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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
ODDZIAŁ WROCŁAWSKI
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
WROCŁAW BRANCH
Kopernika 18, 51-617 Wrocław, Poland

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Packing measures on ultrametric spaces

by

H. HAASE (Greifswald)

Abstract. We introduce packing measures on ultrametric spaces following the ideas of [9]. Since ultrametric spaces have strange properties they are a good object for testing the properties of new classes of measures. Our main concern is to show that packing measures permit similar theorems as for Hausdorff measures [8], for instance the selection problem for subsets of finite positive measure can be attacked by a good Density Theorem. The packing measures are in general a different class from that of Hausdorff measures, i.e. a packing measure cannot be obtained by taking the Hausdorff measure with respect to a different increasing function using another metric which generates the same topology. Furthermore, packing measures seem to be better means for studying sets of non-σ-finite measure. We apply our theorems to prove the existence of Borel measures on the real axis with remarkable properties.

1. Basic notation. Let (X, d) be an ultrametric space, i.e. the usual triangle inequality for d is replaced by the stronger one

$$d(x, y) \leqslant \max(d(x, z), d(z, y)) \quad \text{for all } x, y, z \in X. \tag{1}$$

It is well known that ultrametric spaces have strange properties. Both open and closed balls are clopen sets. Every point of a ball may be its centre. For any two balls, either their intersection is empty or one is contained in the other.

Let $\boldsymbol{H}$ be the family of all *Hausdorff functions*, i.e. $h \in \boldsymbol{H}$ iff $h\colon [0, +\infty[\to [0, +\infty]$ and

$$h(0) = 0, \quad h(q) > 0 \quad \text{for } q > 0, \tag{2}$$

$$q_1 < q_2 \quad \text{implies} \quad h(q_1) \leqslant h(q_2), \tag{3}$$

$$\lim_{q \downarrow 0} h(q) = 0. \tag{4}$$

Let $M \subseteq \{a \in \boldsymbol{R};\ a > 0\}$ and $\inf M = 0$. Put

$$P_M = \{B(x, r);\ x \in X,\ r \in M\},$$

where $B(x, r)$ is the closed ball of radius r and centre x. A countable subset $\{B(x_n, r_n)\}$ of P_M is called a P_M-*packing* for a set $A \subseteq X$ iff

$$x_n \in A, \tag{5}$$

$$B(x_n, r_n) \cap B(x_m, r_m) = \emptyset \quad \text{for } n \neq m. \tag{6}$$

For all $h \in H$ and $A \subseteq X$ put

(7) $$\tau_M^h(A) = \inf_{\delta>0} \sup \{\sum_n h(2r_n);\ \{B(x_n, r_n)\} \text{ is a } P_M\text{-packing for } A \text{ with } r_n \leqslant \delta\},$$

(8) $$p_M^h(E) = \inf \{\sum_n \tau_M^h(E_n);\ E \subseteq \bigcup_n E_n\} \quad [9].$$

We call (7) a *packing premeasure* and (8) *packing measure*.

For a nonempty subset $A \subseteq X$

(9) $$\operatorname{diam}(A) = \sup \{d(x, y);\ x, y \in A\}$$

is the *diameter* of A. Furthermore, for nonempty sets $A, B \subseteq X$ we let

(10) $$\operatorname{dist}(A, B) = \inf \{d(a, b);\ a \in A,\ b \in B\}$$

be the *distance* of A and B, and for a family $\mathscr{A}$ of subsets of X we define

(11) $$\operatorname{mesh}(\mathscr{A}) = \sup \{\operatorname{diam}(A);\ A \in \mathscr{A}\}.$$

The *h-Hausdorff measure* on X for $h \in H$ and $E \subseteq X$ is defined by

(12) $$\mu^h(E) = \sup_{\delta>0} \inf \{\sum_n h(\operatorname{diam}(E_n));\ E \subseteq \bigcup_n E_n,\ \operatorname{diam}(E_n) \leqslant \delta\}.$$

A map $\varphi\colon E \to X$ of some subset E of X into X is called *nonexpanding* iff

(13) $$d(x, y) \geqslant d(\varphi(x), \varphi(y)) \quad \text{for } x, y \in E.$$

A Borel measure μ is *strongly metrically invariant* iff for every Borel set E, each nonexpanding map $\varphi\colon E \to X$ satisfies

(14) $$\mu(E) \geqslant \mu(\varphi(E)).$$

For a Borel measure μ, a set M of positive reals with $\inf M = 0$ and a Hausdorff function h the *lower* (μ, h)*-density* of a point $x \in X$ w.r.t. a Borel set E is

(15) $$\underline{D}_{\mu,M}^h(E, x) = \sup_{\delta>0} \inf \{\mu(E \cap B(x, r))/h(2r);\ r \in M,\ r \leqslant \delta\}.$$

A Borel measure μ on X is *representable as a Hausdorff measure* iff for some Hausdorff function h and a topology-generating metric d' (which is not necessarily an ultrametric)

(16) $$\mu = \mu^{h,d'}.$$

A Borel measure μ is called *tight* iff for all Borel sets B

(17) $$\mu(B) = \sup \{\mu(K);\ K \subseteq B,\ K \text{ is compact}\}.$$

2. Basic properties of packing premeasures and packing measures.

THEOREM 1 (properties of τ_M^h). (i) *$A \subseteq B$ implies $\tau_M^h(A) \leqslant \tau_M^h(B)$.*

(ii) $\tau_M^h(A \cup B) \leqslant \tau_M^h(A) + \tau_M^h(B)$.

