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Generalized convolutions V

by

K. URBANIK (Wrocław)

Abstract. The paper is devoted to the study of limit sets consisting of cluster points of normalized powers under a generalized convolution of probability measures. A relationship between topological and probabilistic properties of these limit sets is established.

1. Notation and preliminaries. For the terminology and notation used here, see [16]. In particular, P denotes the set of all probability measures defined on Borel subsets of the positive half-line $\boldsymbol{R}_+ = [0, \infty)$. The set P is endowed with the metrizable topology of weak convergence. For $\mu \in P$ and $a > 0$ we define the map T_a by setting $(T_a \mu)(E) = \mu(a^{-1} E)$ for all Borel subsets E of $\boldsymbol{R}_+$. By δ_c we denote the probability measure concentrated at the point c. We define the equivalence relation $\sim$ on P by setting $\mu \sim \nu$ whenever $\mu = T_c \nu$ for a certain $c > 0$. According to [16], Lemma 1.1, this relation is continuous on $P \setminus \{\delta_0\}$. Given $A, B \subset P$ we write $A \sim B$ whenever $A/\sim \; = B/\sim$. A set A is said to be a $\sim$ *selector* if the natural map from A onto $A/\sim$ is one-to-one or, in other words, if the relations $\mu, \nu \in A$ and $\mu \sim \nu$ yield $\mu = \nu$. Given $\mu \in P$ and $p > 0$, we shall use the notation $m_p(\mu) = \int_0^\infty x^p \mu(dx)$. A commutative and associative P-valued binary operation $\circ$ on P, continuous in each variable separately, is called a *generalized convolution* if it is distributive with respect to convex combinations and the maps T_a $(a > 0)$ with δ_0 as the unit element. Moreover, we assume the existence of norming constants c_n and a measure $\gamma \in P \setminus \{\delta_0\}$ such that

$$T_{c_n} \delta_1^{\circ n} \to \gamma \tag{1.1}$$

where $\delta_1^{\circ n}$ is the nth power of δ_1 under $\circ$.

Let m_0 be the sum of δ_0 and the Lebesgue measure on $\boldsymbol{R}_+$. By P_0 we shall denote the subset of P consisting of all measures absolutely continuous with respect to m_0. It has been proved in [16] (Theorem 4.1 and Corollary 4.4) that each generalized convolution $\circ$ admits a *weak characteristic function*, i.e. a one-to-one correspondence $\mu \leftrightarrow \hat{\mu}$ between measures μ from P and real-valued Borel functions $\hat{\mu}$ from $L_\infty(m_0)$ such that the functions $\hat{\lambda}$ are continuous for $\lambda \in P_0$, $(c\mu + (1-c)\nu)^\wedge = c\hat{\mu} + (1-c)\hat{\nu}$ $(0 \leqslant c \leqslant 1)$, $(T_a \mu)^\wedge(t) = \hat{\mu}(at)$

$(a > 0)$ and $(\mu \circ \nu)^{\wedge} = \hat{\mu}\hat{\nu}$ for all $\mu, \nu \in P$. Moreover, the weak convergence $\mu_n \to \mu$ is equivalent to the convergence $\hat{\mu}_n \to \hat{\mu}$ in the $L_1(m_0)$-topology of $L_\infty(m_0)$. The weak characteristic function is uniquely determined up to a scale change and for any $\mu \in P$

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\,\mu(dx) \tag{1.2}$$

m_0-almost everywhere. The kernel Ω is a Borel function and $|\Omega(t)| \leqslant 1$. A generalized convolution is said to be *regular* if the kernel Ω is continuous ([16], p. 93).

A measure λ from P is said to be $\circ$-*stable* if $\lambda \neq \delta_0$ and $T_{a_n}\mu^{\circ n} \to \lambda$ for a measure $\mu \in P$ and a norming sequence $\{a_n\}$ of positive numbers; the measure μ which can arise here belongs to the *domain of attraction* of λ. By Theorem 4.2 in [16] there exist a constant $\varkappa$ $(0 < \varkappa \leqslant \infty)$ called the *characteristic exponent* of $\circ$ and a family σ_p $(0 < p \leqslant \varkappa)$ of standard $\circ$-stable measures defined by the condition $\hat{\sigma}_p(t) = \exp(-t^p)$ if $p < \infty$ and $\sigma_\infty = \delta_1$ such that for every $\circ$-stable measure λ we have the relation $\lambda \sim \sigma_p$ for a certain $p \in (0, \varkappa]$. We note that by Proposition 4.5 in [16], $\gamma \sim \sigma_\varkappa$ where γ is defined by (1.1). Moreover, by Lemma 2.1 in [16], $\varkappa = \infty$ if and only if $\circ$ is the max-convolution. The measure $\sigma_\varkappa$ is called the *characteristic measure* of $\circ$.

2. Attraction by compact subsets. We say that a measure μ from P belongs to the *domain of attraction of a compact subset of* $P \setminus \{\delta_0\}$ if there exists a norming sequence $\{a_n\}$ of positive numbers such that the sequence $T_{a_n}\mu^{\circ n}$ is conditionally compact in P and the set $G(a_n, \mu)$ is compact. The set of all norming sequences $\{a_n\}$ with the above properties will be denoted by $N(\mu)$. The set of all measures μ belonging to the domain of attraction of a compact subset of $P \setminus \{\delta_0\}$ will be denoted by Q. It is evident that Q is invariant under the maps T_a $(a > 0)$.

Attraction by compact subsets of $P \setminus \{\delta_0\}$ for the ordinary convolution has been studied by W. Doeblin [2], W. Feller [3], R. A. Maller [11] and W. E. Pruitt [12]. The case of the max-convolution has been considered by L. de Haan, S. J. Resnick and G. Ridder in [4] and [5]. For generalized convolutions a description of the sets Q and $N(\mu)$ in terms of moments and medians of the powers $\mu^{\circ n}$ has been established in [18] and [19].

It is evident that $\mu \in Q$, $\{a_n\} \in N(\mu)$ and $G(a_n, \mu)$ is a one-point set if and only if μ belongs to the domain of attraction of a $\circ$-stable measure σ_p $(0 < p \leqslant \varkappa)$. In this case the limit set will be called $\circ$-*stable*. Now we shall quote less trivial examples of limit sets.

Example 2.1. Put $s_1 = 0$ and for $n \geqslant 2$ let s_n be the positive integer determined by the inequality

$$(s_n - 1)!/2^{s_n - 1} < n \leqslant s_n!/2^{s_n}.$$

Setting $x_n = n2^{s_n}/s_n!$, $y_n = (s_n - 1)!/(n2^{s_n - 1})$ $(n \geqslant 2)$ we have the inequalities $0 < x_n \leqslant 1$ and $0 < y_n < 1$. Moreover, it is easy to check that $[0, 1] \times \{0\} \cup \{0\} \times [0, 1]$ is the set of all cluster points of the sequence $\{(x_n, y_n)\}$. Put $a_n = (1 + y_n)/(s_n!\,x_n)$. Taking the measure μ with the Laplace transform

$$\tilde{\mu}(t) = \exp \sum_{k=0}^{\infty} (e^{-k!t} - 1)\, 2^k/k!$$

we have

$$(T_{a_n}\mu^{*n})^{\sim}(t) = \exp n \sum_{k=0}^{\infty} (e^{-k! a_n t} - 1)\, 2^k/k!$$

where $*$ denotes the ordinary convolution. By standard calculations we conclude that a subsequence $T_{a_{n_k}}\mu^{*n_k}$ converges if and only if the subsequence (x_{n_k}, y_{n_k}) does. Moreover, $T_{a_{n_k}}\mu^{*n_k} \to \lambda_a$ if $(x_{n_k}, y_{n_k}) \to (a, 0)$ and $T_{a_{n_k}}\mu^{*n_k} \to \nu_a$ if $(x_{n_k}, y_{n_k}) \to (0, a)$ where $0 \leqslant a \leqslant 1$. Here we have $\lambda_0 = \nu_0 = \delta_1$ and the Laplace transforms of λ_a and ν_a $(0 < a \leqslant 1)$ are given by the formulae

$$\tilde{\lambda}_a(t) = \exp\big(-t + a(e^{-t/a} - 1)\big),$$

$$\tilde{\nu}_a(t) = \exp\big(-\tfrac{1}{2}(1 + a)t + a^{-1}(e^{-a(1+a)t/2} - 1)\big).$$

Consequently, $\mu \in Q$, $\{a_n\} \in N(\mu)$ and $G(a_n, \mu) = \{\lambda_a, \nu_a\colon 0 \leqslant a \leqslant 1\}$. Since $\lambda_0 = \nu_0$, $\lambda_1 = \nu_1$ and the measures λ_a $(0 \leqslant a \leqslant 1)$ and ν_a $(0 < a < 1)$ are nonequivalent we conclude that $G(a_n, \mu)$ is a $\sim$ selector homeomorphic to a circle.

Example 2.2. Let m be a positive integer and $I_m = \{(c_1, \ldots, c_m)\colon 0 \leqslant c_j < 1,\ j = 1, \ldots, m\}$. Let $q_1, \ldots, q_m$ be an arbitrary m-tuple of positive numbers such that $0 < q = \sum_{j=1}^{m} q_j < 1$ and the numbers $1, q_1, \ldots, q_m$ are linearly independent over the field of rational numbers. Taking the measures μ_c $(c \in I_m)$ with the Laplace transform

$$\tilde{\mu}_c(t) = \exp \int_0^\infty (e^{-tx} - 1) \exp \sum_{j=1}^{m} (c_j - [q_j \log x + c_j])\, dx$$

where the square brackets denote the integral part and setting $a_n = n^{-1/q}$ we get, by simple calculations, $\mu_c \in Q$, $\{a_n\} \in N(\mu_c)$ and $G(a_n, \mu_c) = \{\mu_b\colon b \in I_m\}$. It is clear that the limit set $G(a_n, \mu_c)$ is a $\sim$ selector homeomorphic to the m-dimensional torus.

As a consequence of Lemma 1.1 in [16] we get the following simple statements.

Proposition 2.1. *If* $\mu \in Q$ *and* $\{a_n\}, \{b_n\} \in N(\mu)$, *then* $G(a_n, \mu) \sim G(b_n, \mu)$.

