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Asymptotic periodicity of the iterates of positive contractions on Banach lattices

by

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Abstract. An operator T on a Banach space is called *constrictive* if there is a compact set F such that the iterates $T^n x$ tend to F for any x with $\|x\| \leqslant 1$. We prove that if T is a constrictive positive linear contraction on a real Banach lattice X then there are positive normalized vectors $y_1, \ldots, y_r$ in X and positive functionals $\lambda_1, \ldots, \lambda_r$ on X such that $T^n(x - \sum_{i=1}^{r} \lambda_i(x) y_i) \to 0$ for $x \in X$. As an application, we present a characterization of quasi-compact positive contractions on Banach lattices.

Asymptotic behaviour of the iterates of operators is one of the fundamental problems in ergodic theory. In particular, for constrictive stochastic operators (see (**) below) the asymptotic periodicity of iterates has been investigated by Lasota, Li and Yorke in [4]. Their result can be stated as follows:

$$(*) \qquad P^n x = \sum_{i=1}^{r} \lambda_i(x)\, y_{\alpha^n(i)} + R_n x \qquad (n \geqslant 0,\ x \in L^1)$$

where P denotes a constrictive stochastic operator on L^1, $\lambda_1, \ldots, \lambda_r$ are bounded linear functionals on L^1, $y_1, \ldots, y_r$ are normalized densities with mutually disjoint supports, α denotes a permutation of the set $\{1, \ldots, r\}$ and the remainder R_n converges to zero in the strong operator topology as $n \to \infty$.

The purpose of this note is to present a similar decomposition for a class of operators acting on arbitrary Banach lattices. We will prove:

THEOREM 1. *Let T be a positive linear contraction acting on a real Banach lattice. If T is constrictive then there exists a sequence of positive normalized vectors $y_1, \ldots, y_r$ in X and a sequence of positive (bounded) functionals $\lambda_1, \ldots, \lambda_r$ on X such that*

$$\lim_{n \to \infty} \Big\| T^n \big(x - \sum_{i=1}^{r} \lambda_i(x) y_i\big) \Big\| = 0 \qquad (x \in X).$$

Moreover, there exists a permutation α of the set $\{1, \ldots, r\}$ such that $Ty_i = y_{\alpha(i)}$.

In the second part of the paper we give some applications of this theorem. In particular, we present a characterization of quasi-compact positive contractions on Banach lattices (Theorem 2).

1. Let X be a real Banach lattice. A linear operator $T\colon X \to X$ is called *positive* if $x \geqslant 0 \Rightarrow Tx \geqslant 0$, and is a *contraction* if $\|T\| \leqslant 1$. We say that T is *constrictive* (cf. [4]) if there exists a compact set $F \subseteq X$ such that

$$(**)\qquad \lim_{n\to\infty} d(T^n x, F) = 0 \quad \text{whenever } \|x\| \leqslant 1,$$

where $d(y, F) = \inf\{\|y-f\|\colon f\in F\}$.

The *ω-limit set* of an element $x\in X$ is defined to be the set

$$\omega(x) = \{y\in X\colon y = \lim_{k\to\infty} T^{n_k} x \text{ for some } n_k\to\infty\}.$$

Observe that if T is constrictive, $\omega(x)$ is a nonempty compact T-invariant subset of X. We write $\Omega = \bigcup_{x\in X}\omega(x)$. Clearly Ω is closed and T-invariant. The following lemma is contained in a more general result of Dafermos and Slemrod (see also [8]).

LEMMA 1 ([1], Theorem 1, p. 98).

(i) $$y\in\omega(x) \Rightarrow \omega(y) = \omega(x).$$

(ii) $T|_{\omega(x)}$ *is an invertible isometry.*

In [1], Dafermos and Slemrod have proved (i) and (ii) for an arbitrary nonexpansive (not necessarily constrictive or linear) mapping acting on a complete metric space. Moreover, if (**) is assumed (i.e. every limit set $\omega(x)$ is norm compact), then it can be shown that

(iii) Ω is a finite-dimensional linear subspace of X and T is an invertible isometry of Ω with $T^{-1}\geqslant 0$.

