The dual of every Asplund space admits
a projectional resolution of the identity

by

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Abstract. We show that the dual of every Asplund space admits a projectional resolution of the identity. An application is that any dual space with the Radon-Nikodym property can be linearly and continuously injected into $c_0(I)$, has a Markushevich basis, and admits an equivalent locally uniformly rotund norm. We also show that a nonreflexive Banach space which is an $M$-ideal in its bidual is weakly compactly generated and contains a complemented copy of $c_0$.

Introduction and statement of results. An important and widely open question in geometry of Banach spaces is that of the existence of nontrivial linear continuous mappings on a given space. Among such mappings norm one projections play an important role. In some large classes of Banach spaces the projections do exist, and moreover, they can be organized into a “long sequence” with nice properties. Such a sequence is then called a projectional resolution of the identity (PRI); for the exact definition see the text following Theorem 1. Thus one obtains a powerful tool for studying the structure and for renorming certain spaces.

The first nontrivial PRI were constructed by Lindenstrauss [24], [25] for reflexive spaces and by Amir and Lindenstrauss [1] for weakly compactly generated spaces. Shortly thereafter Tačon [33] constructed a PRI in the dual of a very smooth Banach space.

Let us recall that a very smooth space is Asplund [3, p. 31], [2], [9]. So the natural question about the existence of a PRI on the dual of a general Asplund space has arisen. Papers [18], [6], and [7] are devoted to the construction of a PRI in the dual of an Asplund space $X$ under various additional smoothness conditions imposed on $X$; [7] has provided so far the most general result in this direction.

The present paper solves this problem affirmatively in full generality.

Theorem 1. Let $X$ be an Asplund space, i.e. $X^*$ has the Radon-Nikodym property, and let $\mu$ denote the first ordinal of cardinality $\text{dens } X$.

Then there exist a nondecreasing "long sequence" $\{M_\alpha; \omega \leq \alpha \leq \mu\}$ of subspaces of $X$ and a "long sequence" $\{P_\alpha; \omega \leq \alpha \leq \mu\}$ of linear projections on
such that $M_\mu = X$, $P_\mu$ = identity, and for all $0 < \alpha \leq \mu$ the following conditions hold:

(i) $\text{dens } M_\alpha \subseteq \mathbb{R}$.
(ii) $\bigcup_{\beta < \alpha} M_{\beta + 1}$ is dense in $M_\alpha$.
(iii) $\|P_\beta\| = 1$.
(iv) $P_\beta P_\beta = P_\beta P_\beta = P_\beta$ if $\beta \leq \alpha$.
(v) $\text{dens } P_\alpha X^* \subseteq \mathbb{R}$.
(vi) $\bigcup_{\beta < \alpha} P_{\beta + 1} X^*$ is dense in $P_\alpha X^*$.
(vii) The mapping $R_\alpha f : P_\alpha X^* \to M_\alpha^*$ defined by $R_\alpha f = f|_{M_\alpha}$, $f \in P_\alpha X^*$, is a surjective isometry and $P_\alpha f = R_\alpha^{-1}(f|_{M_\alpha})$ for all $f \in X^*$.

A "long sequence" $\{P_\alpha : 0 \leq \alpha \leq \mu\}$ of linear projections which shares properties (iii)-(vii) is called a PRI. Sometimes, but not in this paper, (iii) is replaced by $\text{sup } \{\|P_\alpha\|\} < +\infty$.

By combining the above theorem with known results and techniques we obtain

**Theorem 2.** Let $X$ be an Asplund space. Then:

(i) There exists a one-to-one linear continuous mapping from $X^*$ into $c_0(F)$.
(ii) $X^*$ has a Markushhevich basis.
(iii) $X^*$ admits an equivalent locally uniformly round (LUR) norm.
(iv) $X^*$ admits an equivalent norm such that every weakly compact convex set in $X^*$ is an intersection of balls.

Of course, this theorem together with other known results yields further consequences. For instance, combining it with a result of Edgar [5, Theorem 1.1], one sees that in the dual space with the Radon–Nikodym property the norm Borel and weak Borel structures coincide.