Proposition 2.2. *If* $\mu \in Q$, $\{a_n\} \in N(\mu)$ *and* $G(a_n, \mu)$ *is a* $\sim$ *selector, then* *for every* $\{b_n\} \in N(\mu)$ *the set* $G(a_n, \mu)$ *is a continuous image of* $G(b_n, \mu)$. *More precisely, there exists a positive continuous function* g *on* $G(b_n, \mu)$ *such that*

$$G(a_n, \mu) = \{T_{g(\lambda)}\lambda\colon\ \lambda \in G(b_n, \mu)\}.$$

PROPOSITION 2.3. *If* $\mu \in Q$, $\{a_n\}$, $\{b_n\} \in N(\mu)$ *and both limit sets* $G(a_n, \mu)$ *and* $G(b_n, \mu)$ *are* $\sim$ *selectors, then* $G(a_n, \mu)$ *and* $G(b_n, \mu)$ *are homeomorphic.*

PROPOSITION 2.4. *If* $\mu \in Q$, $\{a_n\} \in N(\mu)$ *and* f *is a positive continuous function on* $G(a_n, \mu)$, *then there exists* $\{b_n\} \in N(\mu)$ *such that*

$$G(b_n, \mu) \subset \{T_{f(\lambda)}\lambda\colon\ \lambda \in G(a_n, \mu)\}.$$

Proof. Let ϱ be a metric defining the topology of weak convergence on P. Setting

$$A = \{T_{f(\lambda)}\lambda\colon\ \lambda \in G(a_n, \mu)\}$$

we have $c = \varrho(\delta_0, A) > 0$. Consequently, there exists an index n_0 such that $\varrho\big(T_{a_n}\mu^{\circ n}, G(a_n, \mu)\big) < c$ for $n \geqslant n_0$. Since

$$\inf\{\varrho(T_b\,\mu^{\circ n}, A)\colon\ b > 0\} \leqslant \varrho\big(T_{a_n}\mu^{\circ n}, G(a_n, \mu)\big),$$

for every $n \geqslant n_0$ we can find a positive number b_n such that

$$\varrho(T_{b_n}\mu^{\circ n}, A) \leqslant \varrho\big(T_{a_n}\mu^{\circ n}, G(a_n, \mu)\big).$$

Setting $b_n = 1$ for $1 \leqslant n \leqslant n_0$, we infer that $\{b_n\} \in N(\mu)$ and $G(b_n, \mu) \subset A$, which completes the proof.

Put $K_p = \{\mu\colon\ \mu \in P,\ m_p(\mu) = 1\}$ $(p > 0)$. It is clear that the set K_p is a $\sim$ selector. Denote by Q_p the subset of Q consisting of all measures μ admitting a norming sequence $\{a_n\}$ satisfying the condition $G(a_n, \mu) \subset K_p$. The set of all norming sequences with the above property will be denoted by $N_p(\mu)$. By Lemma 1.1 in [16] we have $\lim_{n\to\infty} a_n/b_n = 1$ and $G(a_n, \mu) = G(b_n, \mu)$ for all $\{a_n\}$, $\{b_n\} \in N_p(\mu)$. This enables us to introduce the notation $G_p(\mu)$ for the uniquely determined limit set $G(a_n, \mu)$ with $\{a_n\} \in N_p(\mu)$. It is clear that $G_p(\mu)$ is always a $\sim$ selector.

LEMMA 2.1. *Let* $\mu \in Q$, $\{a_n\} \in N(\mu)$ *and* $0 < p < q$. *If* $s = \sup\{m_q(\lambda)\colon \lambda \in G(a_n, \mu)\} < \infty$, *then* $\mu \in Q_p$.

Proof. Setting $f(\lambda) = m_p(\lambda)^{-1/p}$ for $\lambda \in G(a_n, \mu)$ we get a continuous function satisfying the inequality $f(\lambda) \geqslant s^{-1/q}$. Since

$$\{T_{f(\lambda)}\lambda\colon\ \lambda \in G(a_n, \mu)\} \subset K_p,$$

our assertion is a direct consequence of Proposition 2.4.

The above lemma yields the following statement.

PROPOSITION 2.5. *If* $0 < p < q$, *then* $Q_q \subset Q_p$.

Our next result lies somewhat deeper.

THEOREM 2.1. $Q = \bigcup_{p>0} Q_p$.

Proof. Let $\mu \in Q$. By Theorem 3.1 in [19] there exists a positive number q such that $0 < m_q(\mu^{\circ n}) < \infty$ $(n = 1, 2, \ldots)$ and $\{m_q(\mu^{\circ n})^{-1/q}\} \in N(\mu)$. Setting $a_n = m_q(\mu^{\circ n})^{-1/q}$ we have $m_q(T_{a_n}\mu^{\circ n}) = 1$ $(n = 1, 2, \ldots)$, which yields $m_q(\lambda) \leqslant 1$ for all $\lambda \in G(a_n, \mu)$. Applying Lemma 2.1 we get $\mu \in Q_p$ for $0 < p < q$, which completes the proof.

As a consequence of the above theorem we get the following statement.

COROLLARY 2.1. *For every* $\mu \in Q$ *there exists* $\{a_n\} \in N(\mu)$ *such that the limit set* $G(a_n, \mu)$ *is a* $\sim$ *selector.*

THEOREM 2.2. *The set* Q *is invariant under the generalized convolution* $\circ$, *i.e.* $Q \circ Q \subset Q$.

Proof. Given $\mu, \nu \in Q$, $\{a_n\} \in N(\mu)$, $\{b_n\} \in N(\nu)$ we put $c_n = \min(a_n, b_n)$. It suffices to show that each subsequence of the sequence $T_{c_n}(\mu\circ\nu)^{\circ n}$ has a subsequence converging to a measure other than δ_0. In fact we can always choose a subsequence $T_{c_{n_k}}(\mu\circ\nu)^{\circ n_k}$ with the properties

$$p_k = \frac{c_{n_k}}{a_{n_k}} \to p, \qquad q_k = \frac{c_{n_k}}{b_{n_k}} \to q, \qquad T_{a_{n_k}}\mu^{\circ n_k} \to \lambda, \qquad T_{b_{n_k}}\nu^{\circ n_k} \to \sigma$$

where $0 \leqslant p \leqslant 1$, $0 \leqslant q \leqslant 1$, $\max(p, q) = 1$, $\lambda \neq \delta_0$ and $\sigma \neq \delta_0$. Then we have

$$T_{c_{n_k}}(\mu\circ\nu)^{\circ n_k} = T_{p_k}(T_{a_{n_k}}\mu^{\circ n_k}) \circ T_{q_k}(T_{b_{n_k}}\nu^{\circ n_k}) \to T_p\lambda \circ T_q\sigma.$$

Taking into account Lemma 2.3 in [20] we infer that $T_p\lambda \circ T_q\sigma \neq \delta_0$, which completes the proof.

LEMMA 2.2. *Suppose that* $0 < p < \varkappa$. *Then* $m_p(\mu\circ\nu) \leqslant m_p(\mu) + m_p(\nu)$ *for all* $\mu, \nu \in P$.

Proof. By Lemma 4.4 in [16] for every $\lambda \in P$ we have the inequality $|\hat{\lambda}(t)| \leqslant 1$ m_0-almost everywhere. Moreover, by Theorem 4.1 and Lemma 3.7 in [16] for every p with $0 < p < \varkappa$ we have the formula

$$c_p\, m_p(\lambda) = \int_0^\infty \frac{1-\hat{\lambda}(t)}{t^{1+p}}\,dt$$

where

$$0 < c_p = \int_0^\infty \frac{1-\Omega(t)}{t^{1+p}}\,dt < \infty.$$

Our assertion is an immediate consequence of the inequality $(\mu\circ\nu)^\wedge(t) = \hat{\mu}(t)\,\hat{\nu}(t) \geqslant \hat{\mu}(t) + \hat{\nu}(t) - 1$.

LEMMA 2.3. *Let* $n_1 < n_2 < \ldots$, $\mu_{n_k}, \nu_{n_k} \in P$, $0 < c \leqslant 1$ *and suppose* $(c\mu_{n_k}+(1-c)\nu_{n_k})^{\circ n_k} \to \delta_0$. *Then* $\mu_{n_k}^{\circ n_k} \to \delta_0$.

Proof. By Lemma 1.1 in [19] each subsequence of the sequence $n_1 < n_2 < \ldots$ has a subsequence $m_1 < m_2 < \ldots$ such that

$$\left((c\mu_{m_k}+(1-c)\nu_{m_k})^{\circ m_k}\right)^\wedge(t) \to 1$$

m_0-almost everywhere. Since the left-hand side of the above formula is equal to $(c\hat{\mu}_{m_k}(t)+(1-c)\hat{\nu}_{m_k}(t))^{m_k}$ and, by Lemma 4.4 in [16], $|\hat{\mu}_{m_k}(t)| \leqslant 1$ and $|\hat{\nu}_{m_k}(t)| \leqslant 1$ $(k = 1, 2, \ldots)$ m_0-almost everywhere, we conclude that $|\hat{\mu}_{m_k}(t)|^{m_k} \to 1$ m_0-almost everywhere. Consequently,

$$(\mu_{m_k}^{\circ 2m_k})^\wedge(t) = (\hat{\mu}_{m_k}(t))^{2m_k} \to 1$$

m_0-almost everywhere, which, by Lemma 1.1 in [19], yields $\mu_{n_k}^{\circ 2n_k} \to \delta_0$. Now our assertion is a consequence of Corollary 2.4 in [20].

THEOREM 2.3. *The set Q is convex.*

Proof. Suppose that $\mu, \nu \in Q$. By Theorem 3.1 in [19] there exists a positive number $p < \varkappa$ such that $m_p(\mu^{\circ n}) < \infty$, $m_p(\nu^{\circ n}) < \infty$ $(n = 1, 2, \ldots)$, $\{m_p(\mu^{\circ n})^{-1/p}\} \in N(\mu)$ and $\{m_p(\nu^{\circ n})^{-1/p}\} \in N(\nu)$. Setting

$$a_n = \min\{m_p(\mu^{\circ k})^{-1/p}:\ k = 1, \ldots, n\},$$

$$b_n = \min\{m_p(\nu^{\circ k})^{-1/p}:\ k = 1, \ldots, n\}$$

we have, by Lemma 2 in [18], $\{a_n\} \in N(\mu)$ and $\{b_n\} \in N(\nu)$. Put $c_n = \min(a_n, b_n)$. Then $m_p(\mu^{\circ k}) \leqslant c_n^{-p}$, $m_p(\nu^{\circ k}) \leqslant c_n^{-p}$ $(k = 1, \ldots, n)$ and, by Lemma 2.2, for $0 < c < 1$

$$m_p\left(T_{c_n}(c\mu+(1-c)\nu)^{\circ n}\right) = c_n^p \sum_{k=0}^{n} \binom{n}{k} c^k(1-c)^{n-k} m_p(\mu^{\circ k} \circ \nu^{\circ(n-k)})$$

$$\leqslant c_n^p \sum_{k=0}^{n} \binom{n}{k} c^k(1-c)^{n-k}\left(m_p(\mu^{\circ k})+m_p(\nu^{\circ(n-k)})\right) \leqslant 2.$$

Hence the sequence $T_{c_n}(c\mu+(1-c)\nu)^{\circ n}$ is conditionally compact in P. It remains to show that δ_0 is not its cluster point. Suppose the contrary. Passing to a subsequence if necessary we may assume that there exists a sequence $n_1 < n_2 < \ldots$ of integers such that

$$T_{c_{n_k}}(c\mu+(1-c)\nu)^{\circ n_k} \to \delta_0, \qquad T_{a_{n_k}}\mu^{\circ n_k} \to \lambda, \qquad T_{b_{n_k}}\nu^{\circ n_k} \to \sigma,$$

$c_{n_k}/a_{n_k} \to a$ and $c_{n_k}/b_{n_k} \to b$ where $\lambda \neq \delta_0$, $\sigma \neq \delta_0$, $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$ and $\max(a, b) = 1$. Setting $\mu_{n_k} = T_{c_{n_k}}\mu$, $\nu_{n_k} = T_{c_{n_k}}\nu$ we have

$$(c\mu_{n_k}+(1-c)\nu_{n_k})^{\circ n_k} \to \delta_0, \qquad \mu_{n_k}^{\circ n_k} \to T_a\lambda, \qquad \nu_{n_k}^{\circ n_k} \to T_b\sigma$$

and at least one limit measure $T_a\lambda$, $T_b\sigma$ is not concentrated at the origin. But this contradicts Lemma 2.3, which completes the proof.

