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Ergodic power functions for mean bounded, invertible, positive operators

by

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Abstract. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let T be an invertible positive linear operator on $L^p(X, \mathcal{A}, \mu)$, $1 < p < \infty$, with positive inverse. For each $r > 1$, we consider the ergodic power function

$$P_r = \Big(\sum_{k=0}^{\infty} |T_{0,k+1} - T_{0,k}|^r\Big)^{1/r}, \quad \text{where} \quad T_{0,k} f = (k+1)^{-1} \sum_{i=0}^{k} T^i f.$$

We prove that if T is a mean bounded operator, i.e., $\sup_{k\geq 0} \|T_{0,k}\|_p < \infty$, then P_r is a bounded operator on L^p. This result generalizes one of R. L. Jones and allows us to get another one of R. Sato as an easy consequence.

1. Introduction. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let T be an invertible linear operator on $L^p(X, \mathcal{A}, \mu)$, $1 < p < \infty$, with positive inverse. For each pair of nonnegative integers n and k, we define the operators

$$T_{n,k} f = (n+k+1)^{-1} \sum_{i=-n}^{k} T^i f$$

and for each $r > 1$ the *ergodic power functions* are defined by

$$P_r^+ f = \Big(\sum_{k=0}^{\infty} |T_{0,k+1} f - T_{0,k} f|^r\Big)^{1/r},$$

$$P_r f = \Big(\sum_{k=0}^{\infty} |T_{0,k+1} f - T_{0,k} f|^r + |T_{k+1,0} f - T_{k,0} f|^r\Big)^{1/r}.$$

The operator P_2^+ was introduced by Jones [5] for the case where T is induced by an ergodic, invertible, measure preserving transformation on a nonatomic probability space. He proved that then the operator P_2^+ is

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bounded in L^p, $1 < p < \infty$, and satisfies a weak type inequality for $p = 1$.

The good weights for P_r were studied in [6] under the assumption that T is induced by an invertible measure preserving transformation (not necessarily ergodic). Sato [9] has proved that if T is a positive invertible linear operator on L^p, $1 < p < \infty$, with positive inverse then the uniform boundedness of the averages $T_{k,k}$ is sufficient for the boundedness of P_r in L^p (this result generalizes one in [6]). Other results about ergodic power functions can be seen in [11] and [12].

In this paper we will prove, for the case where T is a positive, invertible, linear operator on L^p with positive inverse, that the uniform boundedness of the averages $T_{0,k}$ is sufficient for the boundedness of P_r^+ in L^p. This generalizes the result in [5] and allows us to get the result in [9] as an easy consequence.

2. Notation and previous results. Throughout this paper, p and r will be numbers greater than 1, q will be the conjugate exponent of p and C will denote a positive constant not necessarily the same at each occurrence.

From now on, T will be a positive, invertible, linear operator on L^p (of a σ-finite measure space $(X, \mathscr{A}, \mu)$) with positive inverse. Then, as is well known [4], T is a Lamperti operator and there exist a positive, multiplicative, linear operator S and two sequences of positive functions $\{g_i\}$ and $\{J_i\}$ such that for every $i \in \mathbf{Z}$ and any f in L^p,

$$T^i f = g_i S^i f, \qquad \int_X f\,d\mu = \int_X J_i S^i f\,d\mu.$$

Therefore, if we set $h_i(x) = g_i^{-p}(x) J_i(x)$ we have

$$\int_X |f|^p\,d\mu = \int_X |T^i f|^p h_i\,d\mu. \tag{2.1}$$

In order to prove our result, we will need a theorem about weights for the one-sided Hardy–Littlewood maximal functions $\vec{F}$ and $\overleftarrow{F}$, associated to functions F on $\mathbf{Z}$, defined by

$$\vec{F}(j) = \sup_{n\geq 0}(n+1)^{-1}\sum_{k=0}^{n}|F(j+k)|, \qquad \overleftarrow{F}(j) = \sup_{n\geq 0}(n+1)^{-1}\sum_{k=0}^{n}|F(j-k)|.$$