For one subclass of the Asplund spaces the PRI from Theorem 1 enjoys nicer properties, so one gets stronger conclusions:

**Theorem 3.** If a nonreflexive Banach space is an $M$-ideal in its bidual, then it is Asplund, weakly compactly generated, and contains a complemented copy of $c_0$.

This theorem can be embellished by the renorming result that a weakly compactly generated Asplund space admits an equivalent LUR norm whose dual norm is also LUR [7], [19], [5, pp. 164, 112, 113].

In what follows we would like to mention the difficulties which we met and had to get around when we tried to prove Theorem 1. First, there is a question of the existence of any nontrivial projection on the dual Banach space. Here a surprising result of Heinrich and Mankiewicz [16, Proposition 3.4, Corollary 3.8] comes. Using model theory, they showed that every sufficiently large dual Banach space admits many norm one projections. An elementary proof of this fact based on ideas of Tacon [33] can be found in Yost and Sims [35]. However, such a general result proves to provide projections too wild for ordering them into a "long sequence" satisfying the conclusion of Theorem 1, or at least properties (iii)-(vi). Mainly, (vi) seems to be violated in general, even in the duals of Asplund spaces. It follows that in order to construct a PRI on the dual of an Asplund space the $P_\alpha$ should be selected more carefully; indeed, they must be controlled somehow. The next proposition, whose origins can be traced back to John an Zizler [18], [19], provides such a construction.

**Proposition ([5, Proposition 1, Remark]).** Let $X$ be a Banach space with a norm-weak lower semicontinuous multivalued mapping $D : X \to 2^X$ such that $Dx$ is a countable set for every $x \in X$ and that

$$(*) \quad \overline{\text{sp } \{x^* \mid x \in Dx, x \in X\}} = V^*$$

for every subspace $V$ of $X$.

Then there exist "long sequences" $\{M_\alpha : 0 \leq \alpha \leq \mu\}$ and $\{P_\alpha : 0 \leq \alpha \leq \mu\}$ as in Theorem 1 such that $P_\alpha X^* = \overline{\text{sp } D(M_\alpha)}$ for all $\alpha$.

So, having a mapping $D$ as in this proposition, we are done. For instance, if $X$ is very smooth, then we can take for $D$ the derivative of $\frac{1}{2}\|\cdot\|^2$; thus we get the result of Tacon [33]. In a general Asplund space the construction of the mapping $D$ proves to be a serious problem. For finding it we use a beautiful theorem of Jayne and Rogers [17, Theorem 8] on the first Baire class selectors for multivalued usco mappings. With the help of this result, it is easy to construct a countable-valued norm-norm lower semicontinuous mapping $D : X \to 2^X$. But we are still far from proving the identity $(*)$ for such a $D$. In order to do it, further powerful levers must be applied. Even for separable $V$ the identity $(*)$ is not obvious. Indeed, its proof needs a result of the second-named author [10, Theorem 1.2] on the so-called boundaries, which in turn is based on a deep lemma of Simons [32, Lemma 2]. Finally, having verified $(*)$ for all separable $V$, we are able to prove it for all $V$ by using a method of separable reduction; to make familiar with this technique, see for instance [9], [31].

The paper is concluded by several remarks, including the proof that the existence of a PRI is neither an isomorphic property nor a space property.

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Notation and recalls. All Banach spaces in this paper are real and infinite-dimensional, mostly nonseparable. If \((X, \| \cdot \|)\) is a Banach space, then \(X^*\) and \(X^{**}\) denote its dual and bidual, respectively. We assume that \(X\) is a subspace of \(X^{**}\). The closed unit ball of \(X\) is denoted by \(B_X\). The symbol \((\langle x^*, x \rangle)\) means the value of \(x^* \in X^*\) at \(x \in X\). If \(V\) is a subspace of \(X\) and \(f \in X^*\), then \(f|_V\) stands for the restriction of \(f\) to \(V\). The concepts such as "closed", "dense", \(\ldots\) are related to the norm topology unless otherwise stated. Given a subset \(S\) of a Banach space, the symbols \(\overline{S}\), \(sp S\), \(co S\), and \(dens S\) are used to denote the closure, closed linear span, convex hull, and density of \(S\), respectively, the latter being defined as the smallest cardinality of a dense subset of \(S\). Given a subspace \(Y\) of \(X\) we put \(Y^* = \{x^* \in X^*: \langle x^*, x \rangle = 0\} \text{ for all } x \in Y\).