(iii) $\operatorname{dist}(A, B) > 0$ *implies* $\tau_M^h(A \cup B) = \tau_M^h(A) + \tau_M^h(B)$.

(iv) *If two Hausdorff functions g and h satisfy*

$$\lim_{q \downarrow 0} g(q)/h(q) = 0$$

then $\tau_M^h(A) < +\infty$ implies $\tau_M^g(A) = 0$.

(v) *If $A = \{x\}$ then $\tau_M^h(A) = 0$.*

(vi) $\tau_M^h(A) = \tau_M^h(\bar{A})$ *($\bar{A}$ stands for the closure of A).*

(vii) *If h is continuous then there is a sequence $r_n \downarrow 0$ such that for $M = \{r_n;\ n \in N\}$*

$$\tfrac{1}{2}\tau^h(A) \leqslant \tau_M^h(A) \leqslant \tau^h(A),$$

where τ^h is obtained by allowing all positive radii for the balls.

Proof. (i) follows from the definition of τ_M^h.

(ii) For all $\delta > 0$

$$\tau_M^h(A \cup B) \leqslant \sup \{\sum_n h(2r_n);\ \{B(x_n, r_n)\} \text{ is a } P_M\text{-packing for } A \cup B \text{ with } r_n \leqslant \delta \text{ for all } n\}.$$

Let $\varepsilon > 0$ and for each $\delta > 0$ choose a packing $\{B(x_n^\delta, r_n^\delta)\}$ satisfying

$$\tau_M^h(A \cup B) - \varepsilon \leqslant \sum_n h(2r_n^\delta).$$

Divide $\{B(x_n^\delta, r_n^\delta)\}$ into two packings, one for A and one for B, by the rule $x_n^\delta \in A$ or $x_n^\delta \in B$. Since

$$\sum_n h(2r_n^\delta) \leqslant \sum_{n, x_n^\delta \in A} h(2r_n^\delta) + \sum_{n, x_n^\delta \in B} h(2r_n^\delta)$$

we thus obtain

$$\tau_M^h(A \cup B) - \varepsilon \leqslant \tau_M^h(A) + \tau_M^h(B).$$

(iii) It is enough by (ii) to verify that

$$\operatorname{dist}(A, B) > 0 \quad \text{implies} \quad \tau_M^h(A \cup B) \geqslant \tau_M^h(A) + \tau_M^h(B).$$

Let $\varepsilon > 0$ and $0 < \delta < \operatorname{dist}(A, B)$. We choose P_M-packings for A and B, called $\{B(x_n^\delta, r_n^\delta)\}$ and $\{B(y_m^\delta, s_m^\delta)\}$ respectively, such that $r_n^\delta \leqslant \delta$ and $s_m^\delta \leqslant \delta$ for all n, m and

$$\tau_M^h(A) - \varepsilon/2 \leqslant \sum_n h(2r_n^\delta), \qquad \tau_M^h(B) - \varepsilon/2 \leqslant \sum_m h(2s_m^\delta).$$

Because of $B(x_n^\delta, r_n^\delta) \cap B(y_m^\delta, s_m^\delta) = \emptyset$ (otherwise $d(x_n^\delta, y_m^\delta) \leqslant \min(r_n^\delta, s_m^\delta)$ would contradict $\delta < \operatorname{dist}(A, B)$), $\{B(x_n^\delta, r_n^\delta)\} \cup \{B(y_m^\delta, s_m^\delta)\}$ is a P_M-packing for $A \cup B$. We conclude that for all $\delta < \operatorname{dist}(A, B)$

$$\tau_M^h(A)+\tau_M^h(B)-\varepsilon \leqslant \sup\{\sum_n h(2t_n);\ \{B(z_n, t_n)\} \text{ is a } P_M\text{-packing}$$

$$\text{for } A \cup B \text{ with } t_n \leqslant \delta \text{ for all } n\},$$

hence

$$\tau_M^h(A)+\tau_M^h(B)-\varepsilon \leqslant \tau_M^h(A \cup B).$$

(iv) and (v) follow easily by the definition.

(vi) It is sufficient to verify $\tau_M^h(A) \geqslant \tau_M^h(\bar{A})$. Let $\{B(x_n, r_n)\}$ be a P_M-packing for $\bar{A}$. Since $x_n \in \bar{A}$ and $B(x_n, r_n) \cap A \neq \emptyset$ we can take some $y_n \in B(x_n, r_n) \cap A$ as a new centre for the ball. Hence $\{B(y_n, r_n)\}$ is a P_M-packing for A and the result immediately follows.

(vii) The inequality $\tau_M^h(A) \leqslant \tau^h(A)$ holds for all M.