Indeed, for any $x, y\in\Omega$, there exists a sequence $n_k\to\infty$ such that $T^{-n_k}x$, $T^{-n_k}y$ converge in Ω to x', y' respectively. By (ii), we have

$$\begin{aligned}\|T^{n_k}(x'+y')-(x+y)\| &= \|T^{n_k}x'-x+T^{n_k}y'-y\|\\ &= \|T^{n_k}(x'-T^{-n_k}x)+T^{n_k}(y'-T^{-n_k}y)\|\\ &\leqslant \|x'-T^{-n_k}x\|+\|y'-T^{-n_k}y\| \to 0.\end{aligned}$$

This proves $x+y\in\Omega$. Clearly if $x\in\Omega$ then $tx\in\Omega$ for every scalar t. It follows that Ω is a linear subspace of X.

Now we show that $T|_\Omega$ is an isometry. Clearly for every $x\in\Omega$ the sequence $\|x\|-\|T^n x\|$ is nondecreasing and nonnegative. Since for some $n_j\to\infty$ we have $T^{n_j}x\to x$ it follows that $\|T^n x\| = \|x\|$ for every n. Since $x-y\in\Omega$ for any $x, y\in\Omega$, by the linearity of T we obtain $\|Tx-Ty\| = \|T(x-y)\| = \|x-y\|$, which means that $T|_\Omega$ is an isometry.

Using (**) again we conclude that the unit ball in Ω is relatively compact, so Ω is finite-dimensional. Since $T^{-1}\omega(x) = \omega(x)$ in Ω, $T\geqslant 0$ implies $T^{-1}x\geqslant 0$ for $0\leqslant x$, i.e. $T^{-1}\geqslant 0$.

In the following lemmas T is a constrictive positive linear contraction on a real Banach lattice X.

LEMMA 2. *If $x\in\Omega$ and $n_k\to\infty$ are such that $T^{n_k}x\to x$ and $T^{n_k}|x|\to y$ for some $y\in\Omega$ then $T^{n_k}y\to y$.*

Proof. Let x_1 be any limit point of the sequence $T^{n_k}(y-|x|)$. Thus $T^{n_{k_l}}y = T^{n_{k_l}}(|x|+y-|x|)\to y+x_1$. By a diagonal procedure we can choose a subsequence m_k of the sequence n_k such that

$$\begin{aligned}&T^{m_k}y\to y+x_1,\\ &T^{m_k}(y+x_1)\to y+x_1+x_2,\\ &\dots\dots\dots\dots\dots\\ &T^{m_k}(y+x_1+\ldots+x_j)\to y+x_1+\ldots+x_{j+1}\end{aligned}$$

for some $x_1, x_2, \ldots\in\Omega$, where each x_{j+1} is taken in $\omega(x_j)$. Observe that since the positive cone X_+ is closed and T is positive we have

$$y = \lim_{k\to\infty} T^{m_k}|x| \geqslant |\lim_{k\to\infty} T^{m_k}x| = |x|,$$

so $x_j\geqslant 0$. For every $j\geqslant 1$, $y+x_1+\ldots+x_j\in\omega(|x|)$. Since $\|x_j\| = \text{const}$ ($T|_\Omega$ is an isometry), this implies $\|x_1\| = 0$ because otherwise the sequence $y+x_1+\ldots+x_j$ would be discrete, which is impossible as $\omega(|x|)$ is compact.

LEMMA 3. *If $\Omega'\subseteq\Omega$ is a T-invariant lattice for the ordering inherited from X and if $\{y_1, \ldots, y_s\}$ is a linear basis in Ω' such that the y_j are positive, normalized, and mutually orthogonal in Ω' then $Ty_j = y_{\alpha(j)}$ for some permutation α of the set $\{1, \ldots, s\}$.*

Proof. The existence of an appropriate basis follows from Theorem 26.11 in [6]. Thus $x = \sum_{j=1}^s t_j y_j\geqslant 0$ if and only if $t_j\geqslant 0$ ($j = 1, \ldots, s$). Let $S\colon \Omega'\to l_s^\infty$ be a positive linear operator such that $Sy_j = e_j$ where $e_1, \ldots, e_s$ denotes the standard basis in l_s^∞. Clearly $Q = S\circ T\circ S^{-1}$ is a positive linear operator on l_s^∞ and Q^{-1} is also positive. It is well known that there exists a permutation α of $\{1, \ldots, s\}$ and a sequence of positive scalars $r_1, \ldots, r_s$ such that $Qe_j = r_j e_{\alpha(j)}$. Hence $Ty_j = S^{-1}\circ S\circ T\circ S^{-1}e_j = S^{-1}(r_j e_{\alpha(j)}) = r_j y_{\alpha(j)}$ for every $j = 1, \ldots, s$. Since T is an isometry on Ω, we get $Ty_j = y_{\alpha(j)}$.

