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The dual of every Asplund space admits a projectional resolution of the identity

by

MARIÁN FABIAN (Prague) and GILLES GODEFROY (Paris)

Abstract. We show that the dual of every Asplund space admits a projectional resolution of the identity. An application is that any dual space with the Radon–Nikodym property can be linearly and continuously injected into $c_0(\Gamma)$, has a Markushevich basis, and admits an equivalent locally uniformly rotund norm. We also show that a nonreflexive Banach space which is an M-ideal in its bidual is weakly compactly generated and contains a complemented copy of c_0.

Introduction and statement of results. An important and widely open question in geometry of Banach spaces is that of the existence of nontrivial linear continuous mappings on a given space. Among such mappings norm one projections play an important role. In some large classes of Banach spaces the projections do exist, and moreover, they can be organized into a "long sequence" with nice properties. Such a sequence is then called a *projectional resolution of the identity* (*PRI*); for the exact definition see the text following Theorem 1. Thus one obtains a powerful tool for studying the structure and for renorming certain spaces.

The first nontrivial PRI were constructed by Lindenstrauss [24], [25] for reflexive spaces and by Amir and Lindenstrauss [1] for weakly compactly generated spaces. Shortly thereafter Tacon [33] constructed a PRI in the dual of a very smooth Banach space.

Let us recall that a very smooth space is Asplund [3, p. 31], [2], [9]. So the natural question about the existence of a PRI on the dual of a general Asplund space has arisen. Papers [18], [6], and [7] are devoted to the construction of a PRI in the dual of an Asplund space X under various additional smoothness conditions imposed on X; [7] has provided so far the most general result in this direction.

The present paper solves this problem affirmatively in full generality.

THEOREM 1. *Let X be an Asplund space, i.e. X^* has the Radon–Nikodym property, and let μ denote the first ordinal of cardinality* dens X.

Then there exist a nondecreasing "long sequence" $\{M_\alpha\colon \omega \leqslant \alpha \leqslant \mu\}$ of subspaces of X and a "long sequence" $\{P_\alpha\colon \omega \leqslant \alpha \leqslant \mu\}$ of linear projections on

X^* such that $M_\mu = X$, P_μ = identity, and for all $\omega < \alpha \leqslant \mu$ the following conditions hold:

(i) dens $M_\alpha \leqslant \bar{\alpha}$.

(ii) $\bigcup_{\beta<\alpha} M_{\beta+1}$ *is dense in* M_α.

(iii) $\|P_\alpha\| = 1$.

(iv) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ *if* $\beta \leqslant \alpha$.

(v) dens $P_\alpha X^* \leqslant \bar{\alpha}$.

(vi) $\bigcup_{\beta<\alpha} P_{\beta+1} X^*$ *is dense in* $P_\alpha X^*$.

(vii) *The mapping* R_α: $P_\alpha X^* \to M_\alpha^*$ *defined by* $R_\alpha f = f|_{M_\alpha}$, $f \in P_\alpha X^*$, *is a surjective isometry and* $P_\alpha f = R_\alpha^{-1}(f|_{M_\alpha})$ *for all* $f \in X^*$.

A "long sequence" $\{P_\alpha\colon \omega \leqslant \alpha \leqslant \mu\}$ of linear projections which shares properties (iii)–(vi) is called a PRI. Sometimes, but not in this paper, (iii) is replaced by $\sup_\alpha \|P_\alpha\| < +\infty$.

By combining the above theorem with known results and techniques we obtain

THEOREM 2. *Let X be an Asplund space. Then:*

(i) *There exists a one-to-one linear continuous mapping from* X^* *into* $c_0(\Gamma)$.

(ii) X^* *has a Markushevich basis.*

(iii) X^* *admits an equivalent locally uniformly rotund (LUR) norm.*

(iv) X^* *admits an equivalent norm such that every weakly compact convex set in* X^* *is an intersection of balls.*

Of course, this theorem together with other known results yields further consequences. For instance, combining it with a result of Edgar [5, Theorem 1.1], one sees that in the dual space with the Radon–Nikodym property the norm Borel and weak Borel structures coincide.