Let $r_n \downarrow 0$ be such that

$$h(2r_{n+1}) = \tfrac{1}{2}h(2r_n)$$

(Such a choice is always possible by the continuity of h.) Put $M = \{r_n;\ n \in N\}$, let $\delta > 0$, $\varepsilon > 0$ and let $\{B(x_m, s_m)\}$ be a P_M-packing for A with $s_m \leqslant \delta$ such that

$$\tau^h(A)-\varepsilon \leqslant \sum_m h(2s_m).$$

For a fixed m we find n_m satisfying

$$r_{n_m} \leqslant s_m < r_{n_m-1}.$$

Then $\{B(x_m, r_{n_m})\}$ is a P_M-packing for A such that $r_{n_m} \leqslant \delta$ and we obtain

$$\tau^h(A)-\varepsilon \leqslant \sum_m h(2r_{n_m-1}) = 2\sum_m h(2r_{n_m}).$$

This proves $\frac{1}{2}\tau^h(A) \leqslant \tau_M^h(A)$.

Remark. It is surprising that we obtain (vi) without requiring h to be continuous. The corresponding result for the Euclidean space [9] or a general metric space [6] requires h to be continuous.

THEOREM 2 (basic properties of p_M^h). (i) *p_M^h is a metric outer measure, i.e.*

$$\operatorname{dist}(A, B) > 0 \quad \textit{implies} \quad p_M^h(A \cup B) = p_M^h(A)+p_M^h(B).$$

(ii) *p_M^h is Borel regular, i.e. for all $E \subseteq X$ there exists a Borel set $B \supseteq E$ such that $p_M^h(B) = p_M^h(E)$.*

(iii) *All Borel sets are p_M^h-measurable.*

(iv) *For $E \subseteq X$, $p_M^h(E) \leqslant \tau_M^h(E)$ and $p_M^h(\{x\}) = 0$ for all $x \in X$.*

(v) *For each sequence $E_n \uparrow E$, $p_M^h(E_n) \uparrow p_M^h(E)$.*

(vi) *If $E \subseteq X$ is p_M^h-measurable, $0 < p_M^h(E) < +\infty$ and $\varepsilon > 0$ then there exists a closed set $F \subseteq E$ such that*

$$p_M^h(F) > p_M^h(E)-\varepsilon.$$

(vii) *For each E*

$$p_M^h(E) = \inf\{\lim_{n\to+\infty} \tau_M^h(E_n);\ E_n \uparrow E\}.$$

Proof. (i) Since p_M^h is certainly an outer measure [8] it suffices to prove that

$$\operatorname{dist}(A, B) > 0 \quad \text{implies} \quad p_M^h(A \cup B) \geqslant p_M^h(A)+p_M^h(B),$$

but this is a consequence of Theorem 1(i), (iii) and [7].

(ii) follows from Theorem 1(vi).

(iii) follows from (i) by [8].

(iv) follows from the definition of p_M^h.

(v) is a consequence of (i)–(iii).

(vi) We find a Borel set $B \supseteq E$ with $p_M^h(B) = p_M^h(E)$ by (iii). Since $p_M^h(B - E) = 0$ there exists again a Borel set $B_1 \supseteq B-E$ with $p_M^h(B_1) = 0$. Then $B_2 = B-B_1$ is a Borel set with $B_2 \subseteq E$ and $p_M^h(B_2) = p_M^h(E)$. The formula

$$\mu(F) = p_M^h(F \cap B_2), \quad F \text{ a Borel set},$$

defines a bounded Borel measure on X. Therefore μ is an inner regular Borel measure, i.e. we can find a closed set $F \subseteq B_2$ with

$$\mu(F) > \mu(B_2)-\varepsilon \quad [4],$$

which gives $p_M^h(F) > p_M^h(E)-\varepsilon$.

(vii) Let

$$\mu^*(E) = \inf\{\lim_{n\to+\infty} \tau_M^h(E_n);\ E_n \uparrow E\}.$$

(iv) and (v) imply

$$p_M^h(E) = \lim_{n\to+\infty} p_M^h(E_n) \leqslant \lim_{n\to+\infty} \tau_M^h(E_n)$$

for all $E_n \uparrow E$, hence $p_M^h(E) \leqslant \mu^*(E)$.

Conversely, let $\varepsilon > 0$. We can cover E by sets F_n such that

$$\sum_{n=1}^{\infty} \tau_M^h(F_n) < p_M^h(E)+\varepsilon.$$

For $E_n = E \cap (\bigcup_{i=1}^n F_i)$ we then obtain using Theorem 1(i), (ii)

$$\tau_M^h(E_n) \leqslant \sum_{i=1}^{n} \tau_M^h(F_i) < p_M^h(E)+\varepsilon$$

and thus $\mu^*(E) \leqslant p_M^h(E)+\varepsilon$.