LEMMA 4. *For every $x\in\Omega$ there exists a sequence $n_k\to\infty$ and a T-*

invariant lattice Ω' with modulus $|\cdot|'$ such that:

(a) $x \in \Omega'$.

(b) $\lim_{k\to\infty} T^{n_k} y = y$ *for every* $y \in \Omega'$.

(c) $\lim_{k\to\infty} T^{n_k}|y|$ *exists where* $|\cdot|$ *denotes the modulus in* X.

Proof. By (i), there exists a sequence m_k such that $T^{m_k} x \to x$. Let

$$\Omega^{(1)} = \{y \in \Omega\colon \lim_{k\to\infty} T^{m_k} y = y\}.$$

Clearly $\Omega^{(1)}$ is a T-invariant linear subspace of Ω. By (**), we can choose a subsequence $m_k^{(1)}$ of m_k such that $\lim_{k\to\infty} T^{m_k^{(1)}}|y|$ exists for every $y \in \Omega^{(1)}$. Let

$$\Omega^{(2)} = \{y \in \Omega\colon \lim_{k\to\infty} T^{m_k^{(1)}} y = y\}.$$

Clearly $\Omega^{(1)} \subseteq \Omega^{(2)}$ and by Lemma 2,

$$\lim_{k\to\infty} T^{m_k^{(1)}}|y| \in \Omega^{(2)} \quad \text{for every } y \in \Omega^{(1)}.$$

By induction we can obtain sequences $\{m_k^{(j)}\}_{k=1}^{\infty}$ where $\{m_k^{(j+1)}\}_{k=1}^{\infty}$ is a subsequence of $\{m_k^{(j)}\}_{k=1}^{\infty}$ for every j, and subspaces $\Omega^{(j)}$ of Ω such that

$$\lim_{k\to\infty} T^{m_k^{(j)}} y = y, \quad \lim_{k\to\infty} T^{m_k^{(j)}}|y| \in \Omega^{(j+1)} \quad \text{for every } y \in \Omega^{(j)}.$$

Since Ω is finite-dimensional, we have $\Omega^{(j)} = \Omega^{(j+1)} = \ldots$ for some j. Put $\Omega' = \Omega^{(j)}$ and $n_k = m_k^{(j)}$, $k = 1, 2, \ldots$ We show that Ω' is a lattice. For every $y \in \Omega'$ define $|y|' = \lim_{k\to\infty} T^{n_k}|y|$. Clearly $|y|' \geqslant |y|$ and $|y|'$ is the least element in Ω' with this property. For if $|y| \leqslant z \in \Omega'$ then for every k we have $T^{n_k} z \geqslant T^{n_k}|y|$ and

$$z = \lim_{k\to\infty} T^{n_k} z \geqslant \lim_{k\to\infty} T^{n_k}|y| = |y|'.$$

Proposition 1. *If T is a constrictive positive contraction on a real Banach lattice X then the limit vectors form a finite-dimensional lattice in X with a normalized, positive, orthogonal basis $y_1, \ldots, y_r$. Moreover, there exists a permutation of $\{1, \ldots, r\}$ such that $Ty_j = y_{\alpha(j)}$.*

Proof. By Lemmas 3 and 4 for every $x \in \Omega$ there exists $i \in N$ such that $T^i x = x$. Since Ω is finite-dimensional, we have $T^d|_\Omega = \mathrm{Id}|_\Omega$ for some natural d. We show that $\lim_{n\to\infty} T^{nd}|x|$ exists for $x \in \Omega$. Indeed, let n_j be such that $T^{n_j d}|x| \to y$. Since $|x| \leqslant T^d|x| \leqslant \ldots$ and $y \geqslant T^{nd}|x|$, we get

$$\limsup_{n\to\infty} \|y - T^{nd} x\| = \lim_{j\to\infty} \|y - T^{n_j d} x\| = 0.$$

As in the proof of Lemma 4, the formula $|x|' = \lim_{n\to\infty} T^{nd}|x|$ defines a modulus in Ω.

