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General theorems of Mazur–Orlicz type

by

JOHANN BOOS (Hagen) and TOIVO LEIGER (Tartu)

Abstract. Generalizing the main results in the papers of the first author [14] and the authors [15], we prove a theorem of Mazur–Orlicz type which has new and known consistency theorems as corollaries, e.g. the bounded consistency theorem of Mazur–Orlicz ([31], [32] and [16]) and its generalizations by A. Jakimovski and A. Livne [24] and J. Tzimbalario [45].

1. Introduction. The main result of this paper says that the implication $Y \cap W_E \subset F \Rightarrow Y \cap W_E \subset W_F$ holds for every separable FK-space F, for every FK-space E containing φ and for every sequence space Y having suitable factor sequences. (Here, φ denotes the space of all finite sequences and W_E is the set of all weakly sectionally convergent elements of E.)

This theorem of Mazur–Orlicz type was proved by the first author [14] under the additional assumption that Y is an FK-AB-space and by both authors [15] under this assumption and in the special case that E is a summability domain. The present theorem also generalizes theorems of Mazur–Orlicz type due to G. Bennett and N. J. Kalton ([5] and [7]) and A. K. Snyder [42]. As immediate corollaries we obtain consistency theorems which contain e.g. the well-known bounded consistency theorem of Mazur–Orlicz ([31], [32] and [16]) and its generalizations due to A. Jakimovski and A. Livne [24] and J. Tzimbalario [45].

2. Notation and preliminaries. Though we need almost the same notation and preliminaries as in [14] we write them down again to make this paper self-contained in notation.

As usual, ω, m, f, f_0, c, c_0 and φ denote the vector spaces of all complex (or real) sequences $x = (x_k) = (x_k)_{k \in N}$, of all bounded sequences, of all almost convergent sequences, of all sequences almost converging to zero, of all convergent sequences, of all null sequences and of all finite sequences, respectively.

For fixed p, $1 \leqslant p \leqslant \infty$, let

$$l^p := \{x = (x_k) \,|\, \sum_{k=1}^{\infty} |x_k|^p < \infty\}, \quad l := l^1,$$

and for fixed $\mu = (\mu_k)$, $0 < \mu_k$ $(k \in N)$, let

$$m_\mu := \{x = (x_k) \mid (x_k/\mu_k) \in m\} \quad (\mu\textit{-bounded sequences}),$$

$$c_{0\mu} := \{x = (x_k) \mid (x_k/\mu_k) \in c_0\},$$

$$l_\mu := \{x = (x_k) \mid (\mu_k x_k) \in l\}.$$

If $p = (p_k)$ and $0 < p_k$ $(k \in N)$, then the following notation will be used:

$$l(p) := \{x = (x_k) \mid (|x_k|^{p_k}) \in l\},$$

$$c_0(p) := \{x = (x_k) \mid (|x_k|^{p_k}) \in c_0\},$$

$$m(p) := \{x = (x_k) \mid (|x_k|^{p_k}) \in m\},$$

$$\operatorname{ces}(p) := \{x = (x_k) \mid \sup_n (n^{-1} \sum_{k=1}^{n} |x_k|)^{p_k} < \infty\}.$$

Furthermore, we consider the sequence spaces

$$\mathrm{bv} := \{x = (x_k) \mid \|x\|_{\mathrm{bv}} := |x_1| + \sum_k |x_k - x_{k+1}| < \infty\},$$

$$\delta := \{x = (x_k) \mid \limsup_k |x_k|^{1/k} = 0\}$$

$$= c_0((1/k)) \quad (\textit{entire sequences}),$$

$$d := \{x = (x_k) \mid \sup_k |x_k|^{1/k} < \infty\} \quad (\textit{analytical sequences}),$$

$$d_r := \{x = (x_k) \mid \limsup_k |x_k|^{1/k} < 1/r\} \quad (0 < r < \infty),$$

$$\Pi_r := \{x = (x_k) \mid \limsup_k |x_k|^{1/k} \leqslant 1/r\} \quad (0 < r < \infty),$$

$$|D| := \{x = (x_k) \mid \sup_n \sum_k |d_{nk} x_k| < \infty\}$$

(*absolute-D-bounded sequences*)

for a matrix $D = (d_{nk})$.

Let $e := (1, 1, \ldots)$ and $e^k := (0, \ldots, 0, 1, 0, \ldots)$, where "1" is in the kth position. For fixed $x = (x_k) \in \omega$ and $n \in N$, the *n-th section* of x is

$$x^{[n]} := \sum_{k=1}^{n} x_k e^k = (x_1, \ldots, x_n, 0, \ldots).$$

For a sequence space E we put

$$E^\beta := \{y \in \omega \mid \forall x \in E: \sum_k y_k x_k \text{ exists}\} \quad (\beta\textit{-dual of } E),$$

$$M(E) := \{y \in \omega \mid \forall x \in E: (x_k y_k) \in E\}$$

(*factor sequences from E into E*).

If $m \subset M(E)$ then E is *solid*.

A locally convex sequence space is called a *K-space* if the coordinate functional $x = (x_k) \to x_j$ is continuous for each $j \in N$. A K-space E which is also a Fréchet space is called an *FK-space*; if in addition the topology is normable, then E is called a *BK-space*. If (E, F) is a dual pair then $\sigma(E, F)$ denotes the weak topology. For a sequence space E and $F := E^\beta$ we consider the natural bilinear form. Furthermore, E' denotes the topological dual of a locally convex space E.

For a fixed K-space E with $\varphi \subset E$ we consider distinguished subsets of E:

$$L_E := \{x \in E \mid \{x^{[n]} \mid n \in N\} \text{ is bounded in } E\}$$

(*sectionally bounded sequences*),

$$F_E := \{x \in E \mid \sum_k x_k f(e^k) \text{ exists for each } f \in E'\}$$

(*sequences with functionally convergent sections*),

$$W_E := \{x \in E \mid x^{[n]} \to x \ (\sigma(E, E'))\}$$

(*sequences with weakly convergent sections*),

$$S_E := \{x \in E \mid x^{[n]} \to x \text{ in } E\}$$

(*sectionally convergent sequences*).

Obviously $\varphi \subset S_E \subset W_E \subset F_E \subset L_E$ for every K-space E with $\varphi \subset E$. An *FK-AB-space* (*BK-AB-space*) and an *FK-AK-space* (*BK-AK-space*) is an FK-space (BK-space) E satisfying $E = L_E$ and $E = S_E$, respectively. An FK-space E containing $\varphi \oplus \langle e \rangle$ is called *conull* if $e \in W_E$.