(2.2) THEOREM. *Let w be a positive function on $\mathbf{Z}$. The operator $F \to \vec{F}$ is of strong type (p, p) with respect to $w dv$, where v is the counting measure on $\mathbf{Z}$, if and only if w satisfies A_p^+, i.e.,*

A_p^+: *There exists a constant $C > 0$ such that for every j and any $n \geq 0$,*

$$\Big(\sum_{k=0}^{n} w(j+k)\Big)\Big(\sum_{k=n}^{2n} w^{1-q}(j+k)\Big)^{p-1} \leq C(2n+1)^p.$$

Remarks. (1) For the proof of Theorem (2.2), just look at the poof in [10] for the maximal function in $\mathbf{R}$ and write it in the integers.

(2) Condition A_p^+ can be expressed in the following equivalent way:

$$\Big(\sum_{k=0}^{n} w(j+k)\Big)\Big(\sum_{k=n}^{m} w^{1-q}(j+k)\Big)^{p-1} \leq C(m+1)^p$$

for any m, n, j with $0 \leq n \leq m$.

(3) The result of Theorem (2.2) holds for $F \to \overleftarrow{F}$ with A_p^- instead of A_p^+, where A_p^- is the following condition:

A_p^-: *There exists $C > 0$ such that for every j and any $n \geq 0$,*

$$\Big(\sum_{k=0}^{n} w(j-k)\Big)\Big(\sum_{k=n}^{2n} w^{1-q}(j-k)\Big)^{p-1} \leq C(2n+1)^p.$$

(4) We will say that a positive function ω on $\mathbf{Z}$ satisfies A_1^+ (A_1^-) if $\overleftarrow{\omega} \leq C\omega$ ($\vec{\omega} \leq C\omega$). Condition A_1^+ (A_1^-) characterizes the weights for which the maximal operator $F \to \vec{F}$ ($F \to \overleftarrow{F}$) is of weak type (1,1) with respect to ωdv (see [7]).

Finally, we will also need a result about the maximal function associated to T:

$$M^+ f = \sup_{n\geq 0} T_{0,n}|f|.$$

(2.3) THEOREM (see [8]). *With the above notation, the following are equivalent:*

(a) *M^+ is bounded in L^p.*

(b) $\sup_{k\geq 0}\|T_{0,k}\|_p < \infty$.

(c) *For almost all x, the function defined on $\mathbf{Z}$ by $i \to h_i(x)$ satisfies A_p^+ with a constant independent of x.*

The operators P_r^+ are related to M^+. More precisely, there exists a constant $C > 0$ (depending only on r) such that

$$P_r^+ \leq CM^+ + Q_r^+, \quad \text{where} \quad Q_r^+ f = \Big(\sum_{k=0}^{\infty}(k+1)^{-r}|T^k f|^r\Big)^{1/r}. \tag{2.4}$$

This inequality will be basic in the proof of the boundedness of P_r^+.

3. Boundedness of the ergodic power functions. Our main result is the following:

(3.1) THEOREM. *If $\sup_{k\geq 0}\|T_{0,k}\|_p < \infty$ then Q_r^+ and P_r^+ are bounded in L^p.*

It follows from (2.3) and (2.4) that it suffices to prove the statement for

Q_r^+. In order to do this, we will study the operator S_r^+, acting on functions on $\mathbf{Z}$, defined by

$$S_r^+ f(i) = \Big(\sum_{k=0}^{\infty} (k+1)^{-r} |f(i+k)|^r\Big)^{1/r}.$$

We will also need some results about functions on $\mathbf{Z}$ satisfying A_p^+.