Remarks. 1. The modulus of x in Ω will be denoted by $|x|'$. If in addition the Banach lattice X has the property that $0 \leqslant x \leqslant y$ and $\|x\| = \|y\|$ imply $x = y$ (e.g. if $X = L^p$) then $|x| = |x|'$ for every $x \in \Omega$, which means that Ω is a sublattice of X. Indeed, $\||x|\| \geqslant \lim_{n\to\infty} \|T^n|x|\| = \||x|'\|$ and $|x| \leqslant |x|'$, so $|x| = |x|'$.

2. If $\|Tx\| = \|x\|$ for all $x \geqslant 0$ then

$$0 = \lim_{n\to\infty} \|T^n(|x|' - |x|)\| = \||x|' - |x|\|$$

and Ω is a sublattice of X.

3. Let $X = C([0, 1])$. The *Markov projection* $T\colon X \to X$ defined by $Tf(x) = (1-x)f(0) + xf(1)$ is constrictive. Now Ω is the space of all affine functions on $[0, 1]$, so Ω is not a sublattice of X.

Now we are in a position to prove Theorem 1. We base the proof on the method of Lasota, Li and Yorke (see [4], Lemma 5.2). For the reader's convenience a complete proof is presented.

Proof of Theorem 1. Let Ω be the lattice of all limit vectors and $\{y_1, \ldots, y_r\}$ a basis as in Proposition 1. By (**) for every $x \in X$ there exist scalars $\lambda_1(x), \ldots, \lambda_r(x)$ such that

$$\Big\|T^n\big(x - \sum_{j=1}^{r} \lambda_j(x) y_j\big)\Big\| \to 0.$$

It remains to show that the λ_j are linear. Let $x, z \in X$. Hence

$$\begin{aligned} 0 &= \lim_{n\to\infty} T^n\big(x + z - \sum_{j=1}^{r} \lambda_j(x+z) y_j\big) \\ &= \lim_{n\to\infty} \Big(T^n\big(x - \sum_{j=1}^{r} \lambda_j(x) y_j\big) + T^n\big(z - \sum_{j=1}^{r} \lambda_j(z) y_j\big) \\ &\quad + T^n\big(\sum_{j=1}^{r} (\lambda_j(x) + \lambda_j(z) - \lambda_j(x+z)) y_j\big)\Big), \end{aligned}$$

so the third component must converge to 0. Therefore

$$\Big\|\sum_{j=1}^{r} (\lambda_j(x) + \lambda_j(z) - \lambda_j(x+z)) y_j\Big\| = \lim_{n\to\infty} \Big\|T^n\big(\sum_{j=1}^{r} (\lambda_j(x) + \lambda_j(z) - \lambda_j(x+z)) y_j\big)\Big\| = 0.$$

By the linear independence of the vectors y_j we obtain the additivity of the

λ_j. Clearly by the linearity and positivity of T we get the positivity of λ_j as well as the homogeneity $\lambda_j(tx) = t\lambda_j(x)$.

2. In this section we consider positive contractions acting on the Banach lattice $L^1(Y, \Sigma, \mu)$. The following corollary has been presented in [4] for a σ-finite space (Y, Σ, μ). It now follows directly from our Theorem 1.

COROLLARY 1. *Let P be a stochastic operator on $L^1(Y, \Sigma, \mu)$ (i.e. $P \geqslant 0$ and $\|Pf\| = \|f\|$ for $f \geqslant 0$). If P is constrictive then P has an asymptotic decomposition* $(*)$.

COROLLARY 2. *Let P be a constrictive stochastic operator on $L^1(Y, \Sigma, \mu)$. If there exists a vector $g \in \Omega$ such that $\operatorname{supp} g = Y$ (equivalently $\sum_{j=1}^r g_j > 0$ a.e. where $g_1, \ldots, g_r$ is a basis as in Proposition 1) then for every $f \in L^1$*

$$\lim_{n\to\infty} \Big\|P^n\Big(f - \sum_{j=1}^{r} \Big(\int_{\operatorname{supp} g_j} f\, d\mu\Big) g_j\Big)\Big\| = 0.$$

Proof. Let $\eta_j \in L^\infty$ be such that $\lambda_j(f) = \int f\eta_j\, d\mu$. Since all λ_j are positive and their norms do not exceed 1 we have $0 \leqslant \eta_j \leqslant 1$. By Theorem 1, P is a permutation on $g_1, \ldots, g_r$, so

$$\lambda_j(g_k) = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\eta_j = 1$ on $\operatorname{supp} g_j$ and $\eta_j = 0$ on $Y - \operatorname{supp} g_j$.