The letters \(\omega\) and \(\Omega\) are reserved for denoting the first countable and uncountable ordinals, respectively. Given an ordinal \(\alpha\), \(\overline{\alpha}\) is its cardinality.

Let \((X, \| \cdot \|)\) be a Banach space. We say that a function \(\varphi: X \to (-\infty, +\infty)\) is Fréchet differentiable at \(x \in X\) if there is \(f \in X^*\) such that
\[
\frac{1}{\|h\|} [\varphi(x+h) - \varphi(x) - \langle f, h \rangle] \to 0 \quad \text{as} \quad \|h\| \to 0,
\]
\(X\) is said to be Asplund if any continuous convex real-valued function on \(X\) is Fréchet differentiable in a dense subset of \(X\) [2, 29]. Further information on Asplund spaces can be found in [4] and [9].

A norm \(\| \cdot \|\) on a Banach space is called LUR if \(\|x_\alpha - x\| \to 0\) whenever \(x_\alpha, x \in X\) and \(2\|x\|^2 + 2\|x_\alpha\|^2 - \|x + x_\alpha\|^2 \to 0\). All the norms on a given Banach space are assumed to be equivalent with the original one; so the adjective "equivalent" is often omitted.

A system \(\{x_\gamma, x_\gamma^*\} : \gamma \in G\) is a Markushevich basis in a Banach space \(X\) if \(sp\{x_\gamma : \gamma \in G\} = X\), the set \(\{x_\gamma^* : \gamma \in G\}\) is total on \(X\), i.e., for every \(0 \neq x \in X\) there is \(\gamma \in G\) such that \(\langle x_\gamma^*, x \rangle \neq 0\), and \(\langle x_\beta^*, x_\gamma \rangle = \delta_{\beta \gamma}\) (the Kronecker delta) for \(\beta, \gamma \in G\). If moreover \(sp\{x_\gamma^* : \gamma \in G\} = X^*\), then we speak about a shrinking Markushevich basis.

The formula \(X = U \oplus_1 V\) means that \(U, V\) are subspaces of a Banach space \(X\) such that \(X = U + V\) and \(\|u + v\| = \|u\| + \|v\|\) for all \(u \in U, v \in V\). According to [15, Proposition 3.1] we say that a Banach space \(X\) is an M-ideal in its bidual \(X^{**}\) if \(X^{**} = X \oplus_1 X^*\).

Proofs.

Proof of Theorem 1. The proof is divided into four steps.

First step. We define the mapping \(J: X \to 2X^*\) by
\[
Jx = \{x^* \in X^*: \|x^*\| \leq 1, \langle x^*, x \rangle = \|x\|\}, \quad x \in X.
\]
Then clearly \(Jx\) is weakly* compact for each \(x \in X\). Further, it is well known and easy to check that \(J\) is norm-weak* upper semicontinuous, i.e., the set
\[
\{x \in X: Jx \cap C \neq \emptyset\}
\]
has diameter less than \(\varepsilon\).

Thus we may apply the theorem of Jayne and Rogers [16, Theorem 8] (unlike [7] we now use this result in full glory), which yields a single valued selector \(D_\varepsilon\) for \(J\) that can be written as a pointwise limit of norm-norm continuous mappings \(D_i: X \to X^*, i = 1, 2, \ldots\) Define the multifunction mapping \(D: X \to 2X^*\) by
\[
Dx = \{D_1x, D_2x, \ldots\}, \quad x \in X.
\]
Hence \(Dx\) is countable for each \(x \in X\). Further, we remark that \(D\) is norm-norm lower semicontinuous, i.e., the set \(\{x \in X: Dx \cap G \neq \emptyset\}\) is open whenever \(G\) is an open subset of \(X^*\).

