Ergodic power functions for mean bounded, invertible, positive operators

by

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Abstract. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and let \(T\) be an invertible positive linear operator on \(L^p(X, \mathcal{A}, \mu)\), \(1 < p < \infty\), with positive inverse. For each \(r > 1\), we consider the ergodic power function

\[ P_r = \left( \sum_{k=0}^{\infty} |T_{0,k+1} f - T_{0,k} f|^{r} \right)^{1/r}, \quad \text{where} \quad T_{0,k} f = (k+1)^{-1} \sum_{i=0}^{k} T^i f. \]

We prove that if \(T\) is a mean bounded operator, i.e., \(\sup_{n \in \mathbb{N}} \|T_n\|_p < \infty\), then \(P_r\) is a bounded operator on \(L^p\). This result generalizes one of R. L. Jones and allows us to get another one of R. Sato as an easy consequence.

1. Introduction. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and let \(T\) be an invertible linear operator on \(L^p(X, \mathcal{A}, \mu)\), \(1 < p < \infty\), with positive inverse. For each pair of nonnegative integers \(n\) and \(k\), we define the operators

\[ T_{n,k} f = (n+k+1)^{-1} \sum_{i=-n}^{k} T^i f \]

and for each \(r > 1\) the ergodic power functions are defined by

\[ P_r^+ f = \left( \sum_{k=0}^{\infty} |T_{0,k+1} f - T_{0,k} f|^{r} \right)^{1/r}, \quad P_r f = \left( \sum_{k=0}^{\infty} |T_{0,k+1} f - T_{0,k} f|^{r} + |T_{k+1,0} f - T_{k,0} f|^{r} \right)^{1/r}. \]

The operator \(P_r^+\) was introduced by Jones [5] for the case where \(T\) is induced by an ergodic, invertible, measure preserving transformation on a nonatomic probability space. He proved that then the operator \(P_r^+\) is

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bounded in $L^p, 1 < p < \infty$, and satisfies a weak type inequality for $p = 1$.

The good weights for $P_r$ were studied in [6] under the assumption that $T$ is induced by an invertible measure preserving transformation (not necessarily ergodic). Sato [9] has proved that if $T$ is a positive invertible linear operator on $L^p, 1 < p < \infty, \text{with positive inverse then }$ the uniform boundedness of the averages $T_n \phi$ is sufficient for the boundedness of $P_r$ in $L^p$ (this result generalizes one in [6]). Other results about ergodic power functions can be seen in [11] and [12].

In this paper we will prove, for the case where $T$ is a positive, invertible, linear operator on $L^q$ with positive inverse, that the uniform boundedness of the averages $T_n \phi$ is sufficient for the boundedness of $P_r^+$ in $L^p$. This generalizes the result in [5] and allows us to get the result in [9] as an easy consequence.

2. Notation and previous results. Throughout this paper, $p$ and $r$ will be numbers greater than 1, $q$ will be the conjugate exponent of $p$ and $C$ will denote a positive constant not necessarily the same at each occurrence.

From now on, $T$ will be a positive, invertible, linear operator on $L^q$ (of a $\sigma$-finite measure space $(X, \mathscr{A}, \mu)$) with positive inverse. Then, as is well known [4], $T$ is a Lamperti operator and there exist a positive, multiplicative, linear operator $S$ and two sequences of positive functions $\{g_i\}$ and $\{J_i\}$ such that for every $i \in Z$ and any $f$ in $L^p$,

$$T^i f = g_i S^i f, \quad \int_X f d\mu = \int_X J_i S^i f d\mu.$$

Therefore, if we set $h_i(x) = g_i^{-\frac{1}{p}}(x) J_i(x)$ we have

$$\int_X f d\mu = \int_X |T^i f|^p h_i d\mu. \tag{2.1}$$

In order to prove our result, we will need a theorem about weights for the one-sided Hardy–Littlewood maximal functions $F$ and $\bar{F}$, associated to functions $F$ on $Z$, defined by

$$\bar{F}(j) = \sup_{\alpha > 0} (n+1)^{-\frac{1}{p}} \sum_{k=0}^n |F(j+k)|, \quad F(j) = \sup_{\alpha > 0} (n+1)^{-\frac{1}{p}} \sum_{k=0}^n |F(j-k)|.$$