Let $B = (b_{nk}) = (b_{nk})_{n,k \in N}$ be a matrix. We put

$$m_B := \{x \in \omega \mid Bx := (\sum_k b_{nk} x_k)_{n \in N} \text{ exists and } Bx \in m\},$$

$$c_B := \{x \in \omega \mid Bx \text{ exists and } Bx \in c\}$$

(*(convergence) domain of B*),

$$\lim_B x := \lim Bx \text{ for each } x \in c_B.$$

Obviously $\varphi \subset c_B$ if and only if each column of B is convergent. In this case

$b_k := \lim_n b_{nk}$ denotes the limit of the kth column of B. Two matrices A and B are called *consistent* on M, $M \subset c_A \cap c_B$, if $\lim_A x = \lim_B x$ for each $x \in M$. Furthermore, we use the following notation:

$$I_B := \{x \in c_B \mid \sum_k b_k x_k \text{ exists}\} \quad (\textit{inset}),$$

$$\Lambda_B\colon I_B \to C, \qquad x \to \Lambda_B(x) := \lim_B x - \sum_k b_k x_k,$$

$$\Lambda_B^{\perp} := \{x \in I_B \mid \Lambda_B(x) = 0\}.$$

It is well known that c_B is an FK-space, and we write L_B, F_B and W_B instead of L_{c_B}, F_{c_B} and W_{c_B}, respectively. The inclusions

$$\varphi \subset W_B = \Lambda_B^{\perp} \cap L_B \subset F_B = I_B \cap L_B \subset L_B$$

were proved by Wilansky [48] in the case of $c \subset c_B$, and they are also true in the general case $\varphi \subset c_B$ ([49]).A sequence space E will be called *pseudo-conull* if every convergence domain containing E is conull.

If E is a vector space and M is a subset of E, then $\operatorname{conv} M$ denotes the convex hull of M in E and $\bar{M}^{\tau} = \bar{M}$ denotes the closure of M relative to a topology τ on E.

In the following an *index sequence* is a sequence (k_ν) in N with $k_\nu < k_{\nu+1}$ $(\nu \in N)$.

3. Main results and corollaries. Generalizing the main result in [14] we prove a very general theorem of Mazur–Orlicz type which contains also Snyder's theorem of Mazur–Orlicz type [42]. The main part of the proof of this theorem which is very technical will be given in the next section. In the second part of this section we consider general consistency theorems. They generalize the consistency theorems of the authors [15] and contain the well-known bounded consistency theorem of Mazur–Orlicz ([31], [32] and [16]) and its generalizations by Jakimovski and Livne [24] and by Tzimbalario [45].

First of all we define a special class $\mathfrak{E}^*$ of "factor sequences" and the "gliding humps property" of sequence spaces.

Definition 1 (see [14]). Let $y = (y_k) \in \omega$; then, by definition, $y \in \mathfrak{E}^*$ if

$$(y_k - y_{k+1}) \in c_0, \qquad y_k \geqslant 0 \quad (k \in N),$$

and if there exists two index sequences (k_j) and (k_j^*) with the following properties:

$$k_j^* < k_j < k_{j+1}^* \qquad (j \in N),$$

$$y_k = \begin{cases} 0 & \text{if } k_{2\mu-1} < k \leqslant k_{2\mu}^* \ (\mu \in N), \\ 1 & \text{if } k_{2\mu} < k \leqslant k_{2\mu+1}^* \ (\mu \in N), \end{cases}$$

$$y_k \leqslant y_n \quad \text{if } k_{2\mu}^* < k \leqslant n \leqslant k_{2\mu} \ (\mu \in N),$$

$$y_n \leqslant y_k \quad \text{if } k_{2\mu+1}^* < k \leqslant n \leqslant k_{2\mu+1} \ (\mu \in N).$$

Definition 2 (see [42]). Let V be a sequence space containing φ. Assume that for each index sequence (p_n) and for each sequence $(y^{(j)})$ in ω satisfying $y_k^{(j)} = 0$ for $k \notin [p_j, p_{j+1}]$ and $(y^{(j)})$ bounded in bv, there exists a subsequence (q_j) of (p_j) such that the pointwise sum $\sum_j y^{(q_j)}$ is an element of V. Then V will be said to have the *gliding humps property*.

Theorem 1. *Let Y be a sequence space containing φ such that $\mathfrak{E}^* \subset M(Y)$ or such that $M(Y)$ has the gliding humps property. Then*

$$Y \cap W_E \subset F \;\Rightarrow\; Y \cap W_E \subset W_F$$

for every FK-space E containing φ and each separable FK-space F.

This result is a part of the following theorem which was proved by A. K. Snyder ([42], Theorem 9) with very restrictive assumptions on Y and E.

Theorem 2. *Consider the following conditions on a sequence space Y containing φ:*

(i) *$\mathfrak{E}^* \subset M(Y)$ or $M(Y)$ has the gliding humps property.*

(ii) *For all conull FK-spaces E, $W_E \cap M(Y)$ is pseudo-conull.*

(iii) *For all FK-spaces E containing φ and summability domains c_B, $W_E \cap Y \subset W_B$ whenever $W_E \cap Y \subset c_B$.*

(iii*) *For all FK-spaces E containing φ and separable FK-spaces F, $W_E \cap Y \subset W_F$ whenever $W_E \cap Y \subset F$.*

(iv) *For all FK-spaces E containing φ, $(W_E \cap Y)^{\beta}$ is $\sigma\big((W_E \cap Y)^{\beta}, W_E \cap Y\big)$-sequentially complete.*

Then (i)$\Rightarrow$(ii), (i)$\Rightarrow$(iii), (iii)$\Leftrightarrow$(iii*)$\Leftrightarrow$(iv), *and with the additional assumption that Y contains an FK-AK-space,* (ii)$\Rightarrow$(iii).

The proof of the implication (i)$\Rightarrow$(iii) will be given in the next section, and in the case $x := e$ it will be the proof of (i)$\Rightarrow$(ii). The equivalence of (iii), (iii*) and (iv) is a well-known result of G. Bennett and N. J. Kalton ([6], Theorem 5). If Y contains an FK-AK-space, then the proof of (ii)$\Rightarrow$(iii) is quite similar to the proof of Lemma 6 in [42].