Second step. We claim that
\[
\text{for every separable subspace} \ V \text{ of} \ X,
\]
\[
\text{proof of the claim. Fix such a} \ V. \text{We remark that the set}
\]
\[
S = \{Dx_\alpha : v \in B_V\}
\]
is a boundary of \(B_{V^*}\) [10], i.e., for any \(v \in V\) there is \(f \in S\) such that \(\langle f, v \rangle = \|v\|\). Assume that \((*)\) is false. Then there are \(v^{**} \in B_{V^{**}}\) and \(f_0 \in B_{V^*}\) such that
\[
\langle v^{**}, f_0 \rangle > 0 \quad \text{and} \quad \langle v^{**}, f_0 \rangle > 0
\]
for all \(f \in sp\{D_\alpha x_\alpha : v \in V, \alpha = 1, 2, \ldots\}\) and, a fortiori, for all \(f \in S\). Now, as \(V\) is separable and Asplund [29, Theorem 12], \(V^*\) is also separable [29, Corollary 10] and so \(B_{V^*}\) endowed with the weak* topology is metrizable. So there is a sequence \(v_\alpha\) in \(B_{V}\) such that \(v_\alpha \to v^{**}\) weakly*. Hence we may assume that
\[
\langle f_0, v_\alpha \rangle = \frac{1}{2} \langle v^{**}, f_0 \rangle \quad \text{for all} \ k = 1, 2, \ldots
\]
Thus by the result of Simons [32, Lemma 2] we have
\[
0 = \sup \{\langle v^{**}, f \rangle : f \in S\} = \sup \{\lim \langle f, v_\alpha \rangle : f \in S\}
\]
\[
\geq \inf \{\|v\| : v \in \text{co} [v_\alpha : \alpha \in \{k, 1, 2, \ldots\}]\}
\]
\[
\geq \inf \{\langle f_0, v \rangle : v \in \text{co} [v_\alpha : \alpha = 1, 2, \ldots]\} \geq \frac{1}{2} \langle v^{**}, f_0 \rangle > 0,
\]
a contradiction. This proves \((*)\) for our separable \(V\).
Third step. We claim that the identity (∗) holds also for nonseparable subspaces \( V \) of \( X \).

Proof of this claim. Let \( V \) be a fixed nonseparable subspace of \( X \). We shall use the method of separable reduction. Let \( \mathcal{U} \) denote the set of all infinite matrices \( a = \{a_{ij}\} \) with rational entries such that \( a_{ij} = 0 \) for all but finitely many \( i, j = 1, 2, \ldots \). Observe that \( \mathcal{U} \) is a countable set.

Let \( f \in V^* \) be arbitrary fixed. Starting from a separable subspace \( Y_1 \neq \{0\} \) of \( V \) we can construct, by induction, separable subspaces \( Y_1, Y_2, \ldots \) of \( V \) sequences \( \{y_j\}, \{y_{j}^*\}, \ldots \), where \( \{y_{j}^*\}, j = 1, 2, \ldots \), \( y_{j} \), \( j = 1, 2, \ldots \), are dense in \( Y_j \), \( n = 1, 2, \ldots \), \( n = 1, 2, \ldots \), and elements \( v(n, a) \in B_{V^*} \) for all \( n = 1, 2, \ldots \) and all \( a \in \mathcal{U} \) such that, for all \( n = 1, 2, \ldots \),

\[
\langle f - \sum_{i,j = 1}^{n} a_{ij} D_i y_j^* v(n, a) \rangle \geq \|f - \sum_{i,j = 1}^{n} a_{ij} D_i y_j^* v(n, a)\| - 1/n
\]

and

\[
Y_{n+1} = \overline{sp}\{Y_n \cup \{v(n, a); a \in \mathcal{U}\}\}.
\]

Then we put \( Y = \bigcup_{n=1}^{\infty} Y_n \). Of course, \( Y \) is a separable subspace of \( V \).

Now, let \( \varepsilon > 0 \) be arbitrary. Since \( f_{y_j} \) is an element of \( Y^* \), according to the second step there are real numbers \( b_1, \ldots, b_m \), elements \( y_1, \ldots, y_m \) in \( Y \) and positive integers \( i_1, \ldots, i_m \) such that

\[
\|f_{y_j} - \sum_{k=1}^{m} b_k D_{i_k} y_k^* v(n, a)\| < \varepsilon/2.
\]