THEOREM 3. *For any Hausdorff function h and M as above, the Hausdorff measure μ^h and the packing measure p_M^h relative to M satisfy $\mu^h \leqslant p_M^h$.*

Proof. It is sufficient to verify

$$\mu^h(E) \leqslant \tau_M^h(E) \quad \text{for } E \subseteq X.$$

Let $\varepsilon > 0$. Then there exists $\delta(\varepsilon) > 0$ such that for all $\delta \leqslant \delta(\varepsilon)$

$$\mu^h(E) - \varepsilon \leqslant \inf\{\sum_n h(\operatorname{diam}(E_n));\ E \subseteq \bigcup_n E_n,\ \operatorname{diam}(E_n) \leqslant \delta\}.$$

Every set E_n is contained in a closed ball with radius $\operatorname{diam}(E_n)$. Furthermore, we may assume $E_n \cap E_m = \emptyset$ for $n \neq m$ and the infimum does not decrease if we assume $\operatorname{diam}(E_n) \in M$ for all $n \in N$. Hence, we may suppose that the E_n are a P_M-packing for E and thus

$$\mu^h(E) - \varepsilon \leqslant \sup\{\sum_n h(2r_n);\ \{B(x_n, r_n)\} \text{ is a } P_M\text{-packing for } E \text{ with } r_n \leqslant \delta \text{ for all } n\},$$

i.e. $\mu^h(E) - \varepsilon \leqslant \tau_M^h(E)$.

Remark. Theorem 3 is not true in a general metric space [6, 9].

THEOREM 4. *p_M^h is a strongly invariant measure.*

Proof. We verify

$$\tau_M^h(E) \geqslant \tau_M^h(\varphi(E))$$

for a subset $E \subseteq X$ and a nonexpanding map $\varphi\colon E \to X$. Let $\{B(y_n, r_n)\}$ be a P_M-packing for $\varphi(E)$. We find $x_n \in E$ satisfying $\varphi(x_n) = y_n$ for all n. We see that $\{B(x_n, r_n)\}$ is a P_M-packing for E. Indeed, otherwise there exist n and m with $n \neq m$ such that

$$B(x_n, r_n) \cap B(x_m, r_m) \neq \emptyset.$$

But this implies $B(x_n, r_n) \subseteq B(x_m, r_m)$ or $B(x_m, r_m) \subseteq B(x_n, r_n)$ and we would obtain

$$d(y_n, y_m) \leqslant d(x_n, x_m) \leqslant \max(r_n, r_m),$$

contradicting $B(y_n, r_n) \cap B(y_m, r_m) = \emptyset$. Hence, we have found for each P_M-packing of $\varphi(E)$ a P_M-packing for E having the same h-sum of double radii.

3. Some selection problems. The next two theorems are basic tools in the sequel.

THEOREM 5 (Density Theorem). *If μ is a Borel measure on X and $E \subseteq X$ is a separable Borel subset with $\mu(E) < +\infty$, then*

$$\inf_{x\in E} \underline{D}_{\mu,M}^h(E, x)\, p_M^h(E) \leqslant \mu(E) \leqslant \sup_{x\in E} \underline{D}_{\mu,M}^h(E, x)\, p_M^h(E).$$

Proof. For the right-hand side inequality we show

$$\mu(F) \leqslant \sup_{x\in F} \underline{D}_{\mu,M}^h(F, x)\, \tau_M^h(F)$$

for an arbitrary Borel subset $F \subseteq E$. The final result then follows since $E = \bigcup_n E_n$, E_n a Borel set, gives

$$\mu(E) \leqslant \sum_n \mu(E_n) \leqslant \sum_n \sup_{x\in E_n} \underline{D}_{\mu,M}^h(E_n, x)\, \tau_M^h(E_n) \leqslant \sup_{x\in E} \underline{D}_{\mu,M}^h(E, x) \sum_n \tau_M^h(E_n)$$

and we may restrict our attention to Borel covers of E by Theorem 1(vi).

Let $a > \sup_{x\in F} \underline{D}_{\mu,M}^h(F, x)$, $\delta > 0$ and

$$V_\delta = \{B(x, r);\ \mu(B(x, r) \cap F)/h(2r) < a,\ r \in M,\ r \leqslant \delta\}.$$

For every $x \in F$ there is some $B(x, r) \in V_\delta$, hence we can cover F by countably many pairwise disjoint $B(x_n, r_n)$, by the Lindelöf property. We obtain

$$\mu(F) = \sum_n \mu(F \cap B(x_n, r_n)) \leqslant a \sum_n h(2r_n) \leqslant a \sup\{\sum_m h(2p_m);\ \{B(y_m, p_m)\} \text{ is a } P_M\text{-packing for } F \text{ with } p_m \leqslant \delta \text{ for all } m\}$$

and conclude that

$$\mu(F) \leqslant \sup_{x\in E} \underline{D}_{\mu,M}^h(F, x)\, \tau_M^h(F).$$

For the remaining inequality let $\inf_{x\in E} \underline{D}_{\mu,M}^h(E, x) > 0$ and take b with $0 < b < \inf_{x\in E} \underline{D}_{\mu,M}^h(E, x)$. Put

$$E_n = \{x \in E;\ r \in M,\ r < 1/n \text{ implies } bh(2r) < \mu(E \cap B(x, r))\}.$$

Since $E_n \uparrow E$ we obtain by Theorem 2(i), (v)

$$p_M^h(E) = \sup_n p_M^h(E) \leqslant \sup_n \tau_M^h(E_n).$$

It follows that

$$\tau_M^h(E_n) \leqslant \sup\{\sum_n h(2p_m);\ \{B(y_m, p_m)\} \text{ is a } P_M\text{-packing for } E_n \text{ with } p_m < 1/n \text{ for all } m\}$$

and we conclude that

$$b\tau_M^h(E_n) \leqslant \sum_m bh(2p_m) < \sum_m \mu(E \cap B(x_m, r_m)) \leqslant \mu(E),$$

hence $bp_M^h(E) \leqslant \mu(E)$.