By (1.1) we have $\delta_c \in Q$ for $c > 0$. Since the convex combinations of these measures form a dense subset of P, as an immediate consequence of Theorem 2.3 we get the following statement.

THEOREM 2.4. *The set Q is dense in P.*

3. Limit sets. We start with the following simple topological property of limit sets.

THEOREM 3.1. *Each* $\sim$ *selector limit set is a continuum.*

Proof. Since $\sim$ selector limit sets are compact it suffices to prove their connectedness. Suppose that $\mu \in Q$, $\{a_n\} \in N(\mu)$ and the limit set $G(a_n, \mu)$ is a $\sim$ selector. If $\limsup_{n\to\infty} a_n > 0$, then, by Lemma 3 in [18], $\{1\} \in N(\mu)$ and $\mu^{\circ n} \to \delta_c$ for some $c > 0$. Consequently, $G(1, \mu) = \{\delta_c\}$ and, by Proposition 2.3, $G(a_n, \mu)$ is also a one-point set. In this case $G(a_n, \mu)$ is obviously connected.

Now consider the case

$$\lim_{n\to\infty} a_n = 0 \tag{3.1}$$

and suppose that $G(a_n, \mu)$ is disconnected. There exists then a decomposition $G(a_n, \mu) = A_1 \cup A_2$ where A_1 and A_2 are compact and disjoint. Denoting by ϱ a metric defining the topology of weak convergence on P we have the inequality $d = \varrho(A_1, A_2) > 0$. We denote by N_i $(i = 1, 2)$ the set of all positive integers n fulfilling the condition $\varrho(T_{a_n}\mu^{\circ n}, A_i) \leqslant d/3$. Since $\varrho(T_{a_n}\mu^{\circ n}, A_2) \geqslant \frac{2}{3}d$ if $n \in N_1$ and $\varrho(T_{a_n}\mu^{\circ n}, A_1) \geqslant \frac{2}{3}d$ if $n \in N_2$, we infer that N_1 and N_2 are disjoint. Moreover, the relation $\varrho(T_{a_n}\mu^{\circ n}, A_1 \cup A_2) \to 0$ as $n \to \infty$ yields the existence of an integer n_0 such that $n \in N_1 \cup N_2$ whenever $n \geqslant n_0$. Since both N_1 and N_2 are infinite, we can find a sequence of integers $n_0 < n_1 < n_2 < \ldots$ such that $n_k \in N_1$ and $n_k+1 \in N_2$ $(k = 1, 2, \ldots)$. Passing to a subsequence if necessary we may assume without loss of generality that

$$T_{a_{n_k}}\mu^{\circ n_k} \to \lambda, \qquad T_{a_{n_k+1}}\mu^{\circ(n_k+1)} \to \nu, \qquad \lambda \in A_1,\ \nu \in A_2.$$

Of course $\lambda \neq \nu$. From (3.1) we get $T_{a_{n_k+1}}\mu \to \delta_0$, which yields $T_{a_{n_k+1}}\mu^{\circ n_k} \to \nu$. Consequently, by Lemma 1.1 in [16], $\lambda \sim \nu$. Since $G(a_n, \mu)$ is a $\sim$ selector the last relation implies $\lambda = \nu$, which yields a contradiction. The theorem is thus proved.

PROPOSITION 3.1. *Let* $\mu \in Q$, $\{a_n\} \in N(\mu)$ *and* $\lambda \in G(a_n, \mu)$. *Then* $\lambda \in Q$ *and there exists* $\{b_n\} \in N(\lambda)$ *such that* $G(b_n, \lambda) \subset G(a_n, \mu)$.

Proof. By Lemma 4 in [18] for every $\lambda \in G(a_n, \mu)$ there exists a sequence $\{b_n\}$ of positive numbers such that $T_{b_k}\lambda^{\circ k} \in G(a_n, \mu)$ $(k = 1, 2, \ldots)$. Hence $\lambda \in Q$, $\{b_n\} \in N(\lambda)$ and $G(b_n, \lambda) \subset G(a_n, \mu)$.

As a consequence of the above proposition we get the following statement.

COROLLARY 3.1. *Let* $\mu \in Q_p$ *and* $\lambda \in G_p(\mu)$. *Then* $\lambda \in Q_p$ *and* $G_p(\lambda) \subset G_p(\mu)$.

Let $\mu \in Q$ and $\{a_n\} \in N(\mu)$. The limit set $G(a_n, \mu)$ is said to be *irreducible* if it is a $\sim$ selector and for every $\lambda \in G(a_n, \mu)$ there exists a sequence $\{b_n\} \in N(\lambda)$ such that $G(a_n, \mu) = G(b_n, \lambda)$. In particular, $G_p(\mu)$ is irreducible if and only if $G_p(\lambda) = G_p(\mu)$ for every $\lambda \in G_p(\mu)$. It is evident that $\circ$-stable limit sets are irreducible. The limit set $G(a_n, \mu_0)$ appearing in Example 2.2 is also irreducible. One can easily check that the limit set from Example 2.1 is not irreducible.

THEOREM 3.2. *Let* $\mu \in Q$, $\{a_n\} \in N(\mu)$ *and suppose that* $G(a_n, \mu)$ *is a* $\sim$ *selector. There exist then* $\lambda \in G(a_n, \mu)$ *and* $\{b_n\} \in N(\lambda)$ *such that* $G(b_n, \lambda) \subset G(a_n, \mu)$ *and the limit set* $G(b_n, \lambda)$ *is irreducible.*

Proof. By Proposition 2.3 and Theorem 2.1 we may restrict our attention to the case $G(a_n, \mu) = G_p(\mu)$ for some $p > 0$. Consider the family of limit sets $F = \{G_p(\nu): \nu \in G_p(\mu)\}$. This is a family of compact sets. Consequently, if $\nu_n \in G_p(\mu)$ and $G_p(\nu_n) \supset G_p(\nu_{n+1})$ $(n = 1, 2, \ldots)$, then $\bigcap_{n=1}^{\infty} G_p(\nu_n) \neq \emptyset$. Let $\sigma \in \bigcap_{n=1}^{\infty} G_p(\nu_n)$. Then, by Corollary 3.1, $G_p(\sigma) \subset \bigcap_{n=1}^{\infty} G_p(\nu_n)$. Applying Brouwer's theorem to the family F ([10], Ch. 4, § 42, IV) we infer that there exists $\lambda \in G_p(\mu)$ fulfilling the condition: $A = G_p(\lambda)$ whenever $A \subset G_p(\lambda)$ and $A \in F$. It is clear that $G_p(\lambda)$ is irreducible, which completes the proof.

Before going on we recall a few simple properties of $\circ$-infinitely divisible probability measures. For regular generalized convolutions $\circ$-infinitely divisible measures were studied in [14] and [15]. Here we consider arbitrary generalized convolutions. We say that a probability measure μ from P is $\circ$-*infinitely divisible* if for every positive integer n there exists a measure $\mu_n \in P$ such that $\mu = \mu_n^{\circ n}$. It is easily seen that in the case of the max-convolution each measure from P is infinitely divisible.

LEMMA 3.1. *Let* $n_1 < n_2 < \ldots$, $\lambda, \mu_k \in P$ $(k = 1, 2, \ldots)$ *and* $\mu^{\circ n_k} \to \lambda$. *Then* λ *is* $\circ$-*infinitely divisible.*

Proof. For the max-convolution our assertion is evident. Consequently, we may assume that $\varkappa < \infty$. Let m be an arbitrary positive integer and $p_k = [n_k/m]$ $(k = 1, 2, \ldots)$. Write $n_k = mp_k + q_k$ where $0 \leqslant q_k < m$. By Lemma 2.3 in [19] we have $\mu_k \to \delta_0$. Consequently,

$$\mu_k^{\circ q_k} \to \delta_0. \tag{3.2}$$

Since $\mu_k^{\circ n_k} = (\mu_k^{\circ p_k})^{\circ m} \circ \mu_k^{\circ q_k}$, we infer, by Corollary 2.3 in [20], that the sequence $\mu_k^{\circ p_k}$ is conditionally compact in P. Let λ_m be its cluster point. Taking into account (3.2) we get $\lambda = \lambda_m^{\circ m}$, which completes the proof.

As an immediate consequence of the above lemma we get the following proposition.

PROPOSITION 3.2. *Let* $\mu \in Q$ *and* $\{a_n\} \in N(\mu)$. *Then each measure from* $G(a_n, \mu)$ *is* $\circ$-*infinitely divisible.*

Let λ be a $\circ$-infinitely divisible probability measure. Since $\lambda = \lambda_2^{\circ 2}$ for some $\lambda_2 \in P$ we infer that $\hat{\lambda}(t) = (\hat{\lambda}_2(t))^2 \geqslant 0$. We note that in the case $\varkappa < \infty$ we have, by Lemma 2.4 in [19], an even stronger result: $\hat{\lambda}(t) > 0$ m_0-almost everywhere.

LEMMA 3.2. *Let* λ *be a* $\circ$-*infinitely divisible measure. Then for every positive number* u, $(\hat{\lambda}(t))^u$ *is the weak characteristic function of a probability measure.*

Proof. Denote by W the set of all rational numbers of the form $r/(2s+1)$ $(s = 0, 1, \ldots;\ r = 1, 2, \ldots)$. It is easily seen that $v - w \in W$ whenever $v > w$ and $v, w \in W$. Suppose that $\lambda = \lambda_n^{\circ n}$ or, equivalently,

$$\hat{\lambda}(t) = (\hat{\lambda}_n(t))^n \quad (n = 1, 2, \ldots) \tag{3.3}$$

m_0-almost everywhere. Given an arbitrary positive number u, we can choose $w_k \in W$ $(k = 1, 2, \ldots)$ and $v \in W$ with the properties $w_k < u < v$ and $w_k \to u$. Setting

$$w_k = \frac{r_k}{2s_k+1}, \qquad v = \frac{p}{2q+1}, \qquad v - w_k = \frac{n_k}{2m_k+1}$$

we have, by virtue of (3.3),

$$(\lambda_{2s_k+1}^{\circ r_k})^\wedge(t) = (\hat{\lambda}(t))^{w_k}, \qquad (\lambda_{2q+1}^{\circ p})^\wedge(t) = (\hat{\lambda}(t))^{v},$$
$$(\lambda_{2m_k+1}^{\circ n_k})^\wedge(t) = (\hat{\lambda}(t))^{v - w_k}$$

m_0-almost everywhere. Hence

$$\lambda_{2s_k+1}^{\circ r_k} \circ \lambda_{2m_k+1}^{\circ n_k} = \lambda_{2q+1}^{\circ p} \qquad (k = 1, 2, \ldots),$$

which, by Corollary 2.3 in [20], shows that the sequence $\lambda_{2s_k+1}^{\circ r_k}$ is conditionally compact in P. Let ν be its cluster point. Then $\hat{\nu}(t) = (\hat{\lambda}(t))^u$, which completes the proof.