Clearly, we may assume the \( b_k \) to be rational. Further, the continuity of \( D_{i_k} \) and the fact that \( Y_1 \subset Y_2 \subset \ldots \subset \bigcup Y_n = Y \) ensure that there are \( n > 2/e \) and positive integers \( j_1, \ldots, j_m \) such that

\[
\|f_{y_j} - \sum_{k=1}^{m} b_k D_{j_k} y_k^* v(n, a)\| < \varepsilon/2.
\]

Hence, putting \( a = \{a_{ij}\} \) where \( a_{i,j_1} = b_1, \ldots, a_{i,j_m} = b_m \), and \( a_{ij} = 0 \) otherwise, we get \( a \in \mathcal{U} \) and

\[
\|f_{y_j} - \sum_{i,j = 1}^{n} a_{ij} D_i y_j^* v(n, a)\| < \varepsilon/2.
\]

Thus by (**) we obtain

\[
\|f - \sum_{i,j = 1}^{n} a_{ij} D_i y_j^* v(n, a)\| < \frac{1}{n}\left(\|f - \sum_{i,j = 1}^{n} a_{ij} D_i y_j^* v(n, a)\| - 1/n\right)
\]

\[
= \frac{1}{n}\left(\|f_{y_j} - \sum_{k=1}^{m} b_k D_{j_k} y_k^* v(n, a)\| - \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{2}.
\]

This means that the distance of \( f \) from

\[
\overline{sp}\{D_i x_i; x_i \in V, i = 1, 2, \ldots\} = \overline{sp}\{x^* y_i; x^* \in D_x, x \in V\}
\]

is less than \( \varepsilon \). And since \( \varepsilon > 0 \) could be taken arbitrarily small, we have verified (∗) for our nonseparable \( V \).

Fourth step. By applying the Proposition the conclusion of Theorem 1 follows.

Proof of Theorem 2. The basic tool we use here is transfinit induction on dense \( X \). Let \( \{M_n\} \) and \( \{P_n\} \) be as in Theorem 1. We note that both \( M_n \) and \( M_{n+1}/M_n \) are Asplund spaces [29, Theorem 12], [2, Proposition 4]. Moreover, (vii) in Theorem 1 easily implies that \( (P_{n+1} - P_n) X^* \) is isometric to \( (M_{n+1}/M_n) X^* \) for all \( \omega \leq \alpha < \mu \).

Now (i) follows by repeating the proof from Tacone [33, pp. 423, 424] word for word. (ii) can be proved as John and Zizler do in [20], (iii) is obtained with the help of Zizler’s extension [36] of the renorming theorem of Troyanski (see the proof of [36, Corollary 1]). Finally, as in [20], we can construct from \( \{P_n\} \) a new “long sequence” \( \{Q_n\}; 0 < \alpha < \nu \) of bounded linear projections on \( X^* \) such that \( Q_0 = 0, Q_1 \neq 0, Q_\alpha = \text{identity}, Q_\beta Q_\alpha = Q_\beta Q_\alpha \) if \( \beta < \alpha \) if \( 0 < \alpha < \nu \) and \( X^* \), and \( (Q_{\nu+1} - Q_\nu) X^* \) is separable if \( \alpha < \nu \). Now (iv) follows from a result of Zizler [37, Theorem 1].

Proof of Theorem 3. Let \( X \) be a nonreflexive Banach space which is an \( M \)-ideal in \( X^{**} \). According to [23, Theorem 2.6], \( X \) is Asplund. So Theorem 1 applies; let \( \{P_n\}; \omega \leq \alpha < \mu \) be the PRI constructed there. Fix any \( \omega \leq \alpha < \mu \). We shall verify that \( P_\alpha \) is weak*-weak* continuous.

In order to do this we shall first show that \( Y \equiv P_\omega X \) is weakly* closed. Since \( X^{***} = X^* \otimes_1 X \) and \( Y \) is norm one complemented in \( X^* \), according to [14] we may write \( Y^{**} = Y \otimes_1 Y \). And this further implies that \( Y_\alpha = Y_{\omega+1} \cap X \). Thus we get \( Y_{\omega+1} = Y \otimes_1 (X_{\omega+1} \cap X) \) and this means that \( Y \) is weakly* closed in \( X^* \).