For one subclass of the Asplund spaces the PRI from Theorem 1 enjoys nicer properties, so one gets stronger conclusions:

THEOREM 3. *If a nonreflexive Banach space is an M-ideal in its bidual, then it is Asplund, weakly compactly generated, and contains a complemented copy of* c_0.

This theorem can be embellished by the renorming result that a weakly compactly generated Asplund space admits an equivalent LUR norm whose dual norm is also LUR [7], [19], [3, pp. 164, 112, 113].

In what follows we would like to mention the difficulties which we met and had to get round when we tried to prove Theorem 1. First, there is a question of the existence of any nontrivial projection on the dual Banach space. Here a surprising result of Heinrich and Mankiewicz [16, Proposition 3.4, Corollary 3.8] comes. Using model theory, they showed that every sufficiently large dual Banach space admits many norm one projections. An elementary proof of this fact based on ideas of Tacon [33] can be found in Yost and Sims [35]. However, such a general result proves to provide projections too wild for ordering them into a "long sequence" satisfying the conclusion of Theorem 1, or at least properties (iii)–(vi). Mainly, (vi) seems to be violated in general, even in the duals of Asplund spaces. It follows that in order to construct a PRI on the dual of an Asplund space the P_α should be selected more carefully; indeed, they must be controlled somehow. The next proposition, whose origins can be traced back to John an Zizler [18], [19], provides such a construction.

PROPOSITION ([6, Proposition 1, Remark]). *Let X be a Banach space with a norm-weak lower semicontinuous multivalued mapping* $D\colon X \to 2^{X^*}$ *such that* Dx *is a countable set for every* $x \in X$ *and that*

$$(*) \qquad \overline{\operatorname{sp}}\,\{x^*|_V\colon\ x^* \in Dx,\ x \in V\} = V^*$$

for every subspace V of X.

Then there exist "long sequences" $\{M_\alpha\colon \omega \leqslant \alpha \leqslant \mu\}$ *and* $\{P_\alpha\colon \omega \leqslant \alpha \leqslant \mu\}$ *as in Theorem 1 such that* $P_\alpha X^* = \overline{\operatorname{sp}}\, D(M_\alpha)$ *for all* α.

So, having a mapping D as in this proposition, we are done. For instance, if X is very smooth, then we can take for D the derivative of $\frac{1}{2}\|\cdot\|^2$; thus we get the result of Tacon [33]. In a general Asplund space the construction of the mapping D proves to be a serious problem. For finding it we use a beautiful theorem of Jayne and Rogers [17, Theorem 8] on the first Baire class selectors for multivalued usco mappings. With the help of this result, it is easy to construct a countable-valued norm-norm lower semicontinuous mapping $D\colon X \to 2^{X^*}$. But we are still far from proving the identity (*) for such a D. In order to do it, further powerful levers must be applied. Even for separable V the identity (*) is not obvious. Indeed, its proof needs a result of the second-named author [10, Theorem I.2] on the so-called boundaries, which in turn is based on a deep lemma of Simons [32, Lemma 2]. Finally, having verified (*) for all separable V, we are able to prove it for all V by using a method of separable reduction; to make familiar with this technique, see for instance [9], [31].

The paper is concluded by several remarks, including the proof that the existence of a PRI is neither an isomorphic property nor a three space property.

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Notation and recalls. All Banach spaces in this paper are real and infinite-dimensional, mostly nonseparable. If $(X, \|\cdot\|)$ is a Banach space, then X^* and X^{**} denote its dual and bidual, respectively. We assume that X is a subspace of X^{**}. The closed unit ball of X is denoted by B_X. The symbol $\langle x^*, x\rangle$ means the value of $x^* \in X^*$ at $x \in X$. If V is a subspace of X and $f \in X^*$, then $f|_V$ stands for the restriction of f to V. The concepts such as "closed", "dense", ... are related to the norm topology unless otherwise stated. Given a subset S of a Banach space, the symbols $\bar{S}$, $\overline{\mathrm{sp}}\, S$, $\mathrm{co}\, S$, and $\mathrm{dens}\, S$ are used to denote the closure, closed linear span, convex hull, and density of S, respectively, the latter being defined as the smallest cardinality of a dense subset of S. Given a subspace Y of X we put $Y^\perp = \{x^* \in X^*\colon \langle x^*, x\rangle = 0 \text{ for all } x \in Y\}$.