In the first remark we demonstrate that the condition "$\mathfrak{E}^* \subset M(Y)$" and "$M(Y)$ has the gliding humps property" is fulfilled by the elements of a large class of sequence spaces Y containing φ. In the second we make some remarks on the bibliography.

Remark 1. (a) By definition, *every solid sequence space Y satisfies the condition $\mathfrak{E}^* \subset M(Y)$*; e.g. this is true if Y equals ω, φ, m, c_0, l^p $(0 < p < \infty)$ or m_μ, $c_{0\mu}$, l_μ $(\mu = (\mu_k),\ 0 < \mu_k\ (k \in N))$ or $m(p)$, $c_0(p)$, $l(p)$, $\operatorname{ces}(p)$ $(\underset{\sim}{p} = (p_k)$ with $p_k > 0$ $(k \in N))$ or δ, d or Π_r, d_r $(r > 0)$ or $|D|$ (for a matrix $\underset{\sim}{D}$). We observe that l^p $(0<p<1)$, d and d_r are not FK-AB-spaces, which is an essential assumption on Y in Theorem 1 of [14].

(b) Let $\{Y_i \mid i\in I\}$ be a family of sequence spaces Y_i fulfilling $\mathfrak{E}^* \subset M(Y_i)$ for each $i\in I$. Then

$$\mathfrak{E}^* \subset M\big(\bigcap_{i\in I} Y_i\big), \qquad \mathfrak{E}^* \subset M\big(\sum_{i\in I} Y_i\big),$$

where $\sum_{i\in I} Y_i$ denotes the linear span of the sequence spaces Y_i $(i\in I)$. Note that $\bigcap_{i\in I} Y_i$ and $\sum_{i\in I} Y_i$ are not necessarily FK-spaces if Y_i is an FK-space for each $i\in I$.

(c) *The sequence space* f_0 *is not solid, but* $\mathfrak{E}^* \subset M(f_0)$ because every $y\in\mathfrak{E}^*$ satisfies $(y_k-y_{k+1})\in c_0\subset f_0$ ([19], Theorem 5); also $M(f_0)$ *has the gliding humps property* (see [42], Theorem 7).

(d) *The sequence space* $Y:=l_\mu\cap m_{C_1}$ $(\mu=(\mu_k)$ with $\mu_k:=k^{-2}$, and C_1 denotes the Cesàro matrix of order 1) *has* the W-hump property (see [45], Theorem 3.4) and therefore *the gliding humps property* (see [45], Theorem 3.3), *but* $\mathfrak{E}^*$ *is not a subset of* $M(Y)$.

Proof of (d). To prove $\mathfrak{E}^* \not\subset M(Y)$ we construct an index sequence (n_r) such that $x=(x_k)$ with

$$x_k := \begin{cases} \sqrt{k} & \text{if } n_{4\nu}\leqslant k<n_{4\nu+1} \\ -\sqrt{k} & \text{if } n_{4\nu+2}\leqslant k<n_{4\nu+3} \\ 0 & \text{otherwise} \end{cases} \qquad (\nu=0,1,\ldots),$$

fulfils $x\in l_\mu\cap m_{C_1}$ but $yx\notin l_\mu\cap m_{C_1}$ for each $y\in\mathfrak{E}^*$ with

$$(*)\qquad y_k=\begin{cases}1 & \text{if } n_{4\nu}\leqslant k<n_{4\nu+1}\\ 0 & \text{if } n_{4\nu+2}\leqslant k<n_{4\nu+3}\end{cases} \qquad (\nu=0,1,\ldots).$$

We start with $n_0:=2$ and $x_1:=x_2:=0$, and we inductively construct (n_r) in the following steps $(\nu=0,1,\ldots)$:

(i) Choose $n_{4\nu+1}>n_{4\nu}$ such that

$$n_{4\nu+1}^{-1}\Big(\sum_{k=1}^{n_{4\nu}-1} x_k+\sum_{k=n_{4\nu}}^{n_{4\nu+1}}\sqrt{k}\Big)>1,$$
$$(n_{4\nu+1}-1)^{-1}\Big(\sum_{k=1}^{n_{4\nu}-1} x_k+\sum_{k=n_{4\nu}}^{n_{4\nu+1}-1}\sqrt{k}\Big)\leqslant 1,$$

and put

$$n_{4\nu+2}:=n_{4\nu+1}+\nu+1, \qquad x_k:=\begin{cases}\sqrt{k} & \text{if } n_{4\nu}\leqslant k<n_{4\nu+1},\\ 0 & \text{if } n_{4\nu+1}\leqslant k<n_{4\nu+2}.\end{cases}$$

(ii) Choose $n_{4\nu+3}>n_{4\nu+2}$ such that

$$n_{4\nu+3}^{-1}\Big(\sum_{k=1}^{n_{4\nu+2}-1} x_k-\sum_{k=n_{4\nu+2}}^{n_{4\nu+3}}\sqrt{k}\Big)<-1,$$
$$(n_{4\nu+3}-1)^{-1}\Big(\sum_{k=1}^{n_{4\nu+2}-1} x_k-\sum_{k=n_{4\nu+2}}^{n_{4\nu+3}-1}\sqrt{k}\Big)\geqslant -1,$$

and put

$$n_{4(\nu+1)}:=n_{4\nu+3}+\nu+1, \qquad x_k:=\begin{cases}-\sqrt{k} & \text{if } n_{4\nu+2}\leqslant k<n_{4\nu+3},\\ 0 & \text{if } n_{4\nu+3}\leqslant k<n_{4(\nu+1)}.\end{cases}$$

Evidently $x\in l_\mu\cap m_{C_1}$. Also, it is a routine calculation to verify $yx\notin l_\mu\cap m_{C_1}$ for each $y\in\mathfrak{E}^*$ fulfilling $(*)$.

Remark 2. (a) In the case of an FK-space E with $c_0\subset E$ and in the case $Y:=m$ and $Y:=f_0$, Theorem 1 was proved by Bennett and Kalton ([5], Theorem 16, and [7], Theorem 9, respectively). The assumption $c_0\subset E$ is a decisive factor for the proofs of Bennett and Kalton.

(b) Theorem 2 was proved by Snyder ([42], Theorem 9), but with restrictive assumptions on Y and E. He considered only the second part in (i) and sequence spaces Y and E such that there exists a semiconservative BK-AK-space with the following properties: φ is dense in K_0^β, $K_0\subset Y\subset M(K_\beta)$ and $K_0\subset E$.