Recall (see [4]) that a stochastic operator P is *asymptotically stable* if there exists a unique normalized vector f_* such that $\lim_{n\to\infty} P^n f = f_*$ for all $f \geqslant 0$, $\|f\| = 1$. Now we give conditions which guarantee the asymptotic stability of P. Let us remark that we are not assuming constrictivity here (cf. Corollary 1.4 from [4] where it is assumed).

PROPOSITION 2. *Let P be a stochastic operator on $L^1(Y, \Sigma, \mu)$ such that $\omega(f) \neq \emptyset$ for every f, and for any $0 \neq f_1, f_2 \geqslant 0$ there exists $n \geqslant 0$ with $P^n f_1 \wedge P^n f_2 \neq 0$. Then P is asymptotically stable.*

Proof. Let $f \in \Omega$ and let $n_j \to \infty$ be such that $P^{n_j} f \to f$. Clearly $P^{n_j} f^{\pm} \to f^{\pm}$ and thus $(Pf - f)^{\pm}$ are also recurrent. Since P is an isometry on "synchronous limit" sets (see [8] for details), for every natural n we have

$$\begin{aligned} \|Pf - f\| &= \|P^n(Pf - f)\| \leqslant \|P^n(Pf-f)^+\| + \|P^n(Pf-f)^-\| \\ &= \|(Pf-f)^+\| + \|(Pf-f)^-\| = \|Pf - f\| \end{aligned}$$

and thus $P^n(Pf-f)^+ \perp P^n(Pf-f)^-$. This means $Pf \leqslant f$ or $Pf \geqslant f$ and by the stochasticity assumption $Pf = f$. Thus for every f the set $\omega(f)$ must be a singleton, and clearly it is a fixed point of P. From our condition it can be easily concluded that the space of fixed points of P is one-dimensional and

thus there exists a unique normalized positive vector g such that for every f, $\lim_{n\to\infty} P^n f = \left(\int f\, d\mu\right) g$.

3. Let K be a compact Hausdorff space and $C(K)$ the Banach lattice of all continuous functions on K. A linear operator $T\colon C(K) \to C(K)$ is called *Markov* if $T1 = 1$ and $f \geqslant 0 \Rightarrow Tf \geqslant 0$. The (nonempty) convex, w^*-compact set of all T-invariant (Radon) probabilities on K is denoted by $P_T(K)$. A nonempty closed subset Z of K is said to be *invariant* if $z \in Z$ implies $T^*\delta_z(Z) = 1$. By the *center* of T we mean

$$M = \operatorname{closure}\Big(\bigcup_{\mu \in P_T(K)} \operatorname{supp} \mu\Big).$$

It is known (see [7]) that M is a nonempty T-invariant subset of K. The main result of this section is the following:

PROPOSITION 3. *Let T be a constrictive Markov operator on $C(K)$. If the center M of T is the whole space K then there exist a partition of K into clopen sets $E_{\varrho,\delta}$ and probability measures $m_{\varrho,\delta}$ with $\operatorname{supp} m_{\varrho,\delta} = E_{\varrho,\delta}$ ($\varrho = 1, \ldots, r$, $\delta = 1, \ldots, d_\varrho$) such that*

$$T^n\Big(f - \sum_{\varrho=1}^{r} \sum_{\delta=1}^{d_\varrho} \Big(\int f\, dm_{\varrho,\delta}\Big) 1_{E_{\varrho,\delta}}\Big) \to 0$$

for every $f \in C(K)$. Moreover, for every $\varrho = 1, \ldots, r$ we have $T1_{E_{\varrho,\delta}} = 1_{E_{\varrho,\delta+1}}$ ($\delta = 1, \ldots, d_\varrho - 1$) and $T1_{E_{\varrho,d_\varrho}} = 1_{E_{\varrho,1}}$.

Proof. Using the constrictivity assumption $(**)$ it is easily seen that the set $\operatorname{ex} P_T(K)$ of extreme points is finite (see [7]). Let $\mu_1, \ldots, \mu_s$ be these extremal probabilities. Since the $\operatorname{supp} \mu_j$ are pairwise disjoint (and T-invariant), without loss of generality we can assume $s = 1$. Now μ denotes the unique T^*-invariant probability measure. Observe that T acts on $L^1(K, \mathscr{B}, \mu)$ as a doubly stochastic operator ($T \geqslant 0$, $T1 = 1$, $T^*1 = 1$ a.e.) and the constrictivity assumption $(**)$ also holds with the norm $\|\cdot\|_1$.