The letters ω and Ω are reserved for denoting the first countable and uncountable ordinals, respectively. Given an ordinal α, $\bar{\bar{\alpha}}$ is its cardinality.

Let $(X, \|\cdot\|)$ be a Banach space. We say that a function $\varphi\colon X \to (-\infty, +\infty)$ is *Fréchet differentiable* at $x \in X$ if there is $f \in X^*$ such that

$$\frac{1}{\|h\|}[\varphi(x+h) - \varphi(x) - \langle f, h\rangle] \to 0 \quad \text{as } \|h\| \to 0.$$

X is said to be *Asplund* if any continuous convex real-valued function on X is Fréchet differentiable in a dense subset of X [2], [29]. Further information on Asplund spaces can be found in [4] and [9].

A norm $\|\cdot\|$ on a Banach space is called *LUR* if $\|x_n - x\| \to 0$ whenever $x, x_n \in X$ and $2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \to 0$. All the norms on a given Banach space are assumed to be equivalent with the original one; so the adjective "equivalent" is often omitted.

A system $\{(x_\gamma, x_\gamma^*)\colon \gamma \in \Gamma\} \subset X \times X^*$ is called a *Markushevich basis* in a Banach space X if $\overline{\mathrm{sp}}\{x_\gamma\colon \gamma \in \Gamma\} = X$, the set $\{x_\gamma^*\colon \gamma \in \Gamma\}$ is total on X, i.e., for every $0 \neq x \in X$ there is $\gamma \in \Gamma$ such that $\langle x_\gamma^*, x\rangle \neq 0$, and $\langle x_\beta^*, x_\gamma\rangle = \delta_{\beta\gamma}$ (the Kronecker delta) for $\beta, \gamma \in \Gamma$. If moreover $\overline{\mathrm{sp}}\{x_\gamma^*\colon \gamma \in \Gamma\} = X^*$, then we speak about a *shrinking Markushevich basis*.

The formula $X = U \oplus_1 V$ means that U, V are subspaces of a Banach space X such that $X = U + V$ and $\|u + v\| = \|u\| + \|v\|$ for all $u \in U$, $v \in V$. According to [15, Proposition 3.1] we say that a Banach space X is an *M-ideal* in its bidual X^{**} if $X^{***} = X^* \oplus_1 X^\perp$.

Proofs.

Proof of Theorem 1. The proof is divided into four steps.

First step. We define the mapping $J\colon X \to 2^{X^*}$ by

$$Jx = \{x^* \in X^*\colon \|x^*\| \leqslant 1,\ \langle x^*, x\rangle = \|x\|\}, \quad x \in X.$$

Then clearly Jx is weakly* compact for each $x \in X$. Further, it is well known and easy to check that J is norm-weak* upper semicontinuous, i.e., the set

$\{x \in X\colon Jx \cap C \neq \emptyset\}$ is closed whenever C is a weakly* closed subset of X^*.

Now, as X is Asplund, it follows from [2, Proposition 5] (see also [29, p. 735]) that X^* is weakly* dentable. This means that for every bounded subset S of X^* and every $\varepsilon > 0$ there are $a > 0$ and $x \in X$ such that the set

$$\{x^* \in S\colon \langle x^*, x\rangle > \sup\langle S, x\rangle - a\}$$

has diameter less than ε.

Thus we may apply the theorem of Jayne and Rogers [17, Theorem 8] (unlike [7] we now use this result in full glory), which yields a singlevalued selector D_0 for J that can be written as a pointwise limit of norm-norm continuous mappings $D_i\colon X \to X^*$, $i = 1, 2, \ldots$ Define the multivalued mapping $D\colon X \to 2^{X^*}$ by

$$Dx = \{D_1 x, D_2 x, \ldots\}, \quad x \in X.$$

Hence Dx is countable for each $x \in X$. Further, we remark that D is norm-norm lower semicontinuous, i.e., the set $\{x \in X\colon Dx \cap G \neq \emptyset\}$ is open whenever G is an open subset of X^*.