(3.2) Lemma. *If ω is a positive function on $\mathbf{Z}$ that satisfies A_r^+ then there exists a constant $C>0$ such that for any j and every $i_0 \geqslant 0$,*

$$\sum_{k=i_0}^{\infty} (i_0+1)^r (k+1)^{-r} \omega(j-k) \leqslant C \sum_{k=0}^{i_0} \omega(j-k). \tag{3.3}$$

Proof. There exists s with $1<s<r$ such that $\omega \in A_s^+$ (see [10]). This implies that there exists a constant $C>0$ such that if $m \geqslant i_0$ then

$$\Big(\sum_{k=i_0}^{m} \omega(j-k)\Big)\Big(\sum_{k=0}^{i_0} \omega^{-1/s-1}(j-k)\Big)^{s-1} \leqslant C(m+1)^s. \tag{3.4}$$

On the other hand, if we apply Hölder's inequality to

$$i_0+1 = \sum_{k=0}^{i_0} \omega^{1/s}(j-k)\omega^{-1/s}(j-k),$$

we have

$$(i_0+1)^s\Big(\sum_{k=0}^{i_0} \omega(j-k)\Big)^{-1} \leqslant \Big(\sum_{k=0}^{i_0} \omega^{-1/s-1}(j-k)\Big)^{s-1}. \tag{3.5}$$

Then (3.4) together with (3.5) give

$$\sum_{k=i_0}^{m} \omega(j-k) \leqslant C(m+1)^s (i_0+1)^{-s} \sum_{k=0}^{i_0} \omega(j-k). \tag{3.6}$$

Multiplying (3.6) by $(m+1)^{-r-1}$ and summing in $m \geqslant i_0$ we get

(3.7)
$$\sum_{m=i_0}^{\infty} (m+1)^{-r-1} \sum_{k=i_0}^{m} \omega(j-k) \leqslant C(i_0+1)^{-s} \sum_{m=i_0}^{\infty} (m+1)^{s-r-1} \sum_{k=0}^{i_0} \omega(j-k).$$

Now, inequality (3.3) follows from (3.7) and the inequalities

$$\sum_{k=i_0}^{\infty} (k+1)^{-r}\omega(j-k) \leqslant C \sum_{m=i_0}^{\infty} (m+1)^{-r-1} \sum_{k=i_0}^{m} \omega(j-k),$$

$$\sum_{m=i_0}^{\infty} (m+1)^{s-r-1} \sum_{k=0}^{i_0} \omega(j-k) \leqslant C(i_0+1)^{s-r} \sum_{k=0}^{i_0} \omega(j-k).$$

Note. Results similar to Lemma (3.2) appear in [1] and [3].

(3.8) Theorem. *If ω is a positive function on $\mathbf{Z}$ that satisfies A_1^+ then S_r^+ is of weak type (1,1) with respect to $\omega d\nu$, where ν is the counting measure on $\mathbf{Z}$.*

Proof of Theorem (3.8). Let f be a function on $\mathbf{Z}$. For $k>0$ fixed, we define

$$f_k(i) = f(i+k),$$
$$g_k(i) = |f_k(i)| \quad \text{if } |f_k(i)| < \lambda(k+1),$$
$$g_k(i) = 0 \quad \text{if } |f_k(i)| \geqslant \lambda(k+1),$$
$$b_k(i) = |f_k(i)| - g_k(i).$$

It is clear that if

$$G_\lambda = \{i \in \mathbf{Z}:\ 2^r \sum_{k=0}^{\infty} (k+1)^{-r} |g_k(i)|^r > \lambda^r\},$$

$$B_\lambda = \{i \in \mathbf{Z}:\ 2^r \sum_{k=0}^{\infty} (k+1)^{-r} |b_k(i)|^r > \lambda^r\}$$

then it suffices to prove

$$\sum_{i\in G_\lambda} \omega(i) \leqslant \frac{C}{\lambda} \sum_{i=-\infty}^{\infty} |f(i)|\,\omega(i), \tag{3.9}$$

$$\sum_{i\in B_\lambda} \omega(i) \leqslant \frac{C}{\lambda} \sum_{i=-\infty}^{\infty} |f(i)|\,\omega(i). \tag{3.10}$$