(2.2) THEOREM. Let $w$ be a positive function on $Z$. The operator $F \to \bar{F}$ is of strong type $(p, p)$ with respect to $w \omega w$, where $\omega$ is the counting measure on $Z$, if and only if $w$ satisfies $A_r^+$, i.e.,

$$A_r^+: \text{ There exists a constant } C > 0 \text{ such that for every } j \text{ and any } n \geq 0,$$

$$\left( \sum_{k=0}^n w(j+k) \right) \left( \sum_{k=0}^{2n} w^{1-q} (j+k) \right)^{p-1} \leq C (2n+1)^q.$$
$Q^\ast$. In order to do this, we will study the operator $S^\ast_r$, acting on functions on $\mathbb{Z}$, defined by

$$S^\ast_r f(i) = \sum_{k=0}^{\infty} (k+1)^{-r} (f(i+k))^r.$$

We will also need some results about functions on $\mathbb{Z}$ satisfying $A^\ast_r$.

(3.2)Lemma. If $\omega$ is a positive function on $\mathbb{Z}$ that satisfies $A^\ast_r$ then there exists a constant $C > 0$ such that for any $j$ and every $i_0 \geq 0$,

$$\sum_{k=0}^{\infty} \omega(j-k) \leq C i_0^{-1/2} (j-k)^{-1}. \quad (3.3)$$

Proof. There exists $s$ with $1 < s < r$ such that $\omega \in A^s$ (see [10]). This implies that there exists a constant $C > 0$ such that if $m \geq i_0$ then

$$\sum_{k=0}^{m} \omega(j-k) \leq C i_0^{-1/2} (j-k)^{-1}. \quad (3.4)$$

On the other hand, if we apply Hölder’s inequality to

$$i_0 + 1 = \sum_{k=0}^{i_0} \omega^{2/3} (j-k) \omega^{-1/3} (j-k),$$

we have

$$i_0 + 1 = \sum_{k=0}^{i_0} \omega^{2/3} (j-k) \omega^{-1/3} (j-k),$$

Then (3.4) together with (3.5) give

$$\sum_{k=0}^{m} \omega(j-k) \leq C (m+1)^{-1/2} \sum_{k=0}^{i_0} \omega(j-k). \quad (3.6)$$

Multiplying (3.6) by $(m+1)^{-1/2}$ and summing in $m \geq i_0$ we get

$$\sum_{m=i_0}^{\infty} (m+1)^{-1/2} \sum_{k=0}^{m} \omega(j-k) \leq C i_0^{-1/2} \sum_{k=0}^{i_0} \omega(j-k). \quad (3.7)$$

Now, inequality (3.3) follows from (3.7) and the inequalities

$$\sum_{k=0}^{i_0} (k+1)^{-r} \omega(j-k) \leq C i_0 \sum_{k=0}^{i_0} (k+1)^{-r} \sum_{k=0}^{i_0} \omega(j-k),$$

$$\sum_{m=i_0}^{\infty} (m+1)^{-r-1} \sum_{k=0}^{m} \omega(j-k) \leq C (i_0+1)^{-r-1} \sum_{k=0}^{i_0} \omega(j-k).$$

Note. Results similar to Lemma (3.2) appear in [1] and [3].

(3.8) Theorem. If $\omega$ is a positive function on $\mathbb{Z}$ that satisfies $A^\ast_r$ then $S^\ast_r$ is of weak type $(1,1)$ with respect to $\omega d\nu$, where $\nu$ is the counting measure on $\mathbb{Z}$.