Second step. We claim that

$$(*) \qquad \overline{\mathrm{sp}}\{x^*|_V\colon x^* \in Dx,\ x \in V\} = V^*$$

for every separable subspace V of X.

Proof of the claim. Fix such a V. We remark that the set

$$S = \{D_0 v|_V\colon v \in B_V\}$$

is a boundary of B_{V^*} [10], i.e., for any $v \in V$ there is $f \in S$ such that $\langle f, v\rangle = \|v\|$. Assume that $(*)$ is false. Then there are $v^{**} \in B_{V^{**}}$ and $f_0 \in B_{V^*}$ such that

$$\langle v^{**}, f_0\rangle > 0 = \langle v^{**}, f\rangle$$

for all $f \in \overline{\mathrm{sp}}\{D_i v|_V\colon v \in V,\ i = 1, 2, \ldots\}$ and, *a fortiori*, for all $f \in S$. Now, as V is separable and Asplund [29, Theorem 12], V^* is also separable [29, Corollary 10] and so $B_{V^{**}}$ endowed with the weak* topology is metrizable. So there is a sequence $\{v_k\}$ in B_V such that $v_k \to v^{**}$ weakly*. Hence we may assume that

$$\langle f_0, v_k\rangle > \tfrac{1}{2}\langle v^{**}, f_0\rangle \quad \text{for all } k = 1, 2, \ldots$$

Thus by the result of Simons [32, Lemma 2] we have

$$\begin{aligned} 0 &= \sup\{\langle v^{**}, f\rangle\colon f \in S\} = \sup\{\lim_k \langle f, v_k\rangle\colon f \in S\} \\ &\geqslant \inf\big\{\|v\|\colon v \in \mathrm{co}\{v_k\colon k = 1, 2, \ldots\}\big\} \\ &\geqslant \inf\big\{\langle f_0, v\rangle\colon v \in \mathrm{co}\{v_k\colon k = 1, 2, \ldots\}\big\} \geqslant \tfrac{1}{2}\langle v^{**}, f_0\rangle > 0, \end{aligned}$$

a contradiction. This proves $(*)$ for our separable V.

Third step. We claim that the identity (*) holds also for nonseparable subspaces V of X.

Proof of this claim. Let V be a fixed nonseparable subspace of X. We shall use the method of separable reduction. Let $\mathfrak{A}$ denote the set of all infinite matrices $a = \{a_{ij}\}$ with rational entries such that $a_{ij} = 0$ for all but finitely many $i, j = 1, 2, \ldots$ Observe that $\mathfrak{A}$ is a countable set.

Let $f \in V^*$ be arbitrary fixed. Starting from a separable subspace $Y_1 \neq \{0\}$ of V we can construct, by induction, separable subspaces $Y_2, Y_3, \ldots$ of V, sequences $\{y_j^1\}, \{y_j^2\}, \ldots$, where $\{y_j^n\}$, $j = 1, 2, \ldots$, are dense in Y_n, $n = 1, 2, \ldots$, and elements $v(n, a) \in B_V$ for all $n = 1, 2, \ldots$ and all $a \in \mathfrak{A}$ such that, for all $n = 1, 2, \ldots$,

$$(**)\qquad \langle f - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_V, v(n, a)\rangle \geqslant \|f - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_V\| - 1/n$$

and

$$Y_{n+1} = \overline{\operatorname{sp}}[Y_n \cup \{v(n, a): a \in \mathfrak{A}\}].$$

Then we put $Y = \overline{\bigcup_{n=1}^{\infty} Y_n}$. Of course, Y is a separable subspace of V.