Consequently, putting

\[
Z = \{x \in X; \langle f, x \rangle = 0 \text{ for all } f \in Y\}
\]

and defining

\[
\langle \eta, x \rangle = \langle g, [x] \rangle, \quad g \in (X/Z)^*, \quad x \in X, \quad [x] = x + Z,
\]

we infer that \( T \) maps \( (X/Z)^* \) onto \( Y = P_\omega X \) isometrically. Then, by (vii) in Theorem 1, \( T^{-1} \circ R_{\omega+1} \) maps \( M_\omega \) onto \( (X/Z)^* \) isometrically. But \( M_\omega \) as well as \( X/Z \) are \( M \)-ideals in their biduals [15, Theorem 3.4]. Thus, putting together the last two facts we conclude by e.g. [15, Proposition 4.2] that \( T^{-1} \circ R_{\omega+1} \) is weak*-weak* continuous. But \( T \) is weak*-weak* continuous.
So, together with the formula $P_\alpha f = R^{-1}(f|_{M_\alpha}) f \in X^*$, we infer that $P_\alpha$ is weak*-weak* continuous as well. This means that $P_\alpha^* X \subset X$; hence by defining $Q_\alpha x = P_\alpha^* x$, $x \in X$, we have $Q_\alpha^* = P_\alpha$.

Moreover, from (vii) in Theorem 1 it easily follows that $Q_\alpha X = M_\alpha$.

Thus, by (i) and (ii) in Theorem 1, $\{Q_\alpha; \omega \leq \alpha \leq \mu\}$ is a PRI on $X$. Now a shrinking Markushevich basis on $X$ can be constructed by transfinite induction on dens $X$ exactly as in the proof of [19, Lemma 4]. Hence $X$ is weakly compactly generated [19, Proposition 2].

Finally, since $X$ is nonreflexive, we can and do take the $M_\alpha = Q_\alpha X$ to be nonreflexive as well. In fact, this can be seen at once by inspecting the proof of [6, Proposition 1]. And since $Q_\alpha X$ is an $M$-ideal in its bidual, it contains an isomorphic copy of $c_0$ [15, Theorem 3.5]. Hence, by Sobczyk's theorem [26, Theorem 2.1.5] there is a projection $P$: $Q_\alpha X \to c_0$. Then $P \circ Q_\alpha$ is a projection showing that $c_0$ is complemented in $X$.

**Remarks, examples, problems.**

1) Let $X$ be an $M$-ideal in $X^{**}$. Then a PRI in $X^*$ can be constructed with the help of a reduced number of tools. In fact, the proof of (5) can be simplified considerably: Fix one subspace $V$ of $X$ and denote by $E$ the set of those points of $B_V$ which are strongly exposed by elements from $V$. Then as Asplund [29, p. 735], $B_V$ is equal to the weak* closure of $c_0 E$. But $V$ is an $M$-ideal in $V^{**}$, which easily implies that the weak* and weak convergences on the unit sphere of $V^*$ coincide. Hence $B_V$ is equal to the norm closure of $c_0 E$, and so $V^* = sp E$. Finally, remarking that $E$ is contained in $\{D_{\alpha\beta}^\alpha; v \in B_V\}$, we get (5).

2) We recall a result of Stegall [4, p. 195] that a dual Banach space has the Radon–Nikodym property if and only if it admits no bounded infinite e-tree for any $\varepsilon > 0$. Thus from Theorems 1 and 2 the following properties arise naturally: If a Banach space admits no bounded infinite e-tree for any $\varepsilon > 0$, does it admit a PRI, a linear continuous injection into $c_0(\ell_1)$, etc.?

3) In [7] it is shown that a weakly countably determined Asplund space admits a Fréchet differentiable norm. By using the mapping $D$ constructed in the proof of Theorem 1, the proof of this result can be made more self-contained. In fact, a renorming theorem of Mercourakis [28, Theorem 4.6] that the dual of a weakly countably determined space has a dual strictly convex norm is then no longer needed.

4) Not every dual has a PRI: see the duals $JT_\mu$ and $JH_\mu$ in Remark 2. On the other hand, if $X^*$ has a PRI, $X$ need not be Asplund. Indeed, if $\ell_1$ is not Asplund, yet its dual, as any abstract $\ell_1$-space, has a PRI.