Now we show that Ω is a sublattice of $C(K)$. Recall that for $g \in \Omega$,

$$|g|' = \lim_{k\to\infty} T^{n_k}|g| \geqslant |g| \quad \text{for some } n_k \to \infty$$

(the limit is in $C(K)$). Since Ω_{L^1} is a sublattice of $L^1(\mu)$, $|g| = |g|'$ μ-a.e. (see Remark 1). From the continuity of $|g|$ and $|g|'$ we get $|g| = |g|'$ since $\operatorname{supp} \mu = K$. By Theorem 1, for every $f \in C(K)$ we have

$$\Big\|T^n\Big(f - \sum_{j=1}^{d} \lambda_j(f) g_j\Big)\Big\| \to 0$$

where the g_j are orthogonal in $C(K)$. Since $T1 = 1$ we get $g_j = 1$ on $E_j = \operatorname{supp} g_j$ and 0 otherwise. Clearly the E_j are clopen sets and $\bigcup_{j=1}^d E_j = K$.

Since λ_j is positive with norm one, there exists a probability measure m_j (with $\operatorname{supp} m_j = E_j$) such that $\lambda_j(f) = \int f\, dm_j$, and the proof is complete.

As a simple consequence we get the following

COROLLARY 3. *Let T be a constrictive Markov operator on $C(K)$ where K is a connected compact space. If the center of T is equal to K then for every $f \in C(K)$*

$$\lim_{n\to\infty} T^n f = \int f\, d\mu \quad \text{uniformly on } K$$

where μ is the unique T-invariant probability.

4. A linear operator $T\colon X \to X$ is called *quasi-compact* if there exists a linear compact operator $Q\colon X \to X$ such that $\|T^m - Q\| < 1$ for some natural m. It is clear that equivalently we can write $\|T^m - Q\| < \varepsilon$ for some m and some compact operator Q where $1 > \varepsilon > 0$ is arbitrary.

LEMMA 5. *If T is a quasi-compact contraction then T is constrictive.*

Proof. First we show that for every $x \in X$ the orbit $\{T^n x\colon n \geqslant 0\}$ is relatively compact. Let $\varepsilon > 0$ be arbitrary and let $m \in N$ be such that $\|T^m - Q\| < \varepsilon$ for some compact operator Q. Thus for some finite set C_ε we have $C_\varepsilon + B_\varepsilon \supseteq T^m B_1 \supseteq T^{(m+1)} B_1 \supseteq \ldots$ (B_r denotes the ball in X with radius r and center 0) and so the orbit of x is relatively compact (here $\|x\| \leqslant 1$).

Now define

$$F = \bigcup_{\|x\| \leqslant 1} \omega(x) \subseteq \bigcap_{j=0}^{\infty} T^j B_1.$$

Since for every $\varepsilon > 0$ there exists $m \geqslant 1$ such that $T^m B_1$ has a finite ε-dense subset, F is compact. Clearly $\lim_{n\to\infty} d(T^n x, F) = 0$ whenever $\|x\| \leqslant 1$.

THEOREM 2. *Let T be a positive linear contraction on a Banach lattice X. Then the following conditions are equivalent:*

(A) *T is quasi-compact.*

(B) *There exists $d \geqslant 1$ such that T^{nd} converges in norm to a finite-dimensional operator P.*

(C) *There exists a compact set $F \subseteq X$ such that*

$$\sup_{\|x\| \leqslant 1} d(T^n x, F) \to 0.$$

Proof. (A)$\Rightarrow$(B). Since T is constrictive (Lemma 5), it follows that $\lim_{n\to\infty} T^{nd} x = Px$ for some $d \geqslant 1$, where P is a finite-dimensional projection.

Clearly for every finite $C \subseteq X$ we have $\sup_{x\in C} \|T^{nd} x - Px\| \to 0$. Next

$$\begin{aligned}\|T^{nd} - P\| &= \sup_{x\in B_1} \|T^{nd} x - Px\| = \sup_{x\in T^{md} B_1} \|T^{(n-m)d} x - Px\| \\ &\leqslant \sup_{x\in C_\varepsilon + B_\varepsilon} \|T^{(n-m)d} x - Px\| \\ &\leqslant \sup_{x\in C_\varepsilon} \|T^{(n-m)d} x - Px\| + \sup_{y\in B_\varepsilon} \|T^{(n-m)d} y - Py\|,\end{aligned}$$

so $\limsup_{n\to\infty} \|T^{nd} - P\| \leqslant 2\varepsilon$, and $\|T^{nd} - P\| \to 0$.

