOREM 2. *Let $k(x)$ be an L^r-Dini singular integral kernel, $1 < r$, and $\tilde f = Kf$ the corresponding singular integral operator. If $w^{r'} \in A_1$ and f and $\tilde f$ belong to L^1_w, then*

(i) $$\|f*k_\varepsilon\|_{L^1_w} \leqslant c(\|f\|_{L^1_w} + \|\tilde f\|_{L^1_w}) \quad \textit{for every } \varepsilon > 0,$$

(ii) $$\lim_{\varepsilon\to 0} \|f*k_\varepsilon - \tilde f\|_{L^1_w} = 0.$$

Proof. Let g belong to $C_0^\infty(\boldsymbol{R}^n)$ and let $\delta_\varepsilon(x)$ be defined as in (4.1). We shall first prove that $\|g*\delta_\varepsilon\|_{L^1_w}$ tends to zero as $\varepsilon \to 0$. Assume that $\operatorname{supp} g$ is contained in $|x| \leqslant N$. By applying Lemma 6 twice, we have

$$(\delta_\varepsilon * g)(x) = (\varphi_\varepsilon * \tilde g)(x) - (k_\varepsilon * g)(x).$$

Therefore, adding and subtracting $\tilde g$, and taking into account Lemma 9, and Lemma 2(iii) of [2], we obtain

$$\|\delta_\varepsilon * g\|_{L^r} \leqslant \|\varphi_\varepsilon * \tilde g - \tilde g\|_{L^r} + \|k_\varepsilon * g - \tilde g\|_{L^r} \to 0$$

as $\varepsilon \to 0$. Then

$$\int_{|x|\leqslant 8N} |(\delta_\varepsilon * g)(x)|\, w(x)\,dx \leqslant \|\delta_\varepsilon * g\|_{L^r} \Big(\int_{|x|\leqslant 8N} w(x)^{r'}\,dx\Big)^{1/r'}, \tag{4.14}$$

which also tends to zero as ε tends to zero.

Using Lemma 6, we have, for $|x| \geqslant 4N$ and $0 < \varepsilon < N$,

$$(g*\delta_\varepsilon)(x) = \int_{|y|\leqslant 2N} k(x-y)\,[(\varphi_\varepsilon * g)(y) - g(y)]\,dy.$$

Since $\int_{|y|\leqslant 2N} [(\varphi_\varepsilon * g)(y) - g(y)]\,dy = 0$, we obtain

$$(g*\delta_\varepsilon)(x) = \int_{|y|\leqslant 2N} [k(x-y) - k(x)]\,[(\varphi_\varepsilon * g)(y) - g(y)]\,dy.$$

Multiplying by $w(x)$ and integrating on $|x| \geqslant 8N$, we get

$$\int_{|x|\geqslant 8N} |(g*\delta_\varepsilon)(x)|\, w(x)\,dx$$
$$\leqslant \int_{|y|\leqslant 2N} \Big(\int_{|x|\geqslant 8N} |k(x-y)-k(x)|\, w(x)\,dx \cdot |(\varphi_\varepsilon * g)(y) - g(y)|\Big)\,dy$$
$$\leqslant \int_{|y|\leqslant 2N} \int_{|x|\geqslant 4|y|} |k(x-y)-k(x)|\, w(x)\,dx \cdot |(\varphi_\varepsilon * g)(y) - g(y)|\,dy.$$

Then, by part (i) of the Corollary of Lemma 2, we have

$$\int_{|x|\geqslant 8N} |(g*\delta_\varepsilon)(x)|\, w(x)\,dx \leqslant c\int |(\varphi_\varepsilon * g)(y) - g(y)|\, w(y)\,dy \to 0$$

as $\varepsilon \to 0$. This, together with (4.14), proves that $\|g*\delta_\varepsilon\|_{L^1_w}$ tends to zero, as we wanted to show.

Now, by the very definition (4.1) of $\delta_\varepsilon(x)$ and Lemmas 6 and 8 we have

$$(f*k_\varepsilon)(x) = (f*\tilde\varphi_\varepsilon)(x) - (f*\delta_\varepsilon)(x) = (\tilde f * \varphi_\varepsilon)(x) - (f*\delta_\varepsilon)(x) \quad \text{a.e.}$$

Then by Lemmas 9 and 5 it follows that

$$\|f*k_\varepsilon\|_{L^1_w} \leqslant \|\tilde f * \varphi_\varepsilon\|_{L^1_w} + \|f*\delta_\varepsilon\|_{L^1_w} \leqslant c(\|\tilde f\|_{L^1_w} + \|f\|_{L^1_w}),$$

which proves part (i) of the present theorem.

In order to prove (ii), we choose $g \in C_0^\infty(\boldsymbol{R}^n)$ such that $\|f-g\|_{L^1_w} < \eta$. Then

$$\|f*k_\varepsilon - \tilde f\|_{L^1_w} \leqslant \|\tilde f * \varphi_\varepsilon - \tilde f\|_{L^1_w} + \|(f-g)*\delta_\varepsilon\|_{L^1_w} + \|g*\delta_\varepsilon\|_{L^1_w}.$$

By Lemma 5, $\|(f-g)*\delta_\varepsilon\|_{L^1_w} \leqslant c\|f-g\|_{L^1_w} \leqslant c\eta$. Thus, the convergence to zero of $\|g*\delta_\varepsilon\|_{L^1_w}$ and Lemma 9 imply

$$\limsup_{\varepsilon\to 0} \|f*k_\varepsilon - \tilde f\|_{L^1_w} \leqslant c\eta.$$

The arbitrariness of η proves part (ii) of the theorem.

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Existence of bases and the dual splitting relation for Fréchet spaces

by

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Abstract. The present article contains a topological condition for nuclear Fréchet spaces E and F which is sufficient for the existence of a basis in the range of every continuous linear operator from E to F. Surprisingly our condition is in some sense dual to the ones describing the splitting of short exact sequences of Fréchet spaces.

The problem of Pełczyński [20] whether complemented subspaces of nuclear Fréchet spaces with basis also have a basis is still unsolved. Positive solutions were obtained by Mityagin and Henkin [16, 17, 18] in the case of finite type power series spaces, by Vogt and Dubinsky [9, 26] in the case of tame power series spaces and by Dubinsky [8] and Fachinger [10] in some more general cases.

The present paper modifies the method of Mityagin and Henkin (see 1.1) so that it can give a necessary and sufficient topological condition for the existence of a basis in complemented subspaces of regular nuclear Köthe spaces (see 1.2). This method leads to a sufficient condition on E and F for the range of every operator $T: E \to F$ to have a basis. Here E and F are nuclear Fréchet spaces or Köthe Schwartz spaces or their dual spaces. Some examples of nuclear Köthe spaces without this property are given in Krone [13, 14]. It is an interesting fact that the above-mentioned sufficient condition can be interpreted as the dual splitting relation (see Apiola [2], Ketonen–Nyberg [12], Krone–Vogt [15], Nyberg [19], Vogt [24, 25]). Hence it is called DS and in the case that E and F are Köthe spaces we have a simpler version called DS* (see 1.3).

The condition DS is easy to check in the standard cases. If both spaces are power series spaces we obtain exactly the above-mentioned well-known results (see 2.1). If one space is a shift-stable power series space the characteristic properties of the DS partner spaces are the conditions $\underline{\mathrm{DN}}$, $\overline{\mathrm{DN}}$, Ω and $\tilde{\Omega}$ which are introduced in Vogt [22, 23] and Wagner [27]. In 2.3 we evaluate the DS condition for Dragilev spaces.