(c) The authors ([15], Satz 1) proved Theorem 1 in the case of FK-AB-spaces Y fulfilling $\mathfrak{E}\subset M(Y)$ and in the case $\varphi\subset E:=c_A$, where A is a matrix and $\mathfrak{E}$ is a special class of sequences with $\mathfrak{E}\subset\mathfrak{E}^*$. (We remark that the usual sequence spaces Y satisfy $\mathfrak{E}^*\subset M(Y)$ if $\mathfrak{E}\subset M(Y)$.)

(d) More generally than the authors in [15], the first author [14] proved Theorem 1 for FK-spaces E containing φ and for FK-AB-spaces Y satisfying $\mathfrak{E}^*\subset M(Y)$.

Undoubtedly, the significance of the theorems lies in the consistency theorems following as corollaries from these results and generalizing the consistency theorems in [15]. First of all, we formulate Theorem 1 in the case of matrix domains and, as an immediate corollary, a limit formula and a consistency theorem.

Corollary 1. *Let* Y *be a sequence space containing* φ *such that* $\mathfrak{E}^*\subset M(Y)$ *or such that* $M(Y)$ *has the gliding humps property, and let* A *and* B *be matrices with* $\varphi\subset Y\cap W_A\subset c_B$. *Then:*

(a) $Y\cap W_A\subset W_B$, *especially* $\lim_B x=\sum b_k x_k$ $(x\in Y\cap W_A)$.

(b) *The consistency of* A *and* B *on* φ *implies the consistency on* $Y\cap W_A$.

More useful and of a common type is

Corollary 2. *Let Y be a sequence space containing φ such that $\mathfrak{E}^* \subset M(Y)$ or such that $M(Y)$ has the gliding humps property, and let A and B be matrices with $\varphi \subset Y \cap F_A \subset c_B$. Choose $u \in \{0\} \cup (Y \cap (F_A \setminus W_A))$ such that $Y \cap F_A = (Y \cap W_A) \oplus \langle u \rangle$. If $u \in I_B$ then*

$$\lim_B x = \alpha(\lim_A x - \sum_{k=1}^{\infty} a_k x_k) + \sum_{k=1}^{\infty} b_k x_k \quad (x \in Y \cap F_A)$$

with $\alpha := 0$ for $u = 0$ and $\alpha := \Lambda_B(u)/\Lambda_A(u)$ for $u \neq 0$.

Consistency Theorem. *The consistency of A and B on $\varphi \oplus \langle u \rangle$* (i.e. $a_k = b_k$ $(k \in N)$ and $\lim_A u = \lim_B u$) *implies the consistency on $Y \cap F_A$.*

Proof. The case $u = 0$ is discussed in Corollary 1. Let $0 \neq u \in I_B$ and $x \in Y \cap F_A$. Then $x = (x - \alpha_x u) + \alpha_x u$ with $\alpha_x := \Lambda_A(x)/\Lambda_A(u)$ and $x - \alpha_x u \in Y \cap W_A \subset W_B$. Therefore

$$\lim_B x = \sum_{k=1}^{\infty} b_k x_k - \alpha_x \sum_{k=1}^{\infty} b_k u_k + \alpha_x \lim_B u$$

$$= \alpha \Lambda_A(x) + \sum_{k=1}^{\infty} b_k x_k \quad \text{with } \alpha := \Lambda_B(u)/\Lambda_A(u).$$

The consistency theorem follows immediately from this formula. (We remark that on account of $a_k = b_k$ $(k \in N)$ and $u \in F_A \subset I_A$ the assumption $u \in I_B$ is fulfilled automatically.)

At the end of this section we give remarks to the bibliography of consistency theorems presented in the corollaries above.

Remark 3. (a) First of all we observe that Corollaries 1 and 2, especially the consistency theorem, are applicable to all sequence spaces Y listed in Remark 1(a). Furthermore, in the case $Y := f$ we refer to the limit formula in [15], Corollaries 4 and 5, which generalizes some results of Bennett and Kalton (see [7], pp. 41–42).

(b) If $Y := m$ and if A and B are regular matrices then the consistency theorem in Corollary 2 is the well-known bounded consistency theorem of Mazur, Orlicz and Brudno: *regular matrices A and B with $m \cap c_A \subset c_B$ are consistent on $m \cap c_A$*. (Note that $m \cap c_A = m \cap F_A = (m \cap W_A) \oplus \langle e \rangle$ in this case). In 1933 Mazur and Orlicz [31] formulated this theorem and proved it later in [32], Theorem 2. Brudno (see [16], Theorem 1) proved it independently of Mazur and Orlicz. Further proofs of the bounded consistency theorem were given for example by Petersen [37] by considering factor sequences (see also [36] and [50]), by Orlicz [33] (see also [34]) using continuous linear functionals in Saks spaces, by Bennett and Kalton [5] with the aid of two-norm spaces and the sequential completeness of $(l, \sigma(l, m \cap W_A))$, by Snyder and Wilansky [43] and Ruckle [40] proving the sequential completeness of $(l, \sigma(l, m \cap W_A))$ and by Snyder and Wilansky [44] reducing it to a well-known theorem of Agnew. The bounded consistency theorem is also an immediate corollary of a quotient theorem of Baumann (see [4], Satz 1, [10], Satz 3 and [13]). The validity of the bounded consistency theorem in the case of conull matrices was characterized by Chang *et al.* [17].

(c) Let A and B be conservative matrices which are consistent on c and let $Y := |A| \cap |B|$. In this special case the consistency theorem included in Corollary 2 was proved for coregular matrices by Volkov [47]; the conull case was studied by the first author [8].

(d) Furthermore, the consistency theorem in Corollary 2 was proved by the first author ([8], Satz 3) for $Y := \omega$ and for coregular matrices A and B consistent on c; the conull case is also discussed in [8].

(e) In 1972 Jakimovski and Livne ([24], Theorem 2.2) gave an extension of the bounded consistency theorem, which was again extended by Tzimbalario in 1973 ([45], Theorem 5.1). A straightforward calculation shows that Tzimbalario's theorem, and therefore the theorem of Jakimovski and Livne, are now generalized by the consistency theorem in Corollary 2.

(f) Concluding these remarks we point out some other papers containing consistency theorems which are related to the bounded consistency theorems: [1]–[3], [9], [11], [12], [18], [20]–[23], [25]–[27], [29], [30], [35]–[39], [41] and [46]. Other references to consistency theorems of Mazur–Orlicz type can be found in [51] and [28].