Now, let $\varepsilon > 0$ be arbitrary. Since $f|_Y$ is an element of Y^*, according to the second step there are real numbers $b_1, \ldots, b_m$, elements $y_1, \ldots, y_m$ in Y and positive integers $i_1, \ldots, i_m$ such that

$$\|f|_Y - \sum_{k=1}^{m} b_k D_{i_k} y_k|_Y\| < \varepsilon/2.$$

Clearly, we may assume the b_k to be rational. Further, the continuity of D_i and the fact that $Y_1 \subset Y_2 \subset \ldots \subset \overline{\bigcup Y_n} = Y$ ensure that there are $n > 2/\varepsilon$ and positive integers $j_1, \ldots, j_m$ such that

$$\|f|_Y - \sum_{k=1}^{m} b_k D_{i_k} y_{j_k}^n|_Y\| < \varepsilon/2.$$

Hence, putting $a = \{a_{ij}\}$ where $a_{i_1 j_1} = b_1, \ldots, a_{i_m j_m} = b_m$, and $a_{ij} = 0$ otherwise, we get $a \in \mathfrak{A}$ and

$$\|f|_Y - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_Y\| < \varepsilon/2.$$

Thus by (**) we obtain

$$\begin{aligned}\|f - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_V\| &< \frac{1}{n} + \langle f - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_V, v(n, a)\rangle \\ &= \frac{1}{n} + \langle f|_Y - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_Y, v(n, a)\rangle \\ &\leqslant \frac{1}{n} + \|f|_Y - \sum_{i,j=1}^{\infty} a_{ij} D_i y_j^n|_Y\| \leqslant \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

This means that the distance of f from

$$\overline{\operatorname{sp}}\{D_i x|_V: x \in V, i = 1, 2, \ldots\} = \overline{\operatorname{sp}}\{x^*|_V: x^* \in Dx, x \in V\}$$

is less than ε. And since $\varepsilon > 0$ could be taken arbitrarily small, we have verified (*) for our nonseparable V.

Fourth step. By applying the Proposition the conclusion of Theorem 1 follows.

Proof of Theorem 2. The basic tool we use here is transfinite induction on dens X. Let $\{M_\alpha\}$ and $\{P_\alpha\}$ be as in Theorem 1. We note that both M_α and $M_{\alpha+1}/M_\alpha$ are Asplund spaces [29, Theorem 12], [2, Proposition 4]. Moreover, (vii) in Theorem 1 easily implies that $(P_{\alpha+1} - P_\alpha)X^*$ is isometric to $(M_{\alpha+1}/M_\alpha)^*$ for all $\omega \leqslant \alpha < \mu$.

Now (i) follows by repeating the proof from Tacon [33, pp. 423, 424] word for word. (ii) can be proved as John and Zizler do in [20]. (iii) is obtained with the help of Zizler's extension [36] of the renorming theorem of Troyanski (see the proof of [36, Corollary 1]). Finally, as in [20], we can construct from $\{P_\alpha\}$ a new "long sequence" $\{Q_\alpha: 0 \leqslant \alpha \leqslant \nu\}$ of bounded linear projections on X^* such that $Q_0 = 0$, $Q_1 \neq 0$, $Q_\nu = \text{identity}$, $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_\beta$ if $\beta \leqslant \alpha$, $Q_\alpha x^* \in \overline{\operatorname{sp}}\{Q_{\beta+1} x^*: \beta < \alpha\}$ if $0 < \alpha$ and $x^* \in X^*$, and $(Q_{\alpha+1} - Q_\alpha)X^*$ is separable if $\alpha < \nu$. Now (iv) follows from a result of Zizler [37, Theorem 1].

Proof of Theorem 3. Let X be a nonreflexive Banach space which is an M-ideal in X^{**}. According to [23, Theorem 2.6], X is Asplund. So Theorem 1 applies; let $\{P_\alpha: \omega \leqslant \alpha \leqslant \mu\}$ be the PRI constructed there. Fix any $\omega \leqslant \alpha \leqslant \mu$. We shall verify that P_α is weak*-weak* continuous.

In order to do this we shall first show that $Y \equiv P_\alpha X^*$ is weakly* closed. Since $X^{***} = X^* \oplus_1 X^\perp$ and Y is norm one complemented in X^*, according to [14] we may write $Y^{**} = Y \oplus_1 Y_s$. And this further implies that $Y_s = Y^{\perp\perp} \cap X^\perp$ [22]. Thus we get $Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap X^\perp)$ and this means that Y is weakly* closed in X^*.