From now on, for every $\circ$-infinitely divisible measure λ and $u > 0$, $\lambda^{\circ u}$ will denote the measure with the weak characteristic function $(\hat{\lambda}(t))^u$.

PROPOSITION 3.3. *Let* $\mu \in Q$ *and* $\{a_n\} \in N(\mu)$. *Then for every* $\lambda \in G(a_n, \mu)$ *we have*

$$\{\lambda^{\circ u}: u > 0\} \subsetneqq G(a_n, \mu).$$

Moreover, if $G(a_n, \mu)$ is a $\sim$ selector, then there exists a positive continuous function $c(u)$ on the half-line $(0, \infty)$ such that $T_{c(u)}\lambda^{\circ u} \in G(a_n, \mu)$.

Proof. Let $\lambda \in G(a_n, \mu)$. By Lemma 4 in [18] for every positive rational number w there exists a positive number c such that $T_c\lambda^{\circ w} \in G(a_n, \mu)$. Now the compactness of $G(a_n, \mu)$ and the continuity of the relation $\sim$ on $P\setminus\{\delta_0\}$ yield the assertion.

Given a $\circ$-infinitely divisible measure λ we put $\Delta(\lambda) = \{u: u > 1, \lambda \sim \lambda^{\circ u}\}$. Since $(T_a\lambda)^{\circ u} = T_a\lambda^{\circ u}$ $(a, u > 0)$, we infer that the set $\Delta(\lambda)$ is either empty or a closed multiplicative subsemigroup of the half-line $(1, \infty)$. Put $d(\lambda) = \inf\Delta(\lambda)$ if $\Delta(\lambda)$ is nonempty and $d(\lambda) = \infty$ otherwise.

PROPOSITION 3.4. *Let λ be a $\circ$-infinitely divisible measure other than δ_0. Then $d(\lambda) = 1$ if and only if λ is $\circ$-stable.*

Proof. If λ is $\circ$-stable, then, by Proposition 3.3, $\lambda \sim \lambda^{\circ u}$ for all $u > 0$. Consequently, $\Delta(\lambda) = (1, \infty)$, which yields $d(\lambda) = 1$. Conversely, suppose that λ is $\circ$-infinitely divisible, $\lambda \neq \delta_0$ and $d(\lambda) = 1$. Then $\Delta(\lambda)$ must be equal to $(1, \infty)$. Thus $\lambda \sim \lambda^{\circ n}$ for $n = 1, 2, \ldots$, which shows that λ is $\circ$-stable.

A measure ν from P is said to be $\circ$-*quasi-stable* if $\nu \neq \delta_0$ and there exists a measure $\mu \in P$, a sequence $n_1 < n_2 < \ldots$ with $\lim_{k\to\infty} n_{k+1}/n_k = r < \infty$ and a norming sequence $\{c_k\}$ such that $T_{c_k}\mu^{\circ n_k} \to \nu$. The measure μ which can arise here belongs to the *domain of partial attraction* of ν. For ordinary convolution this notion has been introduced and studied by V. M. Kruglov in [9] and for generalized ones by R. Jajte in [6]. We note that, by Lemma 3.1, each $\circ$-quasi-stable measure is $\circ$-infinitely divisible. Moreover, each $\circ$-stable measure is $\circ$-quasi-stable.

LEMMA 3.3. *Let λ be a $\circ$-quasi-stable measure, $\mu \in P$, $n_1 < n_2 < \ldots$,*

$$\lim_{k\to\infty} n_{k+1}/n_k = r \tag{3.4}$$

and $T_{c_k}\mu^{\circ n_k} \to \lambda$ for a norming sequence $\{c_k\}$. Then for every sequence $\{m_k\}$ of integers with $n_k \leqslant m_k \leqslant n_{k+1}$ $(k = 1, 2, \ldots)$ and $\lim_{k\to\infty} m_k/n_k = u$ the relation $T_{c_k}\mu^{\circ m_k} \to \lambda^{\circ u}$ holds.

Proof. By (3.4) there exists an integer s such that $n_{k+1} < sn_k$ $(k = 1, 2, \ldots)$. Put $s_k = sn_k - m_k$ $(k = 1, 2, \ldots)$. Then

$$T_{c_k}\mu^{\circ m_k} \circ T_{c_k}\mu^{\circ s_k} = (T_{c_k}\mu^{\circ n_k})^{\circ s} \to \lambda^{\circ s}$$

and, consequently, by Corollary 2.3 in [20] the sequence $T_{c_k}\mu^{\circ m_k}$ is conditionally compact in P. By Lemma 3.1 all its cluster points are $\circ$-infinitely divisible and, consequently, their weak characteristic functions being nonnegative are cluster points in the $L_1(m_0)$-topology of $L_\infty(m_0)$ of the sequence of functions $(T_{c_k}\mu^{\circ m_k})^\wedge(t)$. Since

$$|(T_{c_k}\mu^{\circ n_k})^\wedge(t)| \to \hat{\lambda}(t), \qquad |(T_{c_k}\mu^{\circ m_k})^\wedge(t)| = |(T_{c_k}\mu^{\circ n_k})^\wedge(t)|^{m_k/n_k},$$

we conclude that all cluster points of the sequence $T_{c_k}\mu^{\circ m_k}$ have the same weak characteristic function $(\hat{\lambda}(t))^u$. In other words, $T_{c_k}\mu^{\circ m_k} \to \lambda^{\circ u}$, which completes the proof.

PROPOSITION 3.5. *Let λ be a $\circ$-infinitely divisible measure other than δ_0. Then $d(\lambda) < \infty$ if and only if λ is $\circ$-quasi-stable.*

Proof. Suppose that λ is $\circ$-quasi-stable and for a measure $\mu \in P$ and a norming sequence $\{c_k\}$

$$T_{c_k}\mu^{\circ n_k} \to \lambda \tag{3.5}$$

where $n_1 < n_2 < \ldots$ and $\lim_{k\to\infty} n_{k+1}/n_k = r$. Passing to a subsequence if necessary we may assume without loss of generality that $r > 1$. Setting $m_k = n_{k+1}$ in Lemma 3.3 we get $T_{c_k}\mu^{\circ n_{k+1}} \to \lambda^{\circ r}$. On the other hand, by (3.5), $T_{c_{k+1}}\mu^{\circ n_{k+1}} \to \lambda$. Applying Lemma 1.1 from [16] we get $\lambda \sim \lambda^{\circ r}$. Thus

$$d(\lambda) \leqslant r. \tag{3.6}$$

Conversely, suppose that λ is $\circ$-infinitely divisible, $\lambda \neq \delta_0$ and $d(\lambda) < \infty$. By Proposition 3.4 we may restrict ourselves to the case $d(\lambda) = r > 1$. There exists then a positive constant c such that $\lambda = T_c\lambda^{\circ r}$. Inductively we get

$$\lambda = T_{c^k}\lambda^{\circ r^k} \quad (k = 1, 2, \ldots). \tag{3.7}$$

First we shall prove that $c < 1$. Contrary to this, suppose that $c \geqslant 1$. If $c = 1$, then, by (3.7), $\lambda = \lambda^{\circ r^k}$ $(k = 1, 2, \ldots)$, which yields $\hat{\lambda}(t) = (\hat{\lambda}(t))^{r^k}$ $(k = 1, 2, \ldots)$. Letting $k \to \infty$ we infer that the function $\hat{\lambda}(t)$ takes the values 1 and 0 only. Thus for every $u > 0$, $(\hat{\lambda}(t))^u = \hat{\lambda}(t)$ or, equivalently, $\lambda^{\circ u} = \lambda$. This yields $d(\lambda) = 1$, which contradicts the assumption. Further, if $c > 1$ then, by (3.7), $\lambda \circ \lambda^{\circ(r^k - 1)} = T_{c^{-k}}\lambda \to \delta_0$, which, by Corollary 2.4 in [20], implies the contradiction $\lambda = \delta_0$. This completes the proof of the inequality $c < 1$.

Put $n_k = [r^k]$, $c_k = c^k$ and $p_k = r^k - n_k$ $(k = 1, 2, \ldots)$. Of course $\lim_{k\to\infty} n_{k+1}/n_k = r$ and $0 \leqslant p_k < 1$ $(k = 1, 2, \ldots)$. Since

$$T_{c^k}\lambda^{\circ p_k} \circ T_{c^k}\lambda^{\circ(1-p_k)} = T_{c^k}\lambda \to \delta_0,$$

we have, by Corollary 2.4 in [20], $T_{c^k}\lambda^{\circ p_k} \to \delta_0$. From (3.7) the formula

$$T_{c^k}\lambda^{\circ n_k} \circ T_{c^k}\lambda^{\circ p_k} = \lambda$$

follows. Now taking into account Corollary 2.3 in [20] we have $T_{c_k}\lambda^{\circ n_k} \to \lambda$, which shows that λ is $\circ$-quasi-stable. The proposition is thus proved.

A limit set $G(a_n, \mu)$ is said to be *$\circ$-quasi-stable* if there exists a $\circ$-quasi-stable measure λ called the *generator* of $G(a_n, \mu)$ such that $G(a_n, \mu) \sim \{\lambda^{\circ u}: 1 \leqslant u \leqslant d(\lambda)\}$. By Proposition 3.4 each $\circ$-stable limit set is $\circ$-quasi-stable.

PROPOSITION 3.6. *Each $\circ$-quasi-stable measure is the generator of a $\circ$-quasi-stable limit set.*

Proof. Suppose that λ is $\circ$-quasi-stable and for a measure $\mu \in P$ and a norming sequence $\{c_k\}$, $T_{c_k}\mu^{\circ n_k} \to \lambda$ where $n_1 < n_2 < \ldots$ and $\lim_{k\to\infty} n_{k+1}/n_k = r$. By Propositions 3.4 and 3.5 we have $d(\lambda) < \infty$. Moreover, one can easily prove the relations

$$\{\lambda^{\circ u}: 1 \leqslant u \leqslant d(\lambda)\} \sim \{\lambda^{\circ u}: 1 \leqslant u \leqslant r\} \sim \{\lambda^{\circ u}: u > 0\}. \tag{3.8}$$

Put $a_n = c_k$ if $n_k \leqslant n < n_{k+1}$. Then, by Lemma 3.3, $\mu \in Q$, $\{a_n\} \in N(\mu)$ and $G(a_n, \mu) = \{\lambda^{\circ u}: 1 \leqslant u \leqslant r\}$. Now our assertion is a consequence of (3.8).

Let A be a subset of P. A point μ of A is said to be *regular* if each neighbourhood U of μ contains a neighbourhood V of μ such that the set $A \cap \mathrm{Fr}(V)$ is finite (see [10], Ch. 6, § 51, I). The set of all regular points of A will be denoted by $\mathrm{reg}(A)$.