THEOREM 6. *Let $K \subseteq X$ be compact. Then the following are equivalent:*
(i) *K has non-σ-finite p_M^h-measure.*
(ii) *There is a nonempty compact subset $K_0 \subseteq K$ such that for each subset $U \subseteq K$ relatively open in K, $U \cap K_0 \neq \emptyset$ implies $\tau_M^h(U \cap K_0) = +\infty$.*

Proof. (i)⇒(ii). Let

$$\mathscr{U} = \{U \subseteq K;\ U \text{ relatively open in } K, \text{ nonempty and with } \sigma\text{-finite } p_M^h\text{-measure}\}.$$

Assume that $\mathscr{U} \neq \emptyset$ since otherwise there is nothing to prove. Since (K, d) is a Lindelöf space (just as any other subspace) there are $U_n \in \mathscr{U}$, $n = 1, 2, \ldots$, satisfying $\bigcup \mathscr{U} = \bigcup_n U_n$.

Let $K_0 = K - \bigcup_n U_n$. Then K_0 is a nonempty compact subset, since otherwise K would have σ-finite p_M^h-measure. Let $U \subseteq K$ be open such that $U \cap K_0 \neq \emptyset$. Since

$$U = (U \cap K_0) \cup (U \cap \bigcup_n U_n)$$

and U has non-σ-finite measure, $U \cap K_0$ must have non-σ-finite measure too, hence $\tau_M^h(U \cap K_0) = +\infty$ by Theorem 2(iv).

(ii)⇒(i). If K_0 has non-σ-finite measure then so does K. Suppose that K_0 has σ-finite p_M^h-measure. Then there is a double sequence of compact sets (K_{nm}) such that

(a) $$\sum_m \tau_M^h(K_{nm}) < +\infty \quad \text{for all } n,$$

(b) $$K_0 = \bigcup_{n,m} K_{nm}.$$

Because of (ii) and the completeness of (K, d) there exists, by the Baire Category Theorem, a pair n_0, m_0 such that $K_{n_0 m_0}$ has an inner point w.r.t. K and thus $\tau_M^h(K_{n_0 m_0}) = +\infty$, which contradicts the assumption.

THEOREM 7. *Suppose X is compact and has non-σ-finite p_M^h-measure for $M = \{t_n;\ n \in N\}$ where $t_n \downarrow 0$ and*

$$\prod_{n=1}^{\infty} \big(1 + h(2t_{n+1})/h(2t_n)\big) < +\infty.$$

Then X has a compact subset K such that

$$0 < p_M^h(K) < +\infty.$$

Proof. By Theorem 6 we may suppose $\tau_M^h(U) = +\infty$ for all nonempty open subsets $U \subseteq X$. We find a finite P_M-packing $\{B(x_k, r_k)\}_{k=1}^m$ for X such that:

(i) $$\sum_{k=1}^{m} h(2r_k) > 1,$$

(ii) $$\sum_{k=1}^{m}{}^{(j)}\, h(2r_k) \leqslant 1 \quad \text{for all } j = 1, \ldots, m,$$

where the term $h(2r_j)$ is omitted, and as a condition of minimal choice

(iii) If $\{B(y_l, s_l)\}$ is any other P_M-packing for X with the property: for all l there is $k \in \{1, \ldots, m\}$ with $B(x_k, r_k) \subseteq B(y_l, s_l)$ (equality does not hold for all l) then

$$\sum_l h(2s_l) \leqslant 1.$$

We call the selected balls $\mathscr{S}_1$.

We now proceed by induction. Suppose that the sets of balls $\mathscr{S}_1, \ldots, \mathscr{S}_n$ are already chosen. We define $\mathscr{S}_{n+1}$ as follows. Take any ball $B(x, r) \in \mathscr{S}_n$. Since $\tau_M^h(B(x, r)) = +\infty$ we can find a finite P_M-packing for $B(x, r)$, say $\{B(x_k^{(n)}, r_k^{(n)})\}$, satisfying

(i)′ $$\sum_k h(2r_k^{(n)}) > h(2r),$$

(ii)′ $$\sum_k{}^{(j)}\, h(2r_k^{(n)}) \leqslant h(2r) \quad (h(2r_j^{(n)}) \text{ is omitted}),$$

(iii)′ If $\{B(y_l^{(n)}, s_l^{(n)})\}$ is any other packing for $B(x, r)$ such that for all l there is k with $B(x_k^{(n)}, r_k^{(n)}) \subseteq B(y_l^{(n)}, s_l^{(n)})$ then

$$\sum_l h(2s_l^{(n)}) \leqslant h(2r)$$

as the condition for a minimal choice. Let $\mathscr{S}_{n+1}$ be the collection of all packing balls obtained for all the $B(x, r) \in \mathscr{S}_n$ in this way. Since the radii of all balls are taken from M, $\operatorname{mesh}(\mathscr{S}_n) \to 0$ as $n \to +\infty$, and $\bigcup \mathscr{S}_{n+1} \subseteq \bigcup \mathscr{S}_n$, it follows that

$$K = \bigcap_{n \in N} \bigcup \mathscr{S}_n$$

is a nonempty compact subset.