4. Proof of the main result. Corresponding to the proofs of [15], Satz 1 and [14], Theorem 1, the implication (i)⇒(iii) in Theorem 2 is proved if we show that

$$Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset L_B \Rightarrow Y \cap W_E \subset F_B \Rightarrow Y \cap W_E \subset W_B$$

for each matrix B, every FK-space E containing φ and each sequence space Y containing φ such that $\mathfrak{E}^* \subset M(Y)$ or such that $M(Y)$ has the gliding humps property. The proofs of the second and third implications will run similarly to those in [14] because for these statements in [14] we did not use the FK-AB-space property of Y. In contrast to [14], Y is not necessarily an FK-AB-space in this paper. Therefore we have to prove the first implication with classical arguments too. For this purpose we refine essentially Lemmas 1 and 3 in [14].

Lemma 1. *Let (E, τ) be an FK-space containing φ, and let (η_ν) be an index sequence. If $x \in W_E$, then*

$$x \in \overline{\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in N\}}^{\tau}$$

and there exists a sequence $(x^{(r)})$ of convex combinations

$$(*)\qquad x^{(r)} = \sum_{\nu=s_r}^{t_r} \mu_{r\nu} x^{[\eta_\nu]} \quad (s_r, t_r \in N \text{ with } s_r < t_r < s_{r+1},$$

$$0 \leqslant \mu_{r\nu} \leqslant \frac{1}{r+1},\ \mu_{rs_r} = \frac{1}{r+1},\ \mu_{rt_r} \neq 0,\ \sum_{\nu=s_r}^{t_r} \mu_{r\nu} = 1)$$

of sections $x^{[\eta_\nu]}$ of x such that

$$(**)\qquad x^{(r)} \to x \quad \text{in } (E, \tau).$$

Proof. Applying [14], Lemma 1, we can choose a sequence $(y^{(r)})$ of convex combinations

$$y^{(r)} = \sum_{\nu=s_r}^{t_r} \lambda_{r\nu} x^{[\eta_\nu]} \quad (s_r, t_r \in N \text{ with } s_r < t_r,$$

$$0 \leqslant \lambda_{r\nu} \leqslant \frac{1}{r+1},\ \lambda_{rt_r} \neq 0,\ \sum_{\nu=s_r}^{t_r} \lambda_{r\nu} = 1)$$

of sections $x^{[\eta_\nu]}$ of x such that $y^{(r)} \to x$ in (E, τ). We put

$$\mu_{rs_r} := \frac{1}{r+1}, \qquad \mu_{r\nu} := \lambda_{r\nu} - m_{r\nu}\left(\frac{1}{r+1} - \lambda_{rs_r}\right) \quad (\nu = s_r+1, \ldots, t_r)$$

where $m_{r\nu}$ $(\nu = s_{r+1}, \ldots, t_r)$ is chosen such that $m_{r\nu} \geqslant 0$, $\mu_{r\nu} \geqslant 0$, $\mu_{rt_r} \neq 0$, $\sum_{\nu=s_r}^{t_r} \mu_{r\nu} = 1$. Then we have

$$\sum_{\nu=s_r+1}^{t_r} m_{r\nu} = 1, \quad \text{hence} \quad \sum_{\nu=s_r+1}^{t_r} m_{r\nu} x^{[\eta_\nu]} \in \operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in N\},$$

and therefore

$$x^{(r)} - y^{(r)} = \left(\frac{1}{r+1} - \lambda_{rs_r}\right) x^{[\eta_{s_r}]} + \left(\frac{1}{r+1} - \lambda_{rs_r}\right) \sum_{\nu=s_r+1}^{t_r} \lambda_{r\nu} x^{[\eta_\nu]}$$

$$\to 0 \ (r \to \infty) \quad \text{in } (E, \tau)$$

because $(1/(r+1) - \lambda_{rs_r}) \in c_0$ and $\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in N\}$ is bounded. Thus $(x^{(r)})$ satisfies the conditions $(*)$ and $(**)$.

The next lemma is identical with Lemma 2 in [14] and is due to Snyder [42].

LEMMA 2. *Let E be an FK-space containing φ and let (γ_j) be an index sequence with $\gamma_1 = 1$. Furthermore, let $y = (y_k) \in \omega$ and*

$$y^{(j)} := \sum_{k=\gamma_j}^{\gamma_{j+1}-1} y_k e^k \quad (j \in N) \quad \text{such that} \quad \sup_j \|y^{(j)}\|_{bv} < \infty.$$

Then for each $x \in W_E$ the conditions $yx \in E$ and $yx = \sum_{j=1}^{\infty} y^{(j)} x$ in $(E, \sigma(E, E'))$ imply $yx \in W_E$.

The basis of the next lemma is Lemma 3 in [14], but for the proof of the main result we have to extend the statement of this lemma and to refine essentially the method of proof.

LEMMA 3. *Let (E, τ) be an FK-space with $\varphi \subset E$ and let Y be a sequence space containing φ such that $\mathfrak{E}^* \subset M(Y)$ or such that $M(Y)$ has the gliding humps property. Let $B = (b_{nk})$ be a matrix with $\varphi \subset c_B$. Then for every $x \in W_E \cap c_B$ each of the following statements implies the existence of a sequence $y \in M(Y)$ such that $yx \in W_E$ and $yx \notin c_B$:*

(i) *There exists an index sequence (η_ν) with*

$$\lim_\nu \Big|\sum_{k=1}^{\eta_\nu} b_k x_k\Big| \neq \lim_B x.$$

(ii) $\sup_\nu \left|\sum_{k=1}^{\nu} b_k x_k\right| = \infty$.

(iii) $x \in \Lambda_B^{\perp} \backslash L_B$.

Proof. (i) Let $x \in W_E \cap c_B$ and let (η_ν) be an index sequence with

$$\alpha := \lim_\nu \sum_{k=1}^{\eta_\nu} b_k x_k \neq \lim_B x =: d.$$

We have to prove the existence of a $y \in M(Y)$ such that $yx \in W_E$ and $yx \notin c_B$. In the special case of $x := e$ and $\mathfrak{E}^* \subset M(Y)$ this is Lemma 3 in [14]. For a general $x \in W_E$ the proof is quite similar, but with some more complicated details. For that reason and because we need a similar construction in the second part of the proof, we carry out the construction of y.