The following theorem gives a relationship between topological and probabilistic properties of limit sets.

THEOREM 3.3. *Let A be a limit set. Then the following conditions are equivalent*:

(i) *A is a $\sim$ selector and $\circ$-quasi-stable.*

(ii) *A is irreducible and either a one-point set or homeomorphic to a circle.*

(iii) *A is irreducible and* $\mathrm{reg}(A) \neq \emptyset$.

Proof. By Propositions 2.1 and 2.3 and Theorem 2.1 it suffices to prove the theorem for limit sets of the form $A = G_p(\mu)$ where $p > 0$ and $\mu \in Q$.

(i) $\Rightarrow$ (ii). Suppose that $G_p(\mu)$ is $\circ$-quasi-stable with a generator λ. There exists then a continuous positive function $c(u)$ on the interval $[1, d(\lambda)]$ such that $G_p(\mu) = \{T_{c(u)}\lambda^{\circ u}: 1 \leqslant u \leqslant d(\lambda)\}$. Introducing the notation $\lambda_u = T_{c(u)}\lambda^{\circ u}$ we have $G_p(\mu) = \{\lambda_u: 1 \leqslant u \leqslant d(\lambda)\}$ and $\lambda_1 = \lambda_{d(\lambda)}$. Suppose that $1 \leqslant u < v \leqslant d(\lambda)$ and $\lambda_u = \lambda_v$. Then $\lambda^{\circ u} \sim \lambda^{\circ v}$, which by the formula $(T_a\lambda)^{\circ u} = T_a(\lambda^{\circ u})$, yields $\lambda \sim \lambda^{\circ q}$ where $q = v/u$. Consequently, by the definition of $d(\lambda)$, we have $q \geqslant d(\lambda)$, which implies $u = 1$ and $v = d(\lambda)$. Thus the map $u \to \lambda_u$ is one-to-one on $[1, d(\lambda))$. It follows that $G_p(\mu)$ is a one-point set if $d(\lambda) = 1$ and it is homeomorphic to a circle if $d(\lambda) > 1$. Further, taking into account Proposition 3.3 we have $G_p(\mu) = G_p(\lambda_u)$ for every $u \in [1, d(\lambda)]$, which shows that $G_p(\mu)$ is irreducible.

The implication (ii) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (i). Suppose that $G_p(\mu)$ is irreducible and $\mathrm{reg}(G_p(\mu)) \neq \emptyset$. Let $\lambda \in \mathrm{reg}(G_p(\mu))$. Then

$$G_p(\mu) = G_p(\lambda), \tag{3.9}$$

which yields $\lambda \in G_p(\lambda)$. First we shall prove that $d(\lambda) < \infty$.

Contrary to this, suppose that $d(\lambda) = \infty$. By Proposition 3.3 there exists a positive continuous function $c(u)$ on the half-line $(0, \infty)$ such that $T_{c(u)}\lambda^{\circ u} \in G_p(\lambda)$. Put for brevity $\lambda_u = T_{c(u)}\lambda^{\circ u}$ and $\langle a, b\rangle = \{\lambda_u: a \leqslant u \leqslant b\}$ $(0 < a < b < \infty)$. Since $d(\lambda) = \infty$, we infer that $G_p(\lambda)$ is infinite, $\langle a, b\rangle$ is a homeomorphic image of the interval $[a, b]$ and $\langle a, b\rangle \cap \langle a', b'\rangle = \emptyset$ whenever $[a, b] \cap [a', b'] = \emptyset$. Let $\nu \in G_p(\lambda)$ and $\nu \neq \lambda$. We can choose neighbourhoods U_1 and U_2 of λ and ν respectively such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Since $G_p(\lambda)$ is a $\sim$ selector, we infer that both λ and ν are cluster points of the sequence $\{\lambda_n\}$. Given an arbitrary neighbourhood V of λ, for every positive integer k we can choose a pair n_k, m_k of positive integers with $k \leqslant n_k < m_k$, $\lambda_{n_k} \in V$ and $\lambda_{m_k} \in U_2$. It is clear that $\langle n_k, m_k\rangle \cap \mathrm{Fr}(V) \neq \emptyset$. Taking an infinite subsequence of disjoint intervals $[n_k, m_k]$ we conclude that $G_p(\mu) \cap \mathrm{Fr}(V)$ is infinite, which contradicts the assumption $\lambda \in \mathrm{reg}(G_p(\mu))$. Thus $d(\lambda) < \infty$.

Now taking into account Proposition 3.5 we infer that λ is $\circ$-quasi-stable, which immediately yields the equality $G_p(\lambda) = \{\lambda_u: 1 \leqslant u \leqslant d(\lambda)\}$. Thus, by (3.9), $G_p(\mu)$ is $\circ$-quasi-stable, which completes the proof.

We note that the limit set $G(a_n, \mu)$ in Example 1.1 is a $\sim$ selector homeomorphic to a circle, but is not $\circ$-quasi-stable. This shows that the assumption of irreducibility in conditions (ii) and (iii) in the last theorem is essential and cannot be replaced by the assumption that the limit set is a $\sim$ selector.

4. Regular convolutions. Throughout this section we assume that the convolution in question is regular. Then the kernel Ω of the weak characteristic function (1.2) is continuous and the continuous version of $\hat{\mu}$ is called the *characteristic function* of μ. In this case the convergence in the $L_1(m_0)$-topology of $L_\infty(m_0)$ is equivalent to the uniform convergence on every compact subset of $\boldsymbol{R}_+$ (see [16], Lemma 4.5). From now on $\hat{\mu}$ will denote the characteristic function of μ. By Theorem 7 in [14] the characteristic exponent $\varkappa$ is finite and

$$1 - \Omega(x) = x^{\varkappa} L(x) \tag{4.1}$$

where the function L is slowly varying at the origin. Moreover, the kernel Ω satisfies the inequality $\Omega(x) < 1$ in a neighbourhood $0 < x < x_0$. Changing the scale $\Omega(x) \to \Omega(ax)$ $(a > 0)$ we get another kernel of a characteristic function. Consequently, we may assume without loss of generality that $x_0 > 1$, i.e. $\Omega(x) < 1$ if $x \in (0, 1]$. Let $\omega(x) = 1 - \Omega(x)$ if $x \in [0, 1]$ and $\omega(x)$

$= 1-\Omega(1)$ otherwise. M. Kłosowska proved in [8], Lemma 1, the following formula:

$$\lim_{x\to 0}\frac{\omega(x)}{x^{\varkappa}} = \frac{1}{m_{\varkappa}(\sigma_{\varkappa})}\tag{4.2}$$

where $\sigma_{\varkappa}$ is the characteristic measure of $\circ$. Moreover, it has been proved in [14], Theorem 13, that a measure μ from P is $\circ$-infinitely divisible if and only if its characteristic function is of the form

$$\hat{\mu}(t) = \exp\int_0^{\infty}\frac{\Omega(tx)-1}{\omega(x)}M(dx),$$

where M is a finite Borel measure on R_+ and the integrand is assumed to be $-t^{\varkappa}$ if $x = 0$. By Theorem 1 in [15] this representation is unique. In the sequel we shall use the notation $\mu = e(M)$. In particular, $\sigma_{\varkappa} = e(\delta_0)$ and $\delta_0 = e(0)$. One can easily prove the formulae

$$e(M+N) = e(M)\circ e(N),\qquad e(M)^{\circ u} = e(uM)\quad (u>0),\tag{4.3}$$

$$T_a e(M) = e(M_a)\quad (a>0)\tag{4.4}$$

where $M_a(E) = \int_{a^{-1}E}(\omega(ax)/\omega(x))M(dx)$ and the integrand is assumed to be $a^{\varkappa}$ if $x = 0$. We note that, by Lemma 3 in [18], for regular convolutions we have $\lim_{n\to\infty} a_n = 0$ for all $\mu\in Q$ and all $\{a_n\}\in N(\mu)$.

Given $\mu\in P$ we put $\tilde{\mu}(E) = \int_E \omega(x)\mu(dx)$. By Jurek's theorem on accompanying laws ([7], p. 115) we have the following statement.

PROPOSITION 4.1. *$\mu\in Q$ if and only if $e(\tilde{\mu})\in Q$. Moreover, $N(\mu) = N(e(\tilde{\mu}))$ and $G(a_n,\mu) = G(a_n, e(\tilde{\mu}))$ for any $\{a_n\}\in N(\mu)$.*

Further, taking into account Theorem 2 in [7] on convergence of $\circ$-infinitely divisible measures and the formula $T_a e(\tilde{\mu})^{\circ n} = e(n(T_a\mu)^{\sim})$ $(a>0)$ we obtain the following criterion.

PROPOSITION 4.2. *$\mu\in Q$ and $\{a_n\}\in N(\mu)$ if and only if the sequence of measures $n(T_{a_n}\mu)^{\sim}$ is conditionally compact and the measure identically vanishing is not its cluster point. If $H(a_n,\mu)$ is the set of all cluster points of the sequence $n(T_{a_n}\mu)^{\sim}$, then*

$$G(a_n,\mu) = \{e(M)\colon M\in H(a_n,\mu)\}.$$

LEMMA 4.1. *Let $\mu\in Q$ and $\{a_n\}\in N(\mu)$. Then*

$$\int_0^{\infty}\frac{M(dx)}{\omega(x)} = \infty$$

for all $M\in H(a_n,\mu)$.

Proof. Suppose the contrary, i.e.

$$c = \int_0^{\infty}\frac{M(dx)}{\omega(x)} < \infty$$

for a measure $M\in H(a_n,\mu)$. Since, by Proposition 4.2, M does not vanish identically, we have $c>0$. Put

$$\nu(E) = c^{-1}\int_E\frac{M(dx)}{\omega(x)}.$$

Evidently $\nu\in P$ and the measure λ defined by the formula

$$\lambda = e^{-c}\delta_0 + \sum_{k=1}^{\infty} e^{-c}\frac{c^k}{k!}\nu^{\circ k}$$

also belongs to P. Moreover,

$$\lambda(\{0\}) \geqslant e^{-c}\tag{4.5}$$

and $\hat{\lambda}(t) = \exp(c\hat{\nu}(t)-c) = e(M)^{\wedge}(t)$, which, by Proposition 4.2, yields $\lambda\in G(a_n,\mu)$. From Lemma 5 in [18] we get $\lambda(\{0\}) = 0$, which contradicts (4.5). The lemma is thus proved.