Proof of Theorem (3.8). Let $f$ be a function on $\mathbb{Z}$. For $k > 0$ fixed, we define

$$f_k(i) = f(i+k),$$

$$g_k(i) = |f_k(i)| \quad \text{if} \quad |f_k(i)| < \lambda(k+1),$$

$$g_k(i) = 0 \quad \text{if} \quad |f_k(i)| \geq \lambda(k+1),$$

$$b_k(i) = |f_k(i)| - g_k(i).$$

It is clear that if

$$G_2 = \{ i \in \mathbb{Z}: \sum_{k=0}^{\infty} (k+1)^{-\tau} |g_k(i)|^\tau > \lambda' \},$$

$$B_2 = \{ i \in \mathbb{Z}: \sum_{k=0}^{\infty} (k+1)^{-\tau} |b_k(i)|^\tau > \lambda' \}$$

then it suffices to prove

$$\sum_{i \in B_2} \omega(i) \leq \frac{C}{\lambda} \sum_{i=-\infty}^{\infty} |f(i)| \omega(i), \quad (3.9)$$

$$\sum_{i \in G_2} \omega(i) \leq \frac{C}{\lambda} \sum_{i=-\infty}^{\infty} |f(i)| \omega(i). \quad (3.10)$$

Proof of inequality (3.9). It is clear that

$$\sum_{i \in B_2} \omega(i) \leq \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \omega(i) g_k(i)$$

$$= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \omega(i) \sum_{j=k+1}^{\infty} \omega(j) Z_{G_{k,i}}(t) dt$$

where $G_{k,i} = \{ i \in \mathbb{Z}: g_k(i) > \lambda \}$ and $Z_{G_{k,i}}$ is its characteristic function.

Fix $k$ and $t > 0$; then for $i \in G_{k,i}$ we have $i < |f_k(i)| < \lambda(k+1)$. This implies $k \geq [i/\lambda]$, where $[i/\lambda]$ is the integer part of $i/\lambda$. Therefore,

$$\sum_{i \in G_{k,i}} \omega(i) \leq \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=\lambda(k+1)}^{\infty} \sum_{j=\lambda(k+1)}^{\infty} \omega(i) Z_{G_{k,i}}(t) dt.$$

Replacing $i$ by $j=k$ we get

$$\sum_{i \in G_{k,i}} \omega(i) \leq \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=\lambda(k+1)}^{\infty} \sum_{j=\lambda(k+1)}^{\infty} \omega(j) Z_{G_{k,i}}(j-k) dt$$

$$\leq \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=\lambda(k+1)}^{\infty} \sum_{j=\lambda(k+1)}^{\infty} \omega(j) Z_{G_{k,i}}(j-k) dt$$

where $F_r = \{ i \in \mathbb{Z}: |f(i)| > \lambda \}$. 

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Now Lemma (3.2) and condition $A_1^+$ give
\[
\sum_{i \in D_k} \omega(i) \leq C \lambda^{-r} \left( \int_0^{\lambda^{-r}} \sum_{j \in F_k} \left( \left[ \frac{|j|}{\lambda} \right] + 1 \right)^{-r} \sum_{k=0}^{\left[ \frac{|j|}{\lambda} \right]} \omega(j-k) \right) dt \\
\leq C \lambda^{-r} \left( \int_0^{\lambda^{-r}} \left( \left[ \frac{|j|}{\lambda} \right] + 1 \right)^{-r} \sum_{j \in F_k} \omega(j) dt \right) \\
\leq C \lambda^{-r} \left( \int_0^{\lambda^{-r}} \sum_{j \in F_k} \omega(j) dt = C \lambda^{-r} \sum_{j=-\infty}^{\infty} |f(j)| \omega(j) \right).
\]