Without loss of generality we may assume that

$$\Big|\sum_{k=\eta_\nu+1}^{\eta_{\nu+\mu}} b_k x_k\Big| < 2^{-\nu} \quad (\nu, \mu \in N).$$

According to Lemma 1 we choose a sequence $(x^{(r)})$ in $\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in N\}$ such that

$$(1)\qquad x^{(r)} \to x \quad \text{in } (E, \tau),$$

$$(2)\qquad x^{(r)} = \sum_{\nu=s_r}^{t_r} \mu_{r\nu} x^{[\eta_\nu]} \qquad (s_r,\, t_r \in N \text{ with } s_r < t_r < s_{r+1},$$

$$0 \leqslant \mu_{r\nu} \leqslant \frac{1}{r+1},\ \mu_{rt_r} \neq 0,\ \sum_{\nu=s_r}^{t_r} \mu_{r\nu} = 1).$$

As in the proof of Lemma 3 in [14] we inductively construct index sequences (k_j), (n_j), (r_j) and (k_j^*). For that purpose let $\alpha_j := 2^{-j}$ $(j \in N)$ and let ! ! be a paranorm generating the FK-topology τ of E. We start with $k_1^* := k_1 := 1$, and because of $\varphi \subset c_B$ we may choose an $n_1 \in N$ such that

$$\sum_{k=1}^{k_1} |b_{n_1 k} - b_k|\,|x_k| = |b_{n_1 1} - b_1|\,|x_1| < \alpha_1.$$

In addition, by (1) and $x \in c_B$ we may choose an $r_1 \in N$ such that

$$!x^{(r)} - x^{(r+\mu)}! < \alpha_1 \qquad (r \geqslant r_1,\ \mu \in N),$$

$$\Big|\sum_{k=\nu}^{\nu+\mu} b_{nk} x_k\Big| < \alpha_2 \qquad (n \leqslant n_1,\ \nu \geqslant \eta_{s_{r_1}},\ \mu \in N).$$

We assume that k_{j-1}^*, k_{j-1}, n_{j-1} and r_{j-1} have been chosen. Then we put

$$(3)\qquad k_j^* := \eta_{s_{r_{j-1}}}, \qquad k_j := \eta_{t_{r_{j-1}}}.$$

In particular,

$$(4)\qquad k_{j-1}^* < k_{j-1} < k_j^* < k_j$$

(cf. (2)). Since $\varphi \subset c_B$ we may take $n_j > n_{j-1}$ such that

$$(5)\qquad \sum_{k=1}^{k_j} |b_{n_j k} - b_k|\,|x_k| < \alpha_j.$$

Furthermore, by (1) and $x \in c_B$ we may choose $r_j > r_{j-1}$ so that

$$(6)\qquad !x^{(r)} - x^{(r+\mu)}! < \alpha_{j+1} \qquad (r \geqslant r_j,\ \mu \in N),$$

$$(7)\qquad \Big|\sum_{k=\nu}^{\nu+\mu} b_{nk} x_k\Big| < \alpha_{j+1} \qquad (n \leqslant n_j,\ \nu \geqslant \eta_{s_{r_j}},\ \mu \in N).$$

Now we define

$$(8)\qquad z^{(j)} := x^{(r_{2j})} - x^{(r_{2j-1})} \qquad (j \in N),$$

$$(9)\qquad z := \sum_{j=1}^{\infty} z^{(j)} \qquad \text{(pointwise sum)}.$$

On account of the representation of $x^{(r)}$ in (2) we may choose $y^{(j)}$ $(j \in N)$ with

$z^{(j)} = y^{(j)} x$ and

$$(10)\qquad y_k^{(j)} = \begin{cases} 0 & (k \leqslant k_{2j}^* \text{ or } k > k_{2j+1}), \\ 1 & (k_{2j} < k \leqslant k_{2j+1}^*) \end{cases}$$

because of (3), (8) and (2), and

$$(11)\qquad 0 \leqslant y_{k+1}^{(j)} - y_k^{(j)} \leqslant \frac{1}{r_{2j-1}+1} \qquad (k_{2j}^* \leqslant k < k_{2j+1}^*),$$

$$(12)\qquad 0 \leqslant y_k^{(j)} - y_{k+1}^{(j)} \leqslant \frac{1}{r_{2j}+1} \qquad (k_{2j} \leqslant k < k_{2j+1})$$

because of (2) and (8). In particular, we have

$$(13)\qquad \|y^{(j)}\|_{bv} = 2 \qquad (j \in N).$$

Obviously,

$$(14)\qquad z = yx = \sum_j y^{(j)} x = \sum_j z^{(j)} \qquad \text{where } y := \sum_j y^{(j)}$$

(pointwise sum, cf. (9)) and we may conclude

$$(15)\qquad y_k = \begin{cases} 0 & (k_{2j-1} < k \leqslant k_{2j}^*) \\ 1 & (k_{2j} < k \leqslant k_{2j+1}^*) \end{cases} \qquad (j \in N)$$

because of (10), (4) and (14), and

$$(16)\qquad 0 \leqslant y_k \leqslant 1 \quad (k \in N), \qquad (y_k - y_{k+1}) \in c_0, \qquad y \in \mathfrak{E}^*$$

because of (10)–(12) and (15). In addition, $y \in M(Y)$ if $\mathfrak{E}^* \subset M(Y)$, and if $M(Y)$ has the gliding humps property, we may also assume $y \in M(Y)$ because $(y^{(j)})$ is a sequence as in Definition 2 (cf. (10)–(13)) and we may consider a subsequence of $(y^{(j)})$ (and therefore the corresponding subsequences of (k_j), (n_j), (r_j) and (k_j^*)). Thus, in both cases we have $y \in M(Y)$.

Futhermore, by (6), the sequence $(\sum_{j=1}^{N} z^{(j)})_{N \in N}$ is a Cauchy sequence in (E, τ), and therefore (cf. (14))

$$yx = \sum_j y^{(j)} x \quad \text{in } (E, \tau).$$

Evidently, y and $(y^{(j)})$ satisfy the conditions in Lemma 2 and therefore $yx \in W_E$. It remains to prove $yx \notin c_B$. This part of the proof runs similarly to that of Lemma 3 in [14]. We have only to replace d_k by $b_k x_k$, d_{nk} by $b_{nk} x_k$ and $y \notin c_B$ by $yx \notin c_B$.