Let us also notice that, if $X$ is a separable space not containing $\ell_1$ such that $X^*$ is not separable, then there exists [8, Proposition VII.4] a separable subspace $Z$ in $X^*$ such that $Z$ is not contained in a separable complemented subspace of $X^*$; thus, assuming the continuum hypothesis, we can easily show that $X^*$ has no PRI. Fortunately, in some concrete duals the continuum hypothesis is needless for disproving the existence of PRI; see Remark 7.

5) The projections $P_\alpha$ constructed in Theorem 1 will not be weak*-weak* continuous in some concrete cases. Namely, having a PRI $\{P_\alpha\}$ with the existence of a proper property, a dual version [7] of Zierler’s extension of Troyanski’s theorem would yield a dual LUR norm on $X^*$. But, according to Talagrand [34], we know that the dual of the space $C([0, \Omega])$ of continuous functions on $[0, \Omega]$ admits no dual strictly convex, a fortiori no dual LUR norm; see also the spaces $JL$, $JT^*$ in Remark 7.

6) The existence of a PRI is not an isomorphic property, i.e. it depends on a concrete norm on the space. In order to show this consider once more the space $X = C([0, \Omega])$. Then, putting $P_\alpha x = x \chi_{[0, \alpha]} + x \chi_{(\alpha, \Omega]}$, $x \in X$, $\alpha \leq \Omega$, we can easily verify that $P_\alpha$ is a PRI on $X$ with respect to the supremum norm. On the other hand, if $X$ is equipped with a Fréchet differentiable norm $||\cdot||$ constructed by Talagrand [34], then it does not admit a PRI. Indeed, if a PRI $\{Q_\alpha\}$ existed on $(X, ||\cdot||)$, then by [19, Lemma 3], $Q_\alpha^*$ would be a PRI on $X^*$, each $Q_\alpha^*$ being, of course, weak*-weak* continuous. And this is impossible: see Remark 5.

7) Let $JL$ denote the Johnson–Lindenstrauss space [21]. We shall show, following an idea of Plichko [30], that this space admits no PRI with respect to any equivalent norm. By contradiction, assume that $\{P_\alpha; \omega \leq \alpha \leq \mu\}$ is a PRI on $JL$. Then there exists a subspace $M$ of $JL$ isometric to $c_0$ such that the quotient $M/\ell_1$ is isometric to a Hilbert space $[21]$. Hence $JL$ is Asplund [29, Theorem 14] and it is easy to find $\omega \leq \alpha < \mu$ such that $M \subset P_\alpha(JL) \equiv X$; thus $\text{dens} X \leq \alpha < \mu$. Also, $JL/X$ is a quotient of $JL/\ell_1$ and so $JL$ is isomorphic to the direct sum of $X$ and a Hilbert space $H$. Now recall that $JL^*$ is weakly* separable [21]. Thus, since $JL^*$ is isomorphic to the direct sum of $X^*$ and $H$, it follows that $H$ must be separable. Hence, taking into account the Asplundness of $X$ we conclude that

$$\text{dens} X = \text{dens} X^* = \text{dens} JL^* = \beta > \alpha,$$

a contradiction.

Thus $JL$ does not admit a PRI for any equivalent norm. But both $M$ and Hilbert space have a PRI. So the existence of a PRI is not a three space property. On the other hand, $JL$ is LUR renormable, since, for instance, the existence of a LUR norm is a three space property [12]. Also, the weak* separability of $JL^*$ [21] implies that $JL^*$ endowed with any norm contains no PRI $\{P_\alpha\}$ with all the $P_\alpha$ weak*-weak* continuous. None the less, $JL^*$ admits a dual LUR norm [11] since it is weakly compactly generated. By the way, this implies that the $P_\alpha$ from Theorem 1 are not weak*-weak* continuous in general; see also Remark 5.

Let $JT$ be the James tree space [27] and let $M$ denote its predual $JT_*$. The same method then shows that the dual $JT^*$ enjoys all the properties of
which are quoted in the preceding paragraph. It should be noted that
Plichko found a subspace of $JT^*$ with the same properties [30]. It can also
be shown by a somewhat different method that the dual $JH^*$ of the James-
Hugler space [13] admits no PRI yet is LUR renormable.

References