Proof of inequality (3.10). Let
\[
D = \{ i \in \mathbb{Z} : \sum_{k=0}^{\infty} (k+1)^{-r} |b_k(i) - b_{k+1}(i)| > 0 \}
\]
\[
D_k = \{ i \in \mathbb{Z} : b_k(i) > 0 \}, \quad E_k = \{ i \in \mathbb{Z} : |f(i)| \geq \lambda (k+1) \}.
\]
It is clear that $B_k \subset D_k$, $D = \bigcup_{k=0}^{\infty} D_k$, and $E_k \subset E_{k+1}$ for every $k \geq 1$. Besides, $i \in D_k$ if and only if $i = j-k$ with $j \in E_k$. Therefore,
\[
\sum_{i \in D_k} \omega(i) \leq \sum_{k=0}^{\infty} \left( \sum_{j \in E_k} \omega(j-k) \right) = \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \omega(j-k) \chi_{E_k}(j).
\]
Observe that $j$ belongs to $E_k$ if and only if $|f(j)| \lambda \geq k+1$. Then the last sum with index $k$ is finite and equal to
\[
\sum_{k=0}^{\left[ \frac{|f(j)|}{\lambda} \right]} \omega(j-k) \chi_{E_k}(j).
\]
Therefore, we have
\[
\sum_{i \in D_k} \omega(i) \leq \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\left[ \frac{|f(j)|}{\lambda} \right]} \omega(j-k) \chi_{E_k}(j) \leq \sum_{j=-\infty}^{\infty} \chi_{E_0}(j) \left( \sum_{k=0}^{\left[ \frac{|f(j)|}{\lambda} \right]} \omega(j-k) \right).
\]
Now, condition $A_1^+$ gives
\[
\sum_{i \in E_k} \omega(i) \leq C \sum_{j=-\infty}^{\infty} \chi_{E_0}(j) \left( \left[ \frac{|f(j)|}{\lambda} \right] + 1 \right) \omega(j)
\]
and by the definition of $E_0$,
\[
\sum_{i \in E_k} \omega(i) \leq C \lambda^{-1} \sum_{j=-\infty}^{\infty} |f(j)| \omega(j).
\]

In order to study the boundedness of $S_\omega^+$, we will need a result which is based on an extrapolation theorem (see [2] and [6]).

(3.11) Lemma. Let $\omega$ be a positive function that satisfies $A_\omega^+$ and let $g$ be a nonnegative function in $L^1(\omega dv)$ where $v$ is the counting measure on $\mathbb{Z}$. Then there exists $G \in L^1(\omega dv)$ such that $G \geq g$, $|G|_{\infty,\omega} \leq C |g|_{\infty,\omega}$ and $G \omega$ satisfies $A_\omega^+$ (the constant $C$ is independent of $g$).

Proof. It is clear that $\omega$ satisfies $A_\omega^+$ if and only if $\omega^{-1/\varphi}$ satisfies $A_\omega^-$ and this is equivalent to the strong type $(\varphi, q)$ with respect to $\omega^{-1/\varphi} dv$ of the operator $f \rightarrow \hat{f}$ defined on functions on $\mathbb{Z}$. Thus, the operator $P$ defined by $Pf = \omega^{-1/\varphi} (\hat{u} f)$ is bounded on $L^q(\omega dv)$. The function $G = \sum_{i=-\infty}^{\infty} \omega(i) (2C)^{-1} p f(i) g(i)$, where $|G|_p \leq C$, satisfies the conditions required in the lemma.

(3.12) Theorem. If $\omega$ is a positive function on $\mathbb{Z}$ that satisfies $A_{\omega}^+$ then $S_\omega^+$ is of strong type $(p, p)$ with respect to $\omega dv$ where $v$ is the counting measure on $\mathbb{Z}$. Furthermore, the constant of the strong type inequality depends only on $p$, $r$ and the constant of the $A_\omega^+$ condition.

Proof. First, we will prove that if $\omega \in A^+_\omega$ then $S_\omega^+$ is of weak type $(p, p)$. Let $\lambda > 0$ and $\partial_\lambda = \{ i \in \mathbb{Z} : S_\omega^+(i) > \lambda \}$. Then
\[
\sum_{i \in \partial_\lambda} \omega(i) = \left( \sum_{i=-\infty}^{\infty} g(i) \chi_{\partial_\lambda}(i) \omega(i) \right)^p
\]
where $g$ is a nonnegative function with $|g|_{\infty,\omega} = 1$. By Lemma (3.11), there exists $G \geq g$ with $|G|_{\infty,\omega} \leq C |g|_{\infty,\omega} = C$ and $G \omega \in A^+_\omega$. This fact and Theorem (3.8) give
\[
\sum_{i \in \partial_\lambda} \omega(i) \leq \left( \sum_{i=-\infty}^{\infty} G(i) \omega(i) \right)^p \leq C \lambda^{-\varphi} \left( \sum_{i=-\infty}^{\infty} |f(i)| G(i) \omega(i) \right)^p.
\]
If we apply Hölder's inequality, we obtain
\[
\sum_{i \in \partial_\lambda} \omega(i) \leq C \lambda^{-\varphi} \left( \sum_{i=-\infty}^{\infty} |f(i)|^p \omega(i) \right)^{\frac{p}{q}} \leq C \lambda^{-\varphi} \left( \sum_{i=-\infty}^{\infty} |f(i)|^p \omega(i) \right)^{\frac{p}{q}}.
\]
Therefore, $S_\omega^+$ is of weak type $(p, p)$.