(ii) Now we assume $x \in W_E \cap c_B$ and $\sup_\nu |\sum_{k=1}^{\nu} b_k x_k| = \infty$. Then without loss of generality we may choose an index sequence (η_ν) satisfying

$\operatorname{Re}\sum_{k=1}^{\eta_\nu} b_k x_k \geq 0$ $(\nu \in N)$ and

$$\operatorname{Re}\sum_{k=1}^{\eta_{\nu+1}} b_k x_k - \operatorname{Re}\sum_{k=1}^{\eta_\nu} b_k x_k \geq \nu \quad (\nu \in N). \tag{17}$$

The construction of the desired $y \in M(Y)$ goes as in (i) starting with "According to Lemma 1..." and finishing with ".... therefore $yx \in W_E$".

It remains to prove $yx \notin c_B$. Obviously, we may assume that $\sum_k b_{nk} y_k x_k$ exists for each $n \in N$. We prove the divergence of $(\sum_k b_{n_j k} y_k x_k)_{j \in N}$ considering odd integers j.

Let j be an odd integer, i.e. $j := 2\mu - 1$ for a suitable $\mu \in N$. Then we obtain (cf. (15))

$$\sum_k b_{n_j k} x_k y_k = \sum_{k=1}^{k_{2\mu-1}} (b_{n_j k} - b_k) x_k y_k + \sum_{k=1}^{k_{2\mu-1}} b_k x_k y_k + \sum_{k=k^*_{2\mu}+1}^{\infty} b_{n_j k} x_k y_k =: A_j + A_j^* + C_j.$$

As in the proof of Lemma 3 in [14] one proves $A_j \to 0$ and $C_j \to 0$ $(j \to \infty)$:

$$|A_j| \leq \sum_{k=1}^{k_{2\mu-1}} |b_{n_j k} - b_k| |x_k| y_k < \alpha_{2\mu-1} = \alpha_j \to 0$$

because of (5) and (16), and

$$|C_j| \leq \sum_{\nu=\mu}^{\infty} \Big| \sum_{k=k^*_{2\nu}+1}^{k_{2\nu+1}} b_{n_j k} x_k \Big| \quad \text{(see (15))}$$

$$< 2\sum_{\nu=\mu}^{\infty} \alpha_{2\nu} \quad \text{(because of (3), (7) and (16))}$$

$$\to 0 \quad (\mu \to \infty, \text{ thus } j \to \infty).$$

By the construction of y we obtain

$$\sum_{k=1}^{\eta_\nu} b_k x_k y_k = \sum_{\varrho=1}^{\nu-1} y_{\eta_{\varrho+1}} \sum_{k=\eta_\varrho+1}^{\eta_{\varrho+1}} b_k x_k = \sum_{\varrho=1}^{\nu-1} y_{\eta_{\varrho+1}} (u_{\varrho+1} - u_\varrho) + i \sum_{\varrho=1}^{\nu-1} y_{\eta_{\varrho+1}} (v_{\varrho+1} - v_\varrho)$$

where u_ν and v_ν denote the real part and the imaginary part of $\sum_{k=1}^{\eta_\nu} b_k x_k$, respectively. With $0 \leq y_k \leq 1$ $(k \in N)$, (15) and (17) we conclude that the sequence

$$\Big(\sum_{\varrho=1}^{\nu-1} y_{\eta_{\varrho+1}} (u_{\varrho+1} - u_\varrho)\Big)_{\nu \in N}$$

is increasing and unbounded. Therefore each subsequence of $(\sum_{k=1}^{\eta_\nu} b_k x_k y_k)_{\nu \in N}$, hence (A_j^*), is unbounded. This proves $yx \notin c_B$.

(iii) Let $x \in W_E \cap c_B$ and $x \in \Lambda_B^{\perp} \setminus L_B$, i.e.

$$d := \lim_B x = \sum_k b_k x_k, \qquad \sup_{n,\nu} \Big| \sum_{k=1}^{\nu} b_{nk} x_k \Big| = \infty.$$

Because of the last condition we may choose index sequences (η_ν) and (μ_ν) with

$$\Big| \sum_{k=1}^{\eta_\nu} b_{\mu_\nu k} x_k \Big| \to \infty \quad (\nu \to \infty)$$

and hence

$$\Big| \sum_{k=\eta_\nu+1}^{\infty} b_{\mu_\nu k} x_k \Big| \to \infty \quad (\nu \to \infty).$$

Without loss of generality we may assume

$$\frac{1}{\nu+1} \Big| \sum_{k=\eta_\nu+1}^{\infty} b_{\mu_\nu k} x_k \Big| \to \infty \quad (\nu \to \infty),$$

$$\Big| \sum_{k=l}^{l+p} b_{nk} x_k \Big| < \alpha_\nu \quad (n \leq \mu_\nu,\ l \geq \eta_{\nu+1},\ p \in N) \tag{18}$$

and (since $x \in I_B$)

$$\Big| \sum_{k=\eta_\nu+1}^{\eta_\nu+\mu} b_k x_k \Big| < \alpha_\nu \quad (\nu, \mu \in N) \tag{19}$$

where $\alpha_\nu := 2^{-\nu}$ $(\nu \in N)$. Therefore we obtain

$$\frac{1}{\nu+1} \Big| \sum_{k=\eta_\nu+1}^{\eta_{\nu+1}} b_{\mu_\nu k} x_k \Big| \geq \frac{1}{\nu+1} \Big| \sum_{k=\eta_\nu+1}^{\infty} b_{\mu_\nu k} x_k \Big| - \frac{1}{\nu+1} \alpha_\nu$$

and thus

$$\frac{1}{\nu+1} \Big| \sum_{k=\eta_\nu+1}^{\eta_{\nu+1}} b_{\mu_\nu k} x_k \Big| \to \infty \quad (\nu \to \infty). \tag{20}$$

For (η_ν) we choose according to Lemma 1 a sequence $(x^{(r)})$ in $\operatorname{conv}\{x^{[\eta_\nu]} \mid \nu \in N\}$ satisfying (1), (2) and

$$\mu_{rs_r} = 1/(r+1) \quad (r \in N). \tag{21}$$

Then we inductively construct index sequences (k_j), (k_j^*), (l_j), (r_j) and (n_j). Starting with $l_1 := k_1^* := k_1 := 1$ we choose $r_1 \in N$ with

$$\|x^{(r)} - x^{(r+\mu)}\| < \alpha_1 \quad (r \geq r_1,\ \mu \in N),$$

$$|b_{n_1 k} - b_k| |x_k| < \alpha_1 \quad \text{where } n_1 := \mu_{s_{r_1}}.$$

Furthermore, we put

$$k_2^* := \eta_{s_{r_1}}, \quad l_2 := \eta_{s_{r_1}+1}, \quad k_2 := \eta_{t_{r_1}}.$$