Proof of inequality (3.9). It is clear that

$$\sum_{i\in G_\lambda} \omega(i) \leqslant 2^r\lambda^{-r} \sum_{k=0}^{\infty} (k+1)^{-r} \sum_{i=-\infty}^{\infty} \omega(i)|g_k(i)|^r$$
$$= 2^r\lambda^{-r} \int_0^{\infty} rt^{r-1} \sum_{k=0}^{\infty} (k+1)^{-r} \sum_{i=-\infty}^{\infty} \omega(i)\chi_{G_{k,t}}(i)\,dt$$

where $G_{k,t} = \{i \in \mathbf{Z}:\ g_k(i) > t\}$ and $\chi_{G_{k,t}}$ is its characteristic function.

Fix k and $t>0$; then for $i \in G_{k,t}$ we have $t < |f_k(i)| < \lambda(k+1)$. This implies $k \geqslant [t/\lambda]$, where $[t/\lambda]$ is the integer part of t/λ. Therefore,

$$\sum_{i\in G_\lambda} \omega(i) \leqslant 2^r\lambda^{-r} \int_0^{\infty} rt^{r-1} \sum_{k=[t/\lambda]}^{\infty} (k+1)^{-r} \sum_{i=-\infty}^{\infty} \omega(i)\chi_{G_{k,t}}(i)\,dt.$$

Replacing i by $j-k$ we get

$$\sum_{i\in G_\lambda} \omega(i) \leqslant 2^r\lambda^{-r} \int_0^{\infty} rt^{r-1} \sum_{k=[t/\lambda]}^{\infty} (k+1)^{-r} \sum_{j=-\infty}^{\infty} \omega(j-k)\chi_{G_{k,t}}(j-k)\,dt$$
$$\leqslant 2^r\lambda^{-r} \int_0^{\infty} rt^{r-1} \sum_{j=-\infty}^{\infty} \chi_{F_t}(j) \sum_{k=[t/\lambda]}^{\infty} (k+1)^{-r}\omega(j-k)\,dt$$

where $F_t = \{j \in \mathbf{Z}:\ |f(j)| > t\}$.

Now Lemma (3.2) and condition A_1^+ give

$$\sum_{i\in G_\lambda}\omega(i)\leqslant C\lambda^{-r}\int_0^\infty t^{r-1}\sum_{j\in F_t}([t/\lambda]+1)^{-r}\sum_{k=0}^{[t/\lambda]}\omega(j-k)\,dt$$
$$\leqslant C\lambda^{-1}\int_0^\infty (t/\lambda)^{r-1}([t/\lambda]+1)^{1-r}\sum_{j\in F_t}\bar{\omega}(j)\,dt$$
$$\leqslant C\lambda^{-1}\int_0^\infty \sum_{j\in F_t}\omega(j)\,dt=C\lambda^{-1}\sum_{j=-\infty}^{\infty}|f(j)|\,\omega(j).$$

Proof of inequality (3.10). Let

$$D=\{i\in \boldsymbol{Z}\colon \sum_{k=0}^{\infty}(k+1)^{-r}|b_k(i)|^r>0\},$$

$$D_k=\{i\in \boldsymbol{Z}\colon b_k(i)>0\},\qquad E_k=\{i\in \boldsymbol{Z}\colon |f(i)|\geqslant\lambda(k+1)\}.$$

It is clear that $B_\lambda\subset D$, $D=\bigcup_{k=0}^{\infty}D_k$ and $E_k\subset E_{k-1}$ for every $k\geqslant 1$. Besides, $i\in D_k$ if and only if $i=j-k$ with $j\in E_k$. Therefore,

$$\sum_{i\in B_\lambda}\omega(i)\leqslant\sum_{k=0}^{\infty}\Big(\sum_{j\in E_k}\omega(j-k)\Big)=\sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty}\omega(j-k)\,\chi_{E_k}(j).$$