Given $\mu\in P$ we introduce the notation

$$F_\mu(x) = \int_0^x \omega(y/x)\mu(dx)\qquad (x>0).$$

It is clear that the function F_μ is continuous on the right, the left-hand limit $F_\mu(x-)$ exists and

$$F_\mu(x-) = F_\mu(x) - \omega(1)\mu(\{x\})\qquad (x>0).\tag{4.6}$$

Moreover,

$$\lim_{x\to\infty} F_\mu(x) = 0.\tag{4.7}$$

Put for $\mu\in P\setminus\{\delta_0\}$ and $n = 1, 2, \ldots$

$$B_n(\mu) = \{t\colon t\geqslant 0,\ n(1-\int_0^1\hat{\mu}(tx)\,dx) = 1\},$$

$$D_n(\mu) = \{z\colon z>0,\ nF_\mu(z^{-1})\geqslant 1\}.$$

Since the characteristic function $\hat{\mu}$ is continuous, $\hat{\mu}(0) = 1$, $|\hat{\mu}(t)|\leqslant 1$ for all $t\in R_+$ and $\hat{\mu}$ is not identically zero we infer that the sets $B_n(\mu)$ are nonvoid for n large enough. The sets $D_n(\mu)$ are also nonvoid for sufficiently large n

because F_μ does not vanish identically. Consequently, there exists an index n_0 such that both $B_n(\mu)$ and $D_n(\mu)$ are nonvoid for $n \geq n_0$. Define

$$b_n(\mu) = \min B_n(\mu), \quad d_n(\mu) = \inf D_n(\mu) \quad \text{for } n \geq n_0,$$

$$b_n(\mu) = b_{n_0}(\mu), \quad d_n(\mu) = d_{n_0}(\mu) \quad \text{for } n < n_0.$$

It is clear, by (4.7), that $d_n(\mu) > 0$ $(n = 1, 2, \ldots)$, $d_n(\mu) \to 0$ and, by (4.6),

$$nF_\mu(d_n^{-1}(\mu)) \geq 1 \quad (n \geq n_0). \tag{4.8}$$

LEMMA 4.2. *Let $\mu \neq \delta_0$. Then $d_{n+1}(\mu) < d_n(\mu)$ and $nF_\mu(d_n^{-1}(\mu)) = 1$ for $n \geq n_0$.*

Proof. Given $n \geq n_0$ we have, by the definition of $d_n(\mu)$, the inequality $nF_\mu(z^{-1}) < 1$ for $z \in (0, d_n(\mu))$. Hence, by the continuity of F_μ on the right, $nF_\mu(d_n^{-1}(\mu)) \leq 1$. Comparing this with (4.8) we get $nF_\mu(d_n^{-1}(\mu)) = 1$. Now the inequality $(n+1)F_\mu(d_n^{-1}(\mu)) > 1$ yields $d_n(\mu) > d_{n+1}(\mu)$, which completes the proof.

LEMMA 4.3. *Let $\mu \in Q$ and $\{a_n\} \in N(\mu)$. There exists then a positive integer s such that $a_n > d_{sn}(\mu)$ $(n = 1, 2, \ldots)$.*

Proof. By Proposition 4.2 the sequence of measures $n(T_{a_n}\mu)^\sim$ is conditionally compact and

$$\liminf_{n\to\infty} n(T_{a_n}\mu)^\sim([0, 1)) \geq b$$

where $b = \inf\{M([0, 1)): M \in H(a_n, \mu)\}$. Since the set $H(a_n, \mu)$ is compact, we have, by Lemma 4.1, $b > 0$. Observe that $nF_\mu(a_n^{-1}) = n(T_{a_n}\mu)^\sim([0, 1])$ $(n = 1, 2, \ldots)$. Consequently, we can find a positive integer s such that $nF_\mu(a_n^{-1}) > s^{-1}$ $(n = 1, 2, \ldots)$. For $n \geq n_0$ the inequality $snF_\mu(a_n^{-1}) > 1$ yields $a_n > d_{sn}(\mu)$, which completes the proof.

LEMMA 4.4. *Let $\mu \in P$ and $\mu \neq \delta_0$. There exists then a positive integer r such that $d_n(\mu) \geq b_{rn}(\mu)$ for $n \geq n_0$.*

Proof. Put

$$g(x) = \inf\{\omega(xz)/\omega(z): 0 < z \leq 1\}.$$

Taking into account (4.1) we have $g(x) > 0$ for $x > 0$. Moreover, for $u > 0$ and $x \in (0, 1]$

$$F_\mu(x^{-1}u) \geq \int_0^u \omega(xy/u)\,\mu(dy) \geq g(x)F_\mu(u).$$

Hence, setting for brevity $d_n = d_n(\mu)$, we get

$$n\Big(1 - \int_0^1 \hat\mu(d_n x)\,dx\Big) \geq n\int_0^1 \int_0^{d_n^{-1}x^{-1}} \big(1 - \Omega(d_n xy)\big)\,\mu(dy)\,dx$$

$$= n\int_0^1 F_\mu(d_n^{-1}x^{-1})\,dx \geq nF_\mu(d_n^{-1})\int_0^1 g(x)\,dx.$$

Taking a positive integer r satisfying the inequality $r^{-1} < \int_0^1 g(x)\,dx$ and observing that, by Lemma 4.2, $nF_\mu(d_n^{-1}) = 1$ for $n \geq n_0$ we get

$$rn\Big(1 - \int_0^1 \hat\mu(d_n x)\,dx\Big) > 1 \quad (n \geq n_0),$$

which yields $d_n > b_{rn}(\mu)$ for $n \geq n_0$. The lemma is thus proved.

PROPOSITION 4.3. *If $\mu \in Q$, then $\{d_n(\mu)\} \in N(\mu)$.*

Proof. It follows from Theorem 3.3 in [19] that $\{b_n(\mu)\} \in N(\mu)$. Consequently, by Lemmas 4.3 and 4.4 there exist positive integers s and r such that $b_n(\mu) \geq d_{sn}(\mu)$ and $d_n(\mu) \geq b_{rn}(\mu)$ for $n \geq n_0$. The last inequality implies $d_{sn}(\mu) \geq b_{srn}(\mu)$ $(n \geq n_0)$. Taking into account Corollary 1 in [18] we get $b_{srn}(\mu) \geq ab_n(\mu)$ $(n = 1, 2, \ldots)$ where a is a positive constant. Thus $b_n(\mu) \geq d_{sn}(\mu) \geq ab_n(\mu)$ for $n \geq n_0$, which, by Lemma 1 in [18], yields $\{d_{sn}(\mu)\} \in N(\mu)$. Since, by Lemma 4.2, the sequence $\{d_n(\mu)\}$ is nonincreasing, the last relation and Lemma 3.1 in [19] imply $\{d_n(\mu)\} \in N(\mu)$, which completes the proof.

The following result is an analogue of Feller's criterion established for the ordinary convolution in [3], p. 387.

THEOREM 4.1. *$\mu \in Q$ if and only if $\mu \neq \delta_0$ and*

$$\limsup_{x\to\infty} \mu((x, \infty))\Big/\int_0^x \omega(y/x)\,\mu(dy) < \infty. \tag{4.9}$$

Proof. Necessity. Suppose that $\mu \in Q$. Then $\mu \neq \delta_0$ and, by Proposition 4.3, $\{d_n(\mu)\} \in N(\mu)$. Put for brevity $d_n = d_n(\mu)$ and $x_0 = d_{n_0}^{-1}$ where n_0 is determined in the definition of $d_n(\mu)$. Then for every $x \geq x_0$ we can choose an index $k \geq n_0$ such that $d_k^{-1} \leq x < d_{k+1}^{-1}$. Setting

$$a = \sup\{n\mu((d_n^{-1}, \infty)): n = 1, 2, \ldots\}$$

we have, by Lemma 2.6 in [19], $a < \infty$. Moreover, $k\mu((x, \infty)) \leq k\mu((d_k^{-1}, \infty)) \leq a$. Since, by Lemma 4.2, $kF_\mu(d_k^{-1}) = 1$ the last inequality yields

$$\mu((x, \infty)) \leq aF_\mu(d_k^{-1}). \tag{4.10}$$

Further, by Corollary 1 in [18], we have

$$b = \sup\{d_n/d_{n+1}: n = 1, 2, \ldots\} < \infty.$$

Setting

$$h(u) = \sup\{\omega(zv)/\omega(z):\ 0 < z \leqslant 1,\ 0 \leqslant v \leqslant u\} \tag{4.11}$$

for $u \geqslant 0$ we get, by (4.1), a nondecreasing function with $h(u) < \infty$ and

$$\lim_{u\to 0} h(u) = 0. \tag{4.12}$$

Moreover, it is easy to check that

$$F_\mu(xt) \leqslant h(t^{-1})F_\mu(x) \tag{4.13}$$

for $t \in (0, 1]$. Substituting $t = d_k^{-1}x^{-1}$ in this inequality we obtain

$$F_\mu(d_k^{-1}) \leqslant h(xd_k)F_\mu(x) \leqslant h(b)F_\mu(x),$$

which together with (4.10) yields

$$\mu((x,\infty)) \leqslant ah(b)F_\mu(x)$$

for $x \geqslant x_0$. The necessity of condition (4.9) is thus proved.

Sufficiency. Suppose that (4.9) is fulfilled and $\mu \neq \delta_0$. For simplicity of notation we put $d_n = d_n(\mu)$ and $M_n = n(T_{d_n}\mu)^\sim$ $(n = 1, 2, \ldots)$. By Proposition 4.2 it suffices to prove that the sequence of measures $\{M_n\}$ is conditionally compact and the measure vanishing identically is not its cluster point.

Denote by s the left-hand side of (4.9). Since $\omega(t) \leqslant 2$ for all $t \geqslant 0$, we have

$$M_n(\boldsymbol{R}_+) = n\int_0^\infty \omega(d_n x)\mu(dx) \leqslant nF_\mu(d_n^{-1}) + 2n\mu((d_n^{-1},\infty)),$$

which, by (4.9) and Lemma 4.2, yields $\limsup_{n\to\infty} M_n(\boldsymbol{R}_+) \leqslant 1+2s$. Hence the sequence of measures $\{M_n\}$ is conditionally compact on the compactified half-line $[0, \infty]$. Consequently, each its subsequence contains a subsequence $\{M_{n_k}\}$ converging on $[0, \infty]$.