Then we have

$$\Big|\sum_{k=l+1}^{l+p} b_{nk} x_k\Big| < \alpha_1 \quad (n \leqslant n_1,\ l \geqslant l_2,\ p \in N)$$

because of (18). If $n_{j-1}, r_{j-1}, k_{j-1}^*, l_{j-1}$ and k_{j-1} are defined, then we put

$$k_j^* := \eta_{s_{r_{j-1}}}, \quad l_j := \eta_{s_{r_{j-1}}+1}, \quad k_j := \eta_{t_{r_{j-1}}}.$$

Thus, since (18) is valid, we have

$$\Big|\sum_{k=l+1}^{l+p} b_{nk} x_k\Big| < \alpha_{j-1} \quad (n \geqslant n_{j-1},\ l \geqslant l_j,\ p \in N). \tag{22}$$

Now we choose $r_j \in N$ such that $r_j > r_{j-1}$,

$$\|x^{(r)} - x^{(r+\mu)}\| < \alpha_j \quad (r \geqslant r_j,\ \mu \in N), \tag{23}$$

$$\sum_{k=1}^{k_j} |b_{n_j k} - b_k|\,|x_k| < \alpha_j \quad \text{where } n_j := \mu_{s_{r_j}}. \tag{24}$$

Thus $k_{j+1}^* := \eta_{s_{r_j}}$, $l_{j+1} := \eta_{s_{r_j}+1}$ and $k_{j+1} := \eta_{t_{r_j}}$ are defined and one obtains

$$\Big|\sum_{k=l+1}^{l+p} b_{nk} x_k\Big| < \alpha_j \quad (n \leqslant n_j,\ l \geqslant l_{j+1},\ \mu \in N).$$

In the next step of the proof we define $(z^{(j)})$, $(y^{(j)})$, z and y as in part (i) of the proof and we obtain $y \in M(Y)$ and $yx \in W_E$ as in (i). By (21) we additionally obtain

$$y_k = y_{l_{2\mu}} = 1/(r_{2\mu-1}+1) \quad (k_{2\mu}^* < k \leqslant l_{2\mu}). \tag{25}$$

Now we prove $yx \notin c_B$ and we assume the existence of $\sum_k b_{nk} x_k y_k$ $(n \in N)$. For $j := 2\mu - 1$ $(\mu \in N)$ we obtain (cf. (15))

$$\sum_k b_{n_j k} x_k y_k = \sum_{k=1}^{k_{2\mu-1}} b_{n_j k} x_k y_k + \sum_{k=k_{2\mu}^*+1}^{l_{2\mu}} b_{n_j k} x_k y_k + \sum_{k=l_{2\mu}+1}^{\infty} b_{n_j k} x_k y_k$$
$$=: A_j + A_j' + C_j.$$

First we show that

$$\beta := \lim_\nu \sum_{k=1}^{\eta_\nu} b_k x_k y_k$$

exists; this is easy to check by (19) because

$$y_k = y_\mu \quad (\eta_\nu < k \leqslant \eta_{\nu+1}) \tag{26}$$

for every $\nu \in N$. Then we obtain $A_j \to \beta$ $(j \to \infty)$ because of (19), (24), $0 \leqslant y_k \leqslant 1$ and

$$|A_j - \beta| \leqslant \sum_{k=1}^{k_{2\mu-1}} |b_{n_j k} - b_k|\,|x_k|\, y_k + \Big|\sum_{k=1}^{k_{2\mu-1}} b_k x_k y_k - \beta\Big|.$$

By (22), (26) and the corresponding proof in (i) one may easily prove $C_j \to 0$ $(j \to \infty)$. Furthermore, on account of (25) and (20) we may check that

$$|A_j'| = \frac{1}{r_{2\mu-1}+1}\Big|\sum_{k=k_{2\mu}^*+1}^{l_{2\mu}} b_{n_j k} x_k\Big| \quad (j = 2\mu - 1,\ \mu \in N)$$
$$\to \infty \quad (j \to \infty).$$

This shows $yx \notin c_B$. Altogether, we have proved Lemma 3.

Proof of Theorem 2, (i)⇒(iii). Let Y be a sequence space containing φ such that $\mathfrak{E}^* \subset M(Y)$ or such that $M(Y)$ has the gliding humps property, let B be a matrix and E an FK-space with $\varphi \subset Y \cap W_E \subset c_B$. Then $Y \cap W_E \subset W_B$ is established if we prove the following implications:

(α) $\qquad Y \cap W_E \subset c_B \Rightarrow Y \cap W_E \subset L_B.$

(β) $\qquad Y \cap W_E \subset L_B \Rightarrow Y \cap W_E \subset I_B$ (thus $Y \cap W_E \subset F_B$).

(γ) $\qquad Y \cap W_E \subset F_B \Rightarrow Y \cap W_E \subset \Lambda_B^{\perp}$ (thus $Y \cap W_E \subset W_B$).

We are going to reduce the implications (α), (β) and (γ) to Lemma 3.

(α) Let $Y \cap W_E \subset c_B$, but $Y \cap W_E \not\subset L_B$. Consequently, we may choose an $x \in Y \cap W_E$ so that $x \in c_B \setminus L_B$. Then x satisfies one of the three conditions (i)–(iii) of Lemma 3. Therefore we may choose a $y \in M(Y)$ with $yx \in W_E$, hence $yx \in Y \cap W_E$, and $yx \notin c_B$. This contradicts the assumption $Y \cap W_E \subset c_B$.

(β) Let $Y \cap W_E \subset L_B$, but $Y \cap W_E \not\subset I_B$. Therefore we may choose an $x \in Y \cap W_E$ so that $x \in L_B \setminus I_B$. The last statement implies $(\sum_{k=1}^N b_k x_k)_{N \in N} \in m \setminus c$. Therefore, by Lemma 3(i), there exists a $y \in M(Y)$ with $yx \in Y \cap W_E$, but $yx \notin c_B$, contrary to $Y \cap W_E \subset L_B$.

(γ) Let $Y \cap W_E \subset F_B$, but $Y \cap W_E \not\subset \Lambda_B^{\perp}$. Again, we may apply Lemma 3(i) to each $x \in Y \cap W_E$ with $x \in F_B \setminus \Lambda_B^{\perp}$ and we obtain a $y \in M(Y)$ with $yx \in Y \cap W_E$, but $yx \notin c_B$; this contradicts $Y \cap W_E \subset F_B$ and completes the proof of Theorem 2, (i)⇒(iii).

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