Observe that j belongs to E_k if and only if $|f(j)|/\lambda\geqslant k+1$. Then the last sum with index k is finite and equal to

$$\sum_{k=0}^{[f(j)/\lambda]}\omega(j-k)\,\chi_{E_k}(j).$$

Therefore, we have

$$\sum_{i\in B_\lambda}\omega(i)\leqslant\sum_{j=-\infty}^{\infty}\sum_{k=0}^{[f(j)/\lambda]}\omega(j-k)\,\chi_{E_k}(j)\leqslant\sum_{j=-\infty}^{\infty}\chi_{E_0}(j)\sum_{k=0}^{[f(j)/\lambda]}\omega(j-k)$$
$$\leqslant\sum_{j=-\infty}^{\infty}\chi_{E_0}(j)([f(j)/\lambda]+1)\bar{\omega}(j).$$

Now, condition A_1^+ gives

$$\sum_{i\in B_\lambda}\omega(i)\leqslant C\sum_{j=-\infty}^{\infty}\chi_{E_0}(j)([f(j)/\lambda]+1)\,\omega(j)$$

and by the definition of E_0,

$$\sum_{i\in B_\lambda}\omega(i)\leqslant C\lambda^{-1}\sum_{j=-\infty}^{\infty}|f(j)|\,\omega(j).$$

In order to study the boundedness of S_r^+ we will need a result which is based on an extrapolation theorem (see [2] and [6]).

(3.11) LEMMA. *Let ω be a positive function that satisfies A_p^+ and let g be a nonnegative function in $L^q(\omega d\nu)$ where ν is the counting measure on $\boldsymbol{Z}$. Then there exists $G\in L^q(\omega d\nu)$ such that $G\geqslant g$, $\|G\|_{q,\omega}\leqslant C\|g\|_{q,\omega}$ and $G\omega$ satisfies A_1^+ (the constant C is independent of g).*

Proof. It is clear that ω satisfies A_p^+ if and only if ω^{1-q} satisfies A_q^- and this is equivalent to the strong type (q,q) with respect to $\omega^{1-q}d\nu$ of the operator $f\to\tilde{f}$ defined on functions on $\boldsymbol{Z}$. Thus, the operator P defined by $Pf=\omega^{-1}(f\omega)^{\tilde{}}$ is bounded on $L^q(\omega d\nu)$. The function $G=\sum_{j=0}^{\infty}(2C)^{-j}P^j g$, where $\|P\|\leqslant C$, satisfies the conditions required in the lemma.

(3.12) THEOREM. *If ω is a positive function on $\boldsymbol{Z}$ that satisfies A_p^+ then S_r^+ is of strong type (p,p) with respect to $\omega d\nu$ where ν is the counting measure on $\boldsymbol{Z}$. Furthermore, the constant of the strong type inequality depends only on p, r and the constant of the A_p^+ condition.*

Proof. First, we will prove that if $\omega\in A_p^+$ then S_r^+ is of weak type (p,p). Let $\lambda>0$ and $O_\lambda=\{i\in\boldsymbol{Z}\colon S_r^+f(i)>\lambda\}$. Then

$$\sum_{i\in O_\lambda}\omega(i)=\|\chi_{O_\lambda}\|_{p,\omega}^p=\Big(\sum_{i=-\infty}^{\infty}g(i)\,\chi_{O_\lambda}(i)\,\omega(i)\Big)^p$$

where g is a nonnegative function with $\|g\|_{q,\omega}=1$. By Lemma (3.11), there exists $G\geqslant g$ with $\|G\|_{q,\omega}\leqslant C\|g\|_{q,\omega}=C$ and $G\omega\in A_1^+$. This fact and Theorem (3.8) give

$$\sum_{i\in O_\lambda}\omega(i)\leqslant\Big(\sum_{i\in O_\lambda}G(i)\,\omega(i)\Big)^p\leqslant C\lambda^{-p}\Big(\sum_{i=-\infty}^{\infty}|f(i)|\,G(i)\,\omega(i)\Big)^p.$$