We now construct a probability measure on K. Let $P \in \mathscr{S}_1$, say $P = B(x', r')$. Then we define

$$\mu(P \cap K) = \frac{\mu(K)\, h(2r')}{\sum \{h(2r);\ B(x, r) \in \mathscr{S}_1\}} \quad (\mu(K) = 1).$$

Suppose that $\mu(P\cap K)$ is already defined for all $P\in\mathcal{S}_1\cup\ldots\cup\mathcal{S}_n$. For $P\in\mathcal{S}_{n+1}$ there exists a unique $Q\in\mathcal{S}_n$ with $P\subseteq Q$ (by the construction of K) and we put

$$\mu(P\cap K)=\frac{\mu(Q\cap K)\,h(2r(P))}{\sum\{h(2r(P'));\ P'\in\mathcal{S}_{n+1},\ P'\subseteq Q\}},$$

where $r(B)$ is the radius of the ball B. Observe that

$$\mathcal{M}=\{P\cap K;\ P\in\bigcup_n\mathcal{S}_n\}\cup\{\emptyset,K\}$$

is a base for $\boldsymbol{D}(K)$, the clopen subsets of K which may be obtained by finite unions of members of $\mathcal{M}$. $\boldsymbol{D}(K)$ is an algebra and the measure μ is well defined on $\boldsymbol{D}(K)$ by extending it to the members of $\boldsymbol{D}(K)$ in a natural way. μ is a σ-additive measure on $\boldsymbol{D}(K)$ and the canonical outer measure μ^* reduced to the Borel subsets of K is a Borel measure agreeing with μ on $\boldsymbol{D}(K)$. Denote this Borel measure again by μ.

For each $x\in K$ we now obtain upper and lower bounds for $\underline{D}^h_{\mu,M}(K,x)$. For $x\in K$ there is a sequence $(B(x,s_n))$ of closed balls with:

(a) $B(x,s_n)\in\mathcal{S}_n$ for all n.

(b) $s_n\downarrow 0$ $(s_n\in M)$.

Then we can estimate using (i) and (i)′:

$$\frac{\mu(K\cap B(x,s_n))}{h(2s_n)}=\frac{\mu(K\cap B(x,s_{n-1}))\,h(2s_n)}{h(2s_n)\sum\{h(2r(P));\ P\in\mathcal{S}_n,\ P\subseteq B(x,s_{n-1})\}}\leqslant\frac{\mu(K\cap B(x,s_{n-1}))}{h(2s_{n-1})},$$

$$\frac{\mu(K\cap B(x,s_1))}{h(2s_1)}\leqslant\frac{\mu(K)\,h(2s_1)}{h(2s_1)\sum\{h(2r(P));\ P\in\mathcal{S}_1\}}\leqslant 1.$$

Hence we have

$$\sup_{x\in K}\underline{D}^h_{\mu,M}(K,x)\leqslant 1.$$

To get a positive lower bound for $\sup_{x\in K}\underline{D}^h_{\mu,M}(K,x)$ let $r\in M$ be sufficiently small. There is a unique $n\in N$ such that

$$s_n>r\geqslant s_{n+1}.$$

In the case $r=s_{n+1}$ we consider the fixed P_M-packing for $B(x,s_n)$ whose members belong to $\mathcal{S}_{n+1}$ and can then estimate by (i)′ and (ii)′:

$$\frac{\mu(K\cap B(x,s_{n+1}))}{h(2s_{n+1})}=\frac{\mu(K\cap B(x,s_n))\,h(2s_{n+1})}{h(2s_{n+1})\sum\{h(2r(P));\ P\in\mathcal{S}_{n+1},\ P\subseteq B(x,s_n)\}}\geqslant\frac{\mu(K\cap B(x,s_n))}{h(2s_n)+h(2s_{n+1})}$$

$$=\frac{\mu(K\cap B(x,s_{n-1}))\,h(2s_n)}{(h(2s_n)+h(2s_{n+1}))\sum\{h(2r(P));\ P\in\mathcal{S}_n,\ P\subseteq B(x,s_{n-1})\}}$$

$$\geqslant\frac{\mu(K\cap B(x,s_{n-1}))\,h(2s_n)}{(h(2s_n)+h(2s_{n+1}))(h(2s_{n-1})+h(2s_n))}$$

$$=\frac{1}{(1+h(2s_{n+1})/h(2s_n))}\cdot\frac{\mu(K\cap B(x,s_{n-1}))}{h(2s_{n-1})+h(2s_n)}.$$

Repeating this we obtain

$$\frac{\mu(K\cap B(x,s_{n+1}))}{h(2s_{n+1})}\geqslant\frac{1}{\prod_{i=1}^{n}(1+h(2s_{i+1})/h(2s_i))}\cdot\frac{1}{1+h(2s_1)}.$$

Since $s_i\in M$ for $i=1,\ldots,n$ it follows that

$$\frac{\mu(K\cap B(x,s_{n+1}))}{h(2s_{n+1})}\geqslant\frac{1}{\prod_{i=1}^{\infty}(1+h(2t_{i+1})/h(2t_i))}\cdot\frac{1}{1+h(2t_1)}.$$

and we abbreviate the constant right-hand side by c.