Consequently, putting

$$Z = \{x \in X: \langle f, x\rangle = 0 \text{ for all } f \in Y\}$$

and defining

$$\langle Tg, x\rangle = \langle g, [x]\rangle, \quad g \in (X/Z)^*, \ x \in X, \ [x] = x + Z,$$

we infer that T maps $(X/Z)^*$ onto $Y = P_\alpha X^*$ isometrically. Then, by (vii) in Theorem 1, $T^{-1} \circ R_\alpha^{-1}$ maps M_α^* onto $(X/Z)^*$ isometrically. But M_α as well as X/Z are M-ideals in their biduals [15, Theorem 3.4]. Thus, putting together the last two facts we conclude by e.g. [15, Proposition 4.2] that $T^{-1} \circ R_\alpha^{-1}$ is weak*-weak* continuous. But T is weak*-weak* continuous.

So, together with the formula $P_\alpha f = R_\alpha^{-1}(f|_{M_\alpha})$, $f \in X^*$, we infer that P_α is weak*-weak* continuous as well. This means that $P_\alpha^* X \subset X$; hence by defining $Q_\alpha x = P_\alpha^* x$, $x \in X$, we have $Q_\alpha^* = P_\alpha$.

Moreover, from (vii) in Theorem 1 it easily follows that $Q_\alpha X = M_\alpha$. Thus, by (i) and (ii) in Theorem 1, $\{Q_\alpha: \omega \leqslant \alpha \leqslant \mu\}$ is a PRI on X. Now a shrinking Markushevich basis on X can be constructed by transfinite induction on dens X exactly as in the proof of [19, Lemma 4]. Hence X is weakly compactly generated [19, Proposition 2].

Finally, since X is nonreflexive, we can and do take the $M_\omega = Q_\omega X$ to be nonreflexive as well. In fact, this can be seen at once by inspecting the proof of [6, Proposition 1]. And since $Q_\omega X$ is an M-ideal in its bidual, it contains an isomorphic copy of c_0 [15, Theorem 3.5]. Hence, by Sobczyk's theorem [26, Theorem 2.f.5] there is a projection $P: Q_\omega X \to c_0$. Then $P \circ Q_\omega$ is a projection showing that c_0 is complemented in X.

Remarks, examples, problems. 1) Let X be an M-ideal in X^{**}. Then a PRI in X^* can be constructed with the help of a reduced number of tools. In fact, the proof of (*) can be simplified considerably: Fix one subspace V of X and denote by E the set of those points of B_{V^*} which are strongly exposed by elements from V. Then by Asplund [29, p. 735], B_{V^*} is equal to the weak* closure of co E. But V is an M-ideal in V^{**}, which easily implies that the weak* and weak convergences on the unit sphere of V^* coincide. Hence B_{V^*} is equal to the norm closure of co E, and so $V^* = \overline{\mathrm{sp}}\, E$. Finally, remarking that E is contained in $\{D_0 v|_V: v \in B_V\}$, we get (*).

2) We recall a result of Stegall [4, p. 195] that a dual Banach space has the Radon–Nikodym property if and only if it admits no bounded infinite ε-tree for any $\varepsilon > 0$. Thus from Theorems 1 and 2 the following problems arise naturally: *If a Banach space admits no bounded infinite ε-tree for any $\varepsilon > 0$, does it admit a PRI, a linear continuous injection into $c_0(\Gamma)$, ...?*

3) In [7] it is shown that a weakly countably determined Asplund space admits a Fréchet differentiable norm. By using the mapping D constructed in the proof of Theorem 1, the proof of this result can be made more self-contained. In fact, a renorming theorem of Mercourakis [28, Theorem 4.6] that the dual of a weakly countably determined space has a dual strictly convex norm is then no longer needed.

4) Not every dual has a PRI: see the duals JT^* and JH^* in Remark 7. On the other hand, if X^* has a PRI, X need not be Asplund. Indeed, l_∞ is not Asplund, yet its dual, as any abstract L_1-space, has a PRI.