On the other hand, there exists $\xi > 0$ with $1 < p - \xi$ such that $\omega \in A_{\omega}^{1-\xi}$ (see [10]). Then $S_\omega^+$ is of weak type $(p - \xi, p - \xi)$ with respect to $\omega dv$ and since it is of infinite type, Marcinkiewicz's interpolation theorem ensures that $S_\omega^+$ is of strong type $(p, p)$. The statement about the constant follows easily from our proofs and the results in [10].

Now, we are already prepared to prove Theorem (3.1).

Proof of Theorem (3.1). As we said, it will suffice to prove the boundedness of $Q_\omega^+$. 
By (2.3) we see that for almost all $x$, the function $i \mapsto h_i(x)$ satisfies $A_p^+$ with a constant independent of $x$. Now, let $f$ be a positive function on $X$ and let $N \geq 0$. We define $Q_{i,N}^+$ by

$$Q_{i,N}^+ f = \left( \sum_{k=0}^{N} (k+1)^{-r} (T^k f)(y) \right)^{1/p}.$$

Let $L$ be a positive integer. By (2.1) we have

$$\int_X \left( Q_{i,N}^+ f \right)^p (x) \, d\mu = (L+1)^{-1} \sum_{i=0}^{L} \int_X \left( (T^i (Q_{i,N}^+ f))(x) \right)^p h_i(x) \, d\mu$$

$$= (L+1)^{-1} \int_X \left( \sum_{i=0}^{L} (k+1)^{-r} (T^{k+i} f)(x) \right)^{1/p} h_i(x) \, d\mu.$$

If $0 \leq i \leq L$ then the sum in brackets is bounded by $\left( S_{i,N}^+ (\chi_{0,N+i+1} f^\gamma)(x) \right)^{1/p}$ where $f^\gamma$ is the function on $Z$ defined by

$$f^\gamma(s) = T^s f(x)$$

and $\chi_{0,N+i+1}$ is the characteristic function of the interval $[0, N+i+1]$ in $Z$. Then this observation together with (2.1) and Theorem (3.12) applied to the functions defined on $Z$ by $i \mapsto h_i(x)$ give

$$\int_X \left( Q_{i,N}^+ f(x) \right)^p \, d\mu \leq (L+1)^{-1} \sum_{i=0}^{L} \left( S_{i,N}^+ (\chi_{0,N+i+1} f^\gamma)(x) \right)^{1/p} h_i(x) \, d\mu$$

$$\leq C (L+1)^{-1} \sum_{i=0}^{N+L} (f^\gamma(t)^p h_i(x) \, d\mu = C \frac{N+L+1}{L+1} \int_X f^\gamma(x)^p \, d\mu.$$

Letting $L$ and then $N$ go to infinity we get

$$\int_X \left( Q_{i,+} f \right)^p \, d\mu \leq C \int_X f^p \, d\mu.$$

This finishes the proof of Theorem (3.1).

Now, the result in [9] follows easily from (3.1).

(3.13) **Corollary** (see [9]). If $\sup_{a \geq 0} ||T_a||_p < \infty$ then $P_a$ is bounded in $L_p$.

(3.14) **Note.** The converse of Theorem (3.1) is false even if $T$ is induced by a pointwise transformation on $X$. In fact, if the converse of (3.1) is true, then the converse of Corollary (3.13) is true too. But this is false (see an example in [6]).