If $s_n>r>s_{n+1}$ we have $B(x,s_{n+1})\subseteq B(x,r)$ and can replace all $P\in\mathcal{S}_{n+1}$ with $P\subseteq B(x,r)$ by $B(x,r)$ to obtain a new P_M-packing consisting of $B(x,r)$ and all $P\in\mathcal{S}_{n+1}$, $P\subseteq B(x,s_n)$ but $P\nsubseteq B(x,r)$. The minimal condition (iii)′ permits us to conclude that

$$h(2r)<\sum\{h(2r(P));\ P\subseteq B(x,r)\}.$$

Now we can estimate

$$\frac{\mu(K\cap B(x,r))}{h(2r)}\geqslant\frac{\sum\{\mu(K\cap P);\ P\in\mathcal{S}_{n+1},\ P\subseteq B(x,r)\}}{\sum\{h(2r(P));\ P\in\mathcal{S}_{n+1},\ P\subseteq B(x,r)\}}$$

$$=\sum_{P\in\mathcal{S}_{n+1},P\subseteq B(x,r)}\frac{\mu(K\cap P)}{h(2r(P))}\cdot\frac{h(2r(P))}{\sum\{h(2r(Q));\ Q\in\mathcal{S}_{n+1},\ Q\subseteq B(x,r)\}}$$

$$\geqslant c\sum_{P\in\mathcal{S}_{n+1},P\subseteq B(x,r)}\frac{h(2r(P))}{\sum\{h(2r(Q));\ Q\in\mathcal{S}_{n+1},\ Q\subseteq B(x,r)\}}=c.$$

By Theorem 5 the compact set K has positive finite p^h_M-measure.

THEOREM 8. *Let (X,d) be a complete separable ultrametric space without isolated points and A an analytic subset with non-σ-finite packing measure w.r.t. p^h_M. Then A contains a compact subset with non-σ-finite measure w.r.t. p^h_M.*

Proof. The result is true in a general metric space [6], but note that we have dropped the continuity condition on h since the ultrametric condition on X ensures $\tau^h_M(A)=\tau^h_M(\bar{A})$.

Furthermore, under the same conditions as in the last theorem we have

THEOREM 9. *Every analytic subset A of non-σ-finite packing measure p_M^h contains $2^{\aleph_0}$ pairwise disjoint compact subsets of non-σ-finite measure.*

Proof. [6].

THEOREM 10. *p_M^h is a tight measure on every complete separable metric space without isolated points.*

Proof. The tightness condition for Borel sets B with $p_M^h(B) < +\infty$ follows from Theorem 2(vi) and [4] and for Borel sets with non-σ-finite measure from Theorem 8, since B is an analytic set.

THEOREM 11. *If (X, d) is a complete separable space and p_M^h is representable as a Hausdorff measure then X has σ-finite measure.*

Proof. Suppose that there are $g \in H$ and an equivalent metric d' generating the same topology as d such that

$$p_M^h = \mu^{g,d'}$$

and X has non-σ-finite measure. Let

$$\mathcal{U} = \{U;\ U \text{ nonempty, open and has } \sigma\text{-finite measure}\}.$$

Then

$$X_0 = X - \bigcup \mathcal{U}$$

is a nonempty closed set since otherwise X would have σ-finite measure by the Lindelöf property. If V is an open set with $V \cap X_0 \neq \emptyset$ then $V \cap X_0$ has non-σ-finite measure because

$$V = (V \cap X_0) \cup \big(V \cap \bigcup_n U_n\big)$$

and $\bigcup \mathcal{U} = \bigcup_n U_n$ for countably many $U_n \in \mathcal{U}$ by the Lindelöf property. We conclude that

$$\tau_M^h(V \cap X_0) = +\infty.$$

On the other hand, we can find a dense G_δ-set $X_{00} \subseteq X_0$ such that

$$\mu^{g,d'}(X_{00}) = 0$$

since X_0 must be separable (see [8]). By Baire's Category Theorem using $\tau_M^h(V \cap X_0) = +\infty$ for all open V, $V \cap X_0 \neq \emptyset$, we obtain $p_M^h(X_{00}) = +\infty$, which is a contradiction; hence X has σ-finite measure.

Remarks. 1. As we now see by Theorem 11 and its proof, packing measures need not be G_δ-regular. We know that for any Hausdorff measure μ^h we can always find a G_δ-set $H \supseteq E$ with $\mu^h(H) = \mu^h(E)$.

2. If p_M^h is σ-finite on X it is easy to check by Baire's Category Theorem that p_M^h must be locally finite (i.e. each point has a neighbourhood of finite measure) on a dense open subset Y of X. A result of Bandt [2] tells that p_M^h restricted to Y is representable as a Hausdorff measure.

But in a more general sense we obtain

THEOREM 12. *For any subset $E \subseteq X$, we have*

$$p_M^h(E) = \sup_{\delta > 0} \inf \{\sum_n \tau_M^h(E_n);\ E \subseteq \bigcup_n E_n,\ \operatorname{diam}(E_n) \leqslant \delta\}.$$

Proof. [7].