Let us also notice that, if X is a separable space not containing l_1 such that X^* is not separable, then there exists [8, Proposition VII.4] a separable subspace Z in X^* such that Z is not contained in a separable complemented subspace of X^*; thus, assuming the continuum hypothesis, we can easily show that X^* has no PRI. Fortunately, in some concrete duals the continuum hypothesis is needless for disproving the existence of PRI; see Remark 7.

5) The projections P_α constructed in Theorem 1 will not be weak*-weak* continuous in some concrete cases. Namely, having a PRI $\{P_\alpha\}$ with this continuity property, a dual version [7] of Zizler's extension of Troyanski's theorem would yield a dual LUR norm on X^*. But, according to Talagrand [34], we know that the dual of the space $C([0, \Omega])$ of continuous functions on $[0, \Omega]$ admits no dual strictly convex, *a fortiori* no dual LUR norm; see also the spaces JL, JT^* in Remark 7.

6) *The existence of a PRI is not an isomorphic property*, i.e. it depends on a concrete norm on the space. In order to show this consider once more the space $X = C([0, \Omega])$. Then, putting $P_\alpha x = x\chi_{[0,\alpha]} + x(\alpha)\chi_{(\alpha,\Omega]}$, $x \in X$, $\omega \leqslant \alpha \leqslant \Omega$, we can easily verify that $\{P_\alpha\}$ is a PRI on X with respect to the supremum norm. On the other hand, if X is equipped with a Fréchet differentiable norm $\|\cdot\|$ constructed by Talagrand [34], then it does not admit a PRI. Indeed, if a PRI $\{Q_\alpha\}$ existed on $(X, \|\cdot\|)$, then by [19, Lemma 3], $\{Q_\alpha^*\}$ would be a PRI on X^*, each Q_α^* being, of course, weak*-weak* continuous. And this is impossible: see Remark 5.

7) Let JL denote the Johnson–Lindenstrauss space [21]. We shall show, following an idea of Plichko [30], that this space admits no PRI with respect to any equivalent norm. By contradiction, assume that $\{P_\alpha: \omega \leqslant \alpha \leqslant \mu\}$ is a PRI on JL. There exists a subspace M of JL isometric to c_0 such that the quotient JL/M is isometric to a Hilbert space [21]. Hence JL is Asplund [29, Theorem 14] and it is easy to find $\omega \leqslant \alpha < \mu$ such that $M \subset P_\alpha(JL) \equiv X$; thus dens $X \leqslant \bar{\bar{\alpha}} < \bar{\bar{\mu}}$. Also, JL/X is a quotient of JL/M and so JL is isomorphic to the direct sum of X and a Hilbert space H. Now recall that JL^* is weakly* separable [21]. Thus, since JL^* is isomorphic to the direct sum of X^* and H, it follows that H must be separable. Hence, taking into account the Asplundness of X we conclude that

$$\operatorname{dens} X = \operatorname{dens} X^* = \operatorname{dens} JL^* = \bar{\bar{\mu}} > \bar{\bar{\alpha}},$$

a contradiction.

Thus JL does not admit a PRI for any equivalent norm. But both M and Hilbert space have a PRI. So *the existence of a PRI is not a three space property*. On the other hand, JL is LUR renormable, since, for instance, the existence of a LUR norm is a three space property [12]. Also, the weak* separability of JL^* [21] implies that JL^* endowed with any norm contains no PRI $\{P_\alpha\}$ with all the P_α weak*-weak* continuous. None the less, JL^* admits a dual LUR norm [11] since it is weakly compactly generated. By the way, this implies that the P_α from Theorem 1 are not weak*-weak* continuous in general; see also Remark 5.

Let JT be the James tree space [27] and let M denote its predual JT_*. The same method then shows that the dual JT^* enjoys all the properties of

JL which are quoted in the preceding paragraph. It should be noted that Plichko found a subspace of JT^* with the same properties [30]. It can also be shown by a somewhat different method that the dual JH^* of the James–Hagler space [13] admits no PRI yet is LUR renormable.

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Sibeliova 49, 162 00 Praha 6, Czechoslovakia

and

ÉQUIPE D'ANALYSE
UNIVERSITÉ PARIS VI
4, Pl. Jussieu, 75230 Paris Cedex 05, France

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