Assume that

$$M_{n_k} \to M + q\delta_\infty \tag{4.14}$$

where $q \geqslant 0$ and M is concentrated on $\boldsymbol{R}_+$. Define the auxiliary functions f_a $(a > 1)$ on $\boldsymbol{R}_+$ by setting $f_a(x) = x\omega(a^{-1})$ in $[0, 1]$, $\omega(a^{-1}x)$ in $[1, a]$, $\omega(1)(1+a-x)$ in $(a, 1+a)$, and 0 in $[1+a, \infty)$. It is evident that the functions f_a $(a > 1)$ are continuous, uniformly bounded and

$$\lim_{a\to\infty} f_a(x) = 0 \tag{4.15}$$

for every $x \in \boldsymbol{R}_+$. Moreover, by (4.13),

$$\int_0^\infty f_a(x)M_{n_k}(dx) \to \int_0^\infty f_a(x)M(dx) \tag{4.16}$$

for every $a > 1$. Let h be the function defined by (4.11). Taking into account (4.12) and (4.15), for every $\varepsilon > 0$ we can find $b > 1$ such that

$$h(b^{-1}) < \varepsilon, \qquad \int_0^\infty f_b(x)M(dx) < \varepsilon. \tag{4.17}$$

Since

$$nF_\mu(bd_n^{-1}) = n\int_0^{d_n^{-1}} \omega(d_n xb^{-1})\mu(dx) + \int_1^b \omega(x/b)M_n(dx),$$

$$\int_1^b \omega(x/b)M_n(dx) \leqslant \int_0^\infty f_b(x)M_n(dx)$$

and, by Lemma 4.2,

$$n\int_0^{d_n^{-1}} \omega(d_n xb^{-1})\mu(dx) \leqslant h(b^{-1})nF_\mu(d_n^{-1}) = h(b^{-1}),$$

we have, by (4.16) and (4.17),

$$\limsup_{k\to\infty} n_k F_\mu(bd_{n_k}^{-1}) \leqslant 2\varepsilon,$$

which, by (4.9), implies the inequality

$$\limsup_{k\to\infty} n_k\mu((bd_{n_k}^{-1},\infty)) \leqslant 2\varepsilon s. \tag{4.18}$$

Since $\omega(t) \leqslant 2$ for $t \geqslant 0$ we have

$$M_n((b,\infty)) = n\int_{bd_n^{-1}}^\infty \omega(d_n x)\mu(dx) \leqslant 2n\mu((bd_n^{-1},\infty)).$$

Consequently, form (4.18) the inequality

$$\limsup_{k\to\infty} M_{n_k}((b,\infty)) \leqslant 4\varepsilon s$$

follows. By the arbitrariness of ε this shows that the sequence $\{M_{n_k}\}$ is conditionally compact on $\boldsymbol{R}_+$ and, by (4.14), $M_{n_k} \to M$. Since $M_n([0, 1]) = nF_\mu(d_n^{-1}) = 1$, we conclude that the limit measure M does not vanish identically. This proves that the sequence of measures $\{M_k\}$ is conditionally compact on $\boldsymbol{R}_+$ and the measure vanishing identically is not its cluster point, which completes the proof.

The following theorem has been proved in [1] and [8] under the additional assumption $m_\varkappa(\sigma_\varkappa) < \infty$.

THEOREM 4.2. *A measure μ from P belongs to the domain of attraction of the characteristic measure $\sigma_\varkappa$ if and only if $\mu \neq \delta_0$ and*

$$\lim_{x\to\infty} \mu((x,\infty))/\int_0^x \omega(y/x)\mu(dy) = 0. \tag{4.19}$$

Proof. Sufficiency. Suppose that (4.19) is fulfilled and $\mu \neq \delta_0$. Then, by Theorem 4.1, $\mu \in Q$ and, consequently, by Proposition 4.3, $\{d_n(\mu)\} \in N(\mu)$. Put for brevity $d_n = d_n(\mu)$ and $M_n = n(T_{d_n}\mu)^\sim$ $(n = 1, 2, \ldots)$. Let $a \in (0, 1]$. Then, by (4.13) and Lemma 4.2,

$$nF_\mu(ad_n^{-1}) \leqslant nh(a^{-1})F_\mu(d_n^{-1}) = h(a^{-1}),$$

which together with the inequality

$$M_n((a, \infty)) \leqslant 2n\mu((ad_n^{-1}, \infty))$$

and (4.19) yields $\lim_{n\to\infty} M_n((a, \infty)) = 0$. Consequently, all measures M from $H(d_n, \mu)$ are concentrated at the origin, which, by (4.4), yields $e(M) \sim \sigma_\varkappa$. Applying Proposition 4.2 we obtain $G(d_n, \mu) \sim \{\sigma_\varkappa\}$. By Corollary 2.1 we can find a norming sequence $\{a_n\} \in N(\mu)$ such that $G(a_n, \mu)$ is a $\sim$ selector. Then, by Proposition 2.1, $G(a_n, \mu) = \{T_c\sigma_\varkappa\}$ for some $c > 0$. Thus $G(c^{-1}a_n, \mu) = \{\sigma_\varkappa\}$, which shows that the condition in question is sufficient.

Necessity. Suppose that μ belongs to the domain of attraction of $\sigma_\varkappa$. Obviously, $\mu \in Q$ and, by Proposition 2.1, $G(a_n, \mu) \sim \{\sigma_\varkappa\}$ for every $\{a_n\} \in N(\mu)$. Observe that we can always choose a strictly decreasing norming sequence $\{b_n\}$ from $N(\mu)$. Indeed, by Proposition 4.3 and Lemma 4.2 it suffices to put $b_n = d_n(\mu)$ for $n \geqslant n_0$. Denote by s the left-hand side of (4.9). Since $b_n \to 0$, we can find a sequence $\{x_n\}$ fulfilling the conditions

$$b_n^{-1} \leqslant x_n < b_{n+1}^{-1} \quad (n = 1, 2, \ldots), \tag{4.20}$$

$$\limsup_{n\to\infty} \mu((x_n, \infty))/\int_0^{x_n} \omega(y/x_n)\,\mu(dy) = s. \tag{4.21}$$

By Corollary 1 in [18] the sequence $\{b_{n+1}^{-1}b_n\}$ is bounded. Consequently, from (4.20) it follows that $1 \leqslant x_n b_n$ $(n = 1, 2, \ldots)$ and the sequence $\{x_n b_n\}$ is bounded, which, by Lemma 1 in [18], yields $\{x_n^{-1}\} \in N(\mu)$. Since $G(x_n^{-1}, \mu) \sim \{\sigma_\varkappa\}$, all measures from $H(x_n^{-1}, \mu)$ are concentrated at the origin and

$$c = \inf\{M(R_+)\colon\ M \in H(x_n^{-1}, \mu)\} > 0.$$

Consequently, setting $M_n = n(T_{x_n^{-1}}\mu)^\sim$ $(n = 1, 2, \ldots)$ and taking into account Proposition 4.2 we have $\lim_{n\to\infty} M_n((1, \infty)) = 0$ and $\liminf_{n\to\infty} M_n([0, 1]) \geqslant c$. Since

$$M_n((1, \infty)) = n\int_{x_n}^{\infty} \omega(y/x_n)\,\mu(dy) = n\mu((x_n, \infty))\,\omega(1),$$

$$M_n([0, 1]) = n\int_0^{x_n} \omega(y/x_n)\,\mu(dy),$$

we conclude, by (4.21), that $s = 0$, which completes the proof.

Suppose that $p > \varkappa$, $\mu \in P$, $\mu \neq \delta_0$ and $m_p(\mu) < \infty$. Let L be the slowly varying function appearing in (4.1). Then $\lim_{x\to 0} x^{\varkappa - p}L(x) = \infty$ ([13], p. 18). Consequently, $\lim_{x\to 0} \omega(x)/x^p = \infty$. Put

$$f(x) = \inf\{\omega(y)/y^p\colon\ 0 < y \leqslant x\}.$$

Evidently, $\lim_{x\to 0} f(x) = \infty$. Taking a positive number b satisfying the condition $\int_0^b y^p\,\mu(dy) \geqslant \frac{1}{2}m_p(\mu)$ we have for $x \geqslant b$

$$x^p\int_0^x \omega(y/x)\,\mu(dy) \geqslant \int_0^b f(y/x)\,y^p\,\mu(dy) \geqslant \tfrac{1}{2}m_p(\mu)\,f(b/x),$$

which together with the standard inequality $x^p\mu((x, \infty)) \leqslant m_p(\mu)$ yields

$$\mu((x, \infty))/\int_0^x \omega(y/x)\,\mu(dy) \leqslant \frac{2}{f(b/x)}$$

for $x \geqslant b$. Letting $x \to \infty$ we get condition (4.19). Consequently, Theorem 4.2 yields the following corollary.

Corollary 4.1. *Let $p > \varkappa$. If $\mu \in P$, $\mu \neq \delta_0$ and $m_p(\mu) < \infty$, then μ belongs to the domain of attraction of the characteristic measure $\sigma_\varkappa$.*

From Kłosowska and Bingham's description of the domain of attraction of $\sigma_\varkappa$ in [1] and [8] it follows that under the condition $m_\varkappa(\sigma_\varkappa) < \infty$ all measures μ from $P\setminus\{\delta_0\}$ with $m_\varkappa(\mu) < \infty$ belong to the domain of attraction of $\sigma_\varkappa$ too. The converse implication is also true:

Theorem 4.3. *If all measures μ from $P\setminus\{\delta_0\}$ with $m_\varkappa(\mu) < \infty$ belong to Q, then $m_\varkappa(\sigma_\varkappa) < \infty$.*

Proof. Suppose the contrary. Then, by (4.2), $\lim_{x\to 0} \omega(x)/x^\varkappa = 0$. We define a sequence $c_1 < c_2 < \ldots$ of positive numbers recursively as follows. Put $c_1 = 1$. If $c_1, \ldots, c_n$ are defined, then we take $c_{n+1} > c_n$ satisfying the condition

$$n\max\left\{\omega\left(\frac{c_k}{c_{n+1}}\right)\left(\frac{c_k}{c_{n+1}}\right)^{-\varkappa}\colon\ k = 1, \ldots, n\right\} < \frac{1}{2^{n+1}}. \tag{4.22}$$

Evidently, $c = \sum_{n=1}^{\infty} 2^{-n}c_n^{-\varkappa} < \infty$. Put $p_n = c^{-1}2^{-n}c_n^{-\varkappa}$ $(n = 1, 2, \ldots)$. Then $\sum_{n=1}^{\infty} p_n = 1$ and the measure $\nu = \sum_{n=1}^{\infty} p_n\delta_{c_n}$ belongs to P. Moreover, $m_\varkappa(\nu) = c^{-1} < \infty$. Thus $\nu \in Q$ and, by Theorem 4.1, ν fulfils condition (4.9). It is clear that $c_n \to \infty$ and for sufficiently large n condition (4.9) can be rewritten in the form

$$\sum_{k=n+1}^{\infty} p_k \leqslant b\sum_{k=1}^{n} p_k\,\omega(c_k/x)$$

where $x \in [c_n, c_{n+1})$ and b is a positive constant. Letting $x \to c_{n+1}$ and taking into account (4.22) we get

$$p_{n+1} < \sum_{k=n+1}^{\infty} p_k \leqslant b \sum_{k=1}^{n} p_k \omega(c_k/c_{n+1})$$
$$\leqslant bn^{-1} 2^{-n-1} c_{n+1}^{-\varkappa} \sum_{k=1}^{n} c_k^{\varkappa} p_k \leqslant \frac{b}{n} p_{n+1},$$

which yields a contradiction as $n \to \infty$. The theorem is thus proved.

We say that a $\circ$-infinitely divisible measure λ *has a $\varkappa$-component* if there exist a positive number c and a $\circ$-infinitely divisible measure ν such that $\lambda = T_c \sigma_\varkappa \circ \nu$. Taking the representation $\lambda = e(M)$ we infer that λ has a $\varkappa$-component if and only if $M(\{0\}) > 0$. It follows that each $\circ$-infinitely divisible measure λ with a $\varkappa$-component has a unique decomposition $\lambda = T_c \sigma_\varkappa \circ \nu$ where $c > 0$ and ν is $\circ$-infinitely divisible without a $\varkappa$-component.