4. Examples and an application. 1. Let (X, d) be a complete separable ultrametric space without isolated points. For every $h \in \boldsymbol{H}$ there exists a complete ultrametric d', which generates the original topology, and some $M \subseteq \boldsymbol{R}^+$ such that the packing measure p_M^h based on the ultrametric d' is not representable as a Hausdorff measure. Furthermore, for each Hausdorff measure $\mu^{g,d''}$ (g a Hausdorff function and d'' any other topology-generating metric) there is a compact subset $K \subseteq X$ such that

$$\mu^{g,d''}(K) = 0, \quad 0 < p_M^h(K) < +\infty.$$

To prove this we first choose a sequence $a_n \downarrow 0$ such that

$$\text{(i)} \qquad \prod_{n=1}^{\infty} \big(1 + h(2a_{n+1})/h(2a_n)\big) < +\infty$$

and put $M = \{a_n;\ n \in N\}$. By induction we can find a family of partitons $(\mathcal{P}_n)_{n \in N}$ of X consisting of clopen sets with

$$\text{(ii)} \qquad P \in \mathcal{P}_n, \quad P = \bigcup \{Q;\ P \in \mathcal{P}_{n+1}\},$$

$$\text{(iii)} \qquad \lim_{n \to \infty} \operatorname{mesh}_d(\mathcal{P}_n) = 0,$$

$$\text{(iv)} \qquad \operatorname{card}\{Q;\ Q \in \mathcal{P}_{n+1},\ Q \subseteq P\}\, h(2a_{n+2}) \geqslant n$$

for all $n \in N$ and all $P \in \mathcal{P}_n$.

We define d' as an ultrametric which has the sets contained in $\bigcup_n \mathcal{P}_n$ as balls. Hence, d' generates the same topology and it is clear that d' must be complete since the original metric is. d' is given by

$$d'(x, y) = \sup\{a_n;\ P, Q \in \mathcal{P}_n,\ P \neq Q,\ x \in P,\ y \in Q\} \quad \text{for } x \neq y,$$

$$d'(x, x) = 0.$$

From (iv) we conclude that $\tau_M^h(U) = +\infty$ for all nonempty open sets U. By Theorem 6 which can be extended to sets or spaces satisfying Baire's Category Theorem X has non-σ-finite measure. Theorem 11 yields that p_M^h is not representable as a Hausdorff measure. Condition (i) ensures that for each Hausdorff measure $\mu^{g,d''}$ a compact subset K can be chosen satisfying

$$\mu^{g,d''}(K) = 0, \quad 0 < p_M^h(K) < +\infty.$$

We only have to select a compact subset of non-σ-finite packing measure by Theorem 8 and then apply Theorem 7.

2. If (X, d) is non-σ-totally bounded, for instance the Baire null space $X = N^N$ with the ultrametric

$$d(x, y) = \max\{1/n;\ x_n \neq y_n\} \quad \text{for } x \neq y, \text{ where } x = (x_n),\ y = (y_n),$$

$$d(x, x) = 0,$$

and the domain of positive values of d is a null sequence, we can take those values for the set M and then define a Hausdorff function satisfying (i) as above. The condition that X is non-σ-totally bounded ensures that X has non-σ-finite measure w.r.t. p_M^h. By Theorem 11, p_M^h cannot be a Hausdorff measure. Again for each Hausdorff measure $\mu^{g,d''}$ we can conclude the existence of a compact set $K \subseteq X$ satisfying

$$\mu^{g,d''}(K) = 0, \quad 0 < p_M^h(K) < +\infty.$$

3. Our theorems concerning ultrametric spaces permit us to prove the existence of Borel measures on the real line with remarkable properties:

THEOREM 13. *There is a Borel measure μ on $\mathbf{R}$ with the following properties:*

(a) *$\mu(\{x\}) = 0$ for all $x \in \mathbf{R}$.*

(b) *μ is not G_δ-regular.*

(c) *μ is tight.*

(d) *μ is non-σ-finite.*

(e) *Every Borel set with infinite measure contains a compact subset of positive finite measure.*

(f) *Every Borel set of non-σ-finite measure contains $2^{\aleph_0}$ pairwise disjoint compact subsets, each of non-σ-finite measure.*

(g) *For every Hausdorff measure $\mu^{h,d}$ (depending on h and a metric d generating the Euclidean topology) there is a compact set K satisfying*

$$\mu^{h,d}(K) = 0, \quad 0 < \mu(K) < +\infty.$$

Proof. The space $\mathbf{I}$ of irrationals is homeomorphic to N^N [5] and may be metrized by an ultrametric. Applying the previous remarks, we can find the desired Borel measure μ on $\mathbf{R}$ as a suitable packing measure on $\mathbf{I}$ extended by $\mu(\mathbf{R} - \mathbf{I}) = 0$.

Remark. This theorem shows the type of conditions which can be satisfied by a measure on $\mathbf{R}$ which is not equivalent to any Hausdorff measure.

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SEKTION MATHEMATIK
ERNST-MORITZ-ARNDT UNIVERSITÄT
F.-L. Jahn Str. 15a, DDR-2200 Greifswald, G.D.R.

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