(B)$\Rightarrow$(A) is trivial.

(B)$\Rightarrow$(C). Let $F = P(B_1)$. Then

$$\begin{aligned}&\sup_{\|x\| \leqslant 1} d(T^n x, F) \\ &\quad\leqslant \sup_{\|x\| \leqslant 1} \inf\{\|T^n x - TPx\|, \|T^n x - T^2 Px\|, \ldots, \|T^n x - T^d Px\|\} \to 0.\end{aligned}$$

(C)$\Rightarrow$(B). Since T is constrictive, by Theorem 1 there exist a finite-dimensional operator P and $d \geqslant 1$ such that $T^{nd} x \to Px$. Clearly for every compact $K \subseteq X$ we have $\sup_{x\in K} \|T^{nd} x - Px\| \to 0$. Observe that for every $\varepsilon > 0$ there exists $m \geqslant 1$ such that

$$\sup_{\|x\| \leqslant 1} \|T^{nd} x - Px\| \leqslant \sup_{x\in F + B_\varepsilon} \|T^{(n-m)d} x - Px\|$$

whenever $n \geqslant m$. Thus

$$\limsup_{n\to\infty} \|T^{nd} - P\| \leqslant \lim_{n\to\infty} \sup_{x\in F} \|T^{(n-m)d} x - Px\| + 2\varepsilon = 2\varepsilon,$$

so $\|T^{nd} - P\| \to 0$.

Remark 4. If T is a positive linear contraction on a Banach lattice X then by [5] our conditions (A), (B), (C) are equivalent to

(D) *The Cesàro means $N^{-1} \sum_{j=1}^{N} T^j$ converge in norm to a finite-dimensional projection.*

Remark 5. If T is a positive contraction on a Banach lattice X and (**) holds then T is quasi-compact if and only if $\|R_n\| \to 0$.

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2. After this paper had been written there appeared some new results dealing with this subject. The reader will find them in [2], [3], [9], and [10].

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Packing measures on ultrametric spaces

by

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Abstract. We introduce packing measures on ultrametric spaces following the ideas of [9]. Since ultrametric spaces have strange properties they are a good object for testing the properties of new classes of measures. Our main concern is to show that packing measures permit similar theorems as for Hausdorff measures [8], for instance the selection problem for subsets of finite positive measure can be attacked by a good Density Theorem. The packing measures are in general a different class from that of Hausdorff measures, i.e. a packing measure cannot be obtained by taking the Hausdorff measure with respect to a different increasing function using another metric which generates the same topology. Furthermore, packing measures seem to be better means for studying sets of non-σ-finite measure. We apply our theorems to prove the existence of Borel measures on the real axis with remarkable properties.

1. Basic notation. Let (X, d) be an ultrametric space, i.e. the usual triangle inequality for d is replaced by the stronger one

$$d(x, y) \leqslant \max\big(d(x, z), d(z, y)\big) \quad \text{for all } x, y, z \in X. \tag{1}$$

It is well known that ultrametric spaces have strange properties. Both open and closed balls are clopen sets. Every point of a ball may be its centre. For any two balls, either their intersection is empty or one is contained in the other.

Let $\boldsymbol{H}$ be the family of all *Hausdorff functions*, i.e. $h \in \boldsymbol{H}$ iff $h\colon [0, +\infty[\to [0, +\infty]$ and

$$h(0) = 0, \quad h(q) > 0 \quad \text{for } q > 0, \tag{2}$$

$$q_1 < q_2 \ \text{ implies } \ h(q_1) \leqslant h(q_2), \tag{3}$$

$$\lim_{q \downarrow 0} h(q) = 0. \tag{4}$$

Let $M \subseteq \{a \in \boldsymbol{R};\ a > 0\}$ and $\inf M = 0$. Put

$$P_M = \{B(x, r);\ x \in X,\ r \in M\},$$

where $B(x, r)$ is the closed ball of radius r and centre x. A countable subset $\{B(x_n, r_n)\}$ of P_M is called a P_M-*packing* for a set $A \subseteq X$ iff

$$x_n \in A, \tag{5}$$

$$B(x_n, r_n) \cap B(x_m, r_m) = \emptyset \quad \text{for } n \neq m. \tag{6}$$