LEMMA 4.5. *Suppose that $m_\varkappa(\sigma_\varkappa) < \infty$, $n_1 < n_2 < \ldots$, $c_k > 0$ $(k = 1, 2, \ldots)$, $\limsup_{k\to\infty} n_k c_k^\varkappa > 0$, $\nu_k \to \nu \neq \delta_0$ and $T_{c_k} \nu_k^{\circ n_k} \to \lambda$. Then λ has a $\varkappa$-component.*

Proof. First we observe that, by Lemma 3.1, the measure λ is $\circ$-infinitely divisible and, by Lemma 2.3 in [19], $c_k \to 0$. Taking the representation $\lambda = e(M)$ and setting $M_k = n_k(T_{c_k}\nu_k)$ $(k = 1, 2, \ldots)$ we have, by Theorems 1 and 2 in [7], $M_k \to M$. Further, the assumption $m_\varkappa(\sigma_\varkappa) < \infty$ and formula (4.2) yield the inequality

$$a = \inf\{\omega(y)/y^\varkappa:\ 0 < y \leqslant 1\} > 0.$$

Since $\nu \neq \delta_0$, we can find a sufficiently large number x_0 such that $b = \int_0^{x_0} y^\varkappa \nu(dy) > 0$ and x_0 is not an atom of ν. Given $x \in (0, 1]$ we have for $c_k \leqslant x_0^{-1} x$

$$M_k([0, x]) = n_k \int_0^{xc_k^{-1}} \omega(c_k y) \nu_k(dy)$$
$$\geqslant n_k \int_0^{x_0} \omega(c_k y) \nu_k(dy) \geqslant a n_k c_k^\varkappa \int_0^{x_0} y^\varkappa \nu_k(dy),$$

which yields

$$M([0, x]) \geqslant \limsup_{k\to\infty} M_k([0, x]) \geqslant ab \limsup_{k\to\infty} n_k c_k^\varkappa.$$

Hence $M(\{0\}) > 0$, which shows that λ has a $\varkappa$-component.

THEOREM 4.4. *The set $Q_\varkappa$ is contained in the domain of attraction of the characteristic measure $\sigma_\varkappa$.*

Proof. Suppose that $Q_\varkappa \neq \emptyset$, $\mu \in Q_\varkappa$ and $\lambda \in G_\varkappa(\mu)$. It suffices to show that $\lambda \sim \sigma_\varkappa$. By Lemma 4 in [18] we can choose a sequence $\{a_n\}$ of positive numbers such that $T_{a_n} \lambda^{\circ n} \in G_\varkappa(\mu)$. Of course, $\{a_n\} \in N_\varkappa(\lambda)$. Taking the representation $\lambda = e(M)$ and setting $M_n = n(T_{a_n}\lambda)^\sim$ we have $T_{a_n}\lambda^{\circ n} = e(M_n)$ $(n = 1, 2, \ldots)$. Further, applying Theorem 1 from [17] we get

$$1 = m_\varkappa(T_{a_n}\lambda^{\circ n}) = n a_n^\varkappa \quad (n = 1, 2, \ldots). \tag{4.23}$$

From formula (4.2) the inequality

$$b = \sup\{\omega(y)/y^\varkappa:\ y > 0\} < \infty$$

follows. Given $x > 0$ we have, by (4.23),

$$M_n([x, \infty)) = n \int_{xa_n^{-1}}^{\infty} \omega(a_n y) \lambda(dy) \leqslant b \int_{xa_n^{-1}}^{\infty} y^\varkappa \lambda(dy) \quad (n = 1, 2, \ldots).$$

Thus $\lim_{n\to\infty} M_n([x, \infty)) = 0$ and, consequently, by Proposition 4.2, $H(a_n, \lambda)$ consists of measures concentrated at the origin. In other words, $G_\varkappa(\lambda) \sim \{\sigma_\varkappa\}$, which yields $m_\varkappa(\sigma_\varkappa) < \infty$. By Lemma 4 in [18] we have the existence of a sequence $\{c_n\}$ of positive numbers and a sequence $\{\nu_n\}$ of measures from $G_\varkappa(\mu)$ such that

$$\lambda = T_{c_n} \nu_n^{\circ n} \quad (n = 1, 2, \ldots). \tag{4.24}$$

By Theorem 1 in [17] we have

$$1 = m_\varkappa(T_{c_n}\nu_n^{\circ n}) = n c_n^\varkappa \quad (n = 1, 2, \ldots). \tag{4.25}$$

Taking the representation $\nu_n = e(N_n)$ we have, by (4.3), (4.4) and (4.24),

$$M(E) = n \int_{c_n^{-1}E} \frac{\omega(c_n x)}{\omega(x)} N_n(dx),$$

which, by (4.25), yields $M(\{0\}) = N_n(\{0\})$ $(n = 1, 2, \ldots)$. Setting $\sigma = e(M(\{0\})\delta_0)$, $M'(E) = M(E \cap (0, \infty))$, $N'_n(E) = N_n(E \cap (0, \infty))$ $(n = 1, 2, \ldots)$, we have the formulae

$$\lambda = \sigma \circ e(M'), \tag{4.26}$$

$$\nu_n = \sigma \circ e(N'_n) \quad (n = 1, 2, \ldots) \tag{4.27}$$

and, by (4.24),

$$e(M') = T_{c_n} e(N'_n)^{\circ n} \quad (n = 1, 2, \ldots). \tag{4.28}$$

Now we shall prove that

$$e(N'_n) \to \delta_0. \tag{4.29}$$

Suppose the contrary. Since $\nu_n \in G_\varkappa(\mu)$, the sequence $\{\nu_n\}$ is conditionally compact. From (4.27) and Corollary 2.3 in [20] we obtain the conditional compactness of the sequence $\{e(N'_n)\}$. Consequently, there exists a subsequence $n_1 < n_2 < \ldots$ such that $e(N'_{n_k})$ tends to a measure other than δ_0. Taking into account (4.25), (4.28) and applying Lemma 4.5 we infer that the measure $e(M')$ has a $\varkappa$-component. But this contradicts the equality $M'(\{0\}) = 0$, which completes the proof of (4.29).

From (4.27) and (4.29) we get $\nu_n \to \sigma$, which, by the compactness of $G_\varkappa(\mu)$ and the relations $\nu_n \in G_\varkappa(\mu)$, yields $\sigma \in G_\varkappa(\mu)$. Thus

$$\sigma \sim \sigma_\varkappa \tag{4.30}$$

and $m_\varkappa(\sigma) = 1$. On the other hand, by (4.26) and Theorem 1 in [17],

$$1 = m_\varkappa(\lambda) = m_\varkappa(\sigma) + m_\varkappa(e(M')),$$

which yields $m_\varkappa(e(M')) = 0$. Consequently, $e(M') = \delta_0$ and $\lambda = \sigma$, which together with (4.30) completes the proof.

The following statement is an analogue of Maller's theorem ([11], p. 267) for the ordinary convolution.

THEOREM 4.5. *Suppose that $m_\varkappa(\sigma_\varkappa) < \infty$. Let $\mu \in Q$ and $\{a_n\} \in N(\mu)$. Then the following conditions are equivalent:*

(i) *$G(a_n, \mu)$ contains a measure λ with $m_\varkappa(\lambda) < \infty$.*

(ii) *$G(a_n, \mu)$ contains a measure with a $\varkappa$-component.*

(iii) *$\{\sigma_\varkappa\} \subseteq G(a_n, \mu)$.*

Proof. (i)$\Rightarrow$(iii). From Kłosowska and Bingham's description of the domain of attraction of $\sigma_\varkappa$ in [1] and [8] it follows that each measure λ from $G(a_n, \mu)$ with $m_\varkappa(\lambda) < \infty$ belongs to this domain. In other words, there exists a norming sequence $\{b_n\} \in N(\lambda)$ such that $G(b_n, \lambda) = \{\sigma_\varkappa\}$. Condition (iii) is now an immediate consequence of Propositions 2.1 and 3.1.

(ii)$\Rightarrow$(iii). Suppose that $G(a_n, \mu)$ contains a measure with a $\varkappa$-component. Put

$$a = \sup\{N(\{0\}):\ N \in H(a_n, \mu)\}.$$

Evidently, $a > 0$ and, by the compactness of $H(a_n, \mu)$, $a < \infty$ and there exists a measure $\lambda = e(M) \in G(a_n, \mu)$ with $M(\{0\}) = a$. By Lemma 4 in [18] we can find a sequence $\{c_n\}$ of positive numbers and a sequence $\{M_n\}$ of measures from $H(a_n, \mu)$ such that $\lambda = T_{c_n} e(M_n)^{\circ n}$ $(n = 1, 2, \ldots)$. Of course,

$$M_n(\{0\}) \leqslant M(\{0\}) \quad (n = 1, 2, \ldots) \tag{4.31}$$

and, by (4.3) and (4.4),

$$M(E) = n \int_{c_n^{-1}E} \frac{\omega(c_n x)}{\omega(x)} M_n(dx) \quad (n = 1, 2, \ldots). \tag{4.32}$$

Consequently, $M(\{0\}) = nc_n^\varkappa M_n(\{0\})$, which, by (4.31), yields

$$nc_n^\varkappa \geqslant 1 \quad (n = 1, 2, \ldots). \tag{4.33}$$

Setting $M'(E) = M(E \cap (0, \infty))$, $M'_n(E) = M_n(E \cap (0, \infty))$ and $\nu_n = e(M'_n)$ we have

$$e(M_n) = e(M_n(\{0\})\delta_0) \circ \nu_n \quad (n = 1, 2, \ldots) \tag{4.34}$$

and, by (4.32),

$$e(M') = T_{c_n} \nu_n^{\circ n} \quad (n = 1, 2, \ldots). \tag{4.35}$$

Since $e(M_n) \in G(a_n, \mu)$ we conclude, by the compactness of $G(a_n, \mu)$, that the sequence $\{e(M_n)\}$ is conditionally compact and all its cluster points belong to $G(a_n, \mu)$. Moreover, by (4.34) and Corollary 2.3 in [20], the sequences $\{e(M_n(\{0\})\delta_0)\}$ and $\{\nu_n\}$ are also conditionally compact. Consequently, taking into account inequality (4.31) we can choose a subsequence $n_1 < n_2 < \ldots$ such that $e(M_{n_k}) \to \sigma$, $e(M_{n_k}(\{0\})\delta_0) \to e(b\delta_0)$ and $\nu_{n_k} \to \nu$ where $\nu \in P$, $0 \leqslant b \leqslant M(\{0\})$,

$$\sigma \in G(a_n, \mu), \tag{4.36}$$

$$\sigma = e(b\delta_0) \circ \nu. \tag{4.37}$$

Suppose that $\nu \neq \delta_0$. Then, by (4.33), (4.35) and Lemma 4.5, the measure $e(M')$ has a $\varkappa$-component, which contradicts the equality $M'(\{0\}) = 0$. Thus $\nu = \delta_0$ and, by (4.37), $\sigma = e(b\delta_0)$, which, by (4.36), yields $b > 0$. Consequently, $\sigma \sim \sigma_\varkappa$, which shows that condition (iii) is fulfilled.

The remaining implications (iii)$\Rightarrow$(i) and (iii)$\Rightarrow$(ii) are evident, which completes the proof.

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INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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