If we apply Hölder's inequality, we obtain

$$\sum_{i\in O_\lambda}\omega(i)\leqslant C\lambda^{-p}\sum_{i=-\infty}^{\infty}|f(i)|^p\,\omega(i)\Big(\sum_{i=-\infty}^{\infty}|G(i)|^q\,\omega(i)\Big)^{p/q}$$
$$\leqslant C\lambda^{-p}\sum_{i=-\infty}^{\infty}|f(i)|^p\,\omega(i).$$

Therefore, S_r^+ is of weak type (p,p).

On the other hand, there exists $\xi>0$ with $1<p-\xi$ such that $\omega\in A_{p-\xi}^+$ (see [10]). Then S_r^+ is of weak type $(p-\xi,p-\xi)$ with respect to $\omega d\nu$ and since it is of infinite type, Marcinkiewicz's interpolation theorem ensures that S_r^+ is of strong type (p,p). The statement about the constant follows easily from our proofs and the results in [10].

Now, we are already prepared to prove Theorem (3.1).

Proof of Theorem (3.1). As we said, it will suffice to prove the boundedness of Q_r^+.

By (2.3) we see that for almost all x, the function $i \to h_i(x)$ satisfies A_p^+ with a constant independent of x. Now, let f be a positive function on X and let $N \geqslant 0$. We define $Q_{r,N}^+$ by

$$Q_{r,N}^+ f = \Big(\sum_{k=0}^{N} (k+1)^{-r} (T^k f)^r\Big)^{1/r}.$$

Let L be a positive integer. By (2.1) we have

$$\int_X (Q_{r,N}^+ f)^p(x)\, d\mu = (L+1)^{-1} \sum_{i=0}^{L} \int_X \big(T^i (Q_{r,N}^+ f)(x)\big)^p h_i(x)\, d\mu$$
$$= (L+1)^{-1} \int_X \sum_{i=0}^{L} \Big(\sum_{k=0}^{N} (k+1)^{-r} \big(T^{i+k} f(x)\big)^r\Big)^{p/r} h_i(x)\, d\mu.$$

If $0 \leqslant i \leqslant L$ then the sum in brackets is bounded by $\big(S_r^+ (\chi_{[0,N+L]} f^x)(i)\big)^r$ where f^x is the function on $\boldsymbol{Z}$ defined by

$$f^x(s) = T^s f(x)$$

and $\chi_{[0,N+L]}$ is the characteristic function of the interval $[0, N+L]$ in $\boldsymbol{Z}$. Then this observation together with (2.1) and Theorem (3.12) applied to the functions defined on $\boldsymbol{Z}$ by $i \to h_i(x)$ give

$$\int_X (Q_{r,N}^+ f(x))^p\, d\mu \leqslant (L+1)^{-1} \int_X \sum_{i=0}^{L} \big(S_r^+ (\chi_{[0,N+L]} f^x)(i)\big)^p h_i(x)\, d\mu$$
$$\leqslant C(L+1)^{-1} \int_X \sum_{i=0}^{N+L} (f^x(i))^p h_i(x)\, d\mu = C\frac{N+L+1}{L+1} \int_X (f(x))^p\, d\mu.$$

Letting L and then N go to infinity we get

$$\int_X (Q_r^+ f)^p\, d\mu \leqslant C \int_X f^p\, d\mu.$$

This finishes the proof of Theorem (3.1).

Now, the result in [9] follows easily from (3.1).

(3.13) Corollary (see [9]). *If* $\sup_{k \geqslant 0} \|T_{k,k}\|_p < \infty$ *then* P_r *is bounded in* L^p.

(3.14) Note. The converse of Theorem (3.1) is false even if T is induced by a pointwise transformation on X. In fact, if the converse of (3.1) is true, then the converse of Corollary (3.13) is true too. But this is false (see an example in [6]).

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