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Topologies on measure spaces and the Radon–Nikodym theorem

by

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Abstract. Let $M(X)$ be the space of complex Borel measures on a compact metric space X. If $\sigma \in M(X)$, the Radon–Nikodym theorem identifies $L^1(\sigma)$ with $L(\sigma)$, the measures which vanish on those sets where $|\sigma|$ vanishes. Let $\mathcal{T}$ be a topology on $M(X)$ and $L^{\mathcal{T}}(\sigma)$ the $\mathcal{T}$-closure of $L(\sigma)$. Analogously to the Radon–Nikodym theorem, we show that for certain $\mathcal{T}$, $L^{\mathcal{T}}(\sigma)$ is characterized by its common null sets. This unifies previous work of the author [5].

Let $M(X)$ be the space of complex Borel measures on a compact metric space X. If $\sigma \in M(X)$, we let $L^1(\sigma) = \{f \cdot \sigma\colon \int |f|\, d|\sigma| < \infty\}$ and $L(\sigma) = \{\mu \in M(X)\colon \mu \ll \sigma\}$. A set E has measure zero for all $\mu \in L^1(\sigma)$ iff $|\sigma|(E) = 0$. Conversely, the Radon–Nikodym theorem says that $\mu \in L^1(\sigma)$ iff $\mu(E) = 0$ for all E of $|\sigma|$-measure 0, i.e., that $L^1(\sigma) = L(\sigma)$. Now let $\mathcal{T}$ be a topology on $M(X)$ which is weaker than the usual norm topology and let $L^{\mathcal{T}}(\sigma)$ denote the $\mathcal{T}$-closure of $L^1(\sigma)$. Given a class $\mathcal{C} \subset M(X)$, we denote by $\mathcal{C}^{\perp}$ the class of Borel sets $E \subset X$ such that $|\mu|(E) = 0$ for all $\mu \in \mathcal{C}$. Likewise, if $\mathcal{E}$ is a class of Borel sets, $\mathcal{E}^{\perp}$ denotes the measures μ such that $|\mu|(E) = 0$ for all $E \in \mathcal{E}$. Thus, the Radon–Nikodym theorem asserts that $L^1(\sigma)^{\perp\perp} = L^1(\sigma)$. Here we shall investigate the question of whether $L^{\mathcal{T}}(\sigma)^{\perp\perp} = L^{\mathcal{T}}(\sigma)$, of which the Radon–Nikodym theorem is the case where $\mathcal{T}$ is the norm topology.

A prime example is given by the pseudomeasure topology PM on the circle $\boldsymbol{T}$; this is defined by the norm

$$\|\mu\|_{\mathrm{PM}} = \sup_{n \in \boldsymbol{Z}} |\hat{\mu}(n)|.$$

Thus, if λ denotes Lebesgue measure, we see that

$$L^{\mathrm{PM}}(\lambda) = M_0(\boldsymbol{T}) \overset{\text{def}}{=} \{\mu \in M(\boldsymbol{T})\colon \lim_{|n| \to \infty} \hat{\mu}(n) = 0\}.$$

The fact that $M_0(\boldsymbol{T})^{\perp\perp} = M_0(\boldsymbol{T})$ was only proved recently [4]. Another interesting topology is the "Wiener-norm" topology, defined in [5] by

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$$\|\mu\|_{WN} = \sup_{n\geq 0}\left(\frac{1}{2N+1}\sum_{|n|\leq N}|\hat{\mu}(n)|^2\right)^{1/2}.$$

For example, $L^{WN}(\lambda) = M_c(T)$, the class of continuous measures. For this topology and for the weak* topology, we showed in [5] that $L^{\mathcal{T}}(\sigma)^{\perp\perp} = L^{\mathcal{T}}(\sigma)$ for all σ. After finding an example [5] of a norm topology $\mathcal{T}$ for which $L^{\mathcal{T}}(\sigma)^{\perp\perp} \neq L^{\mathcal{T}}(\sigma)$ for $\sigma = \lambda$ or $\sigma \in M_d(T)$ (the class of discrete measures), we felt that it was merely wishful thinking to hope for a general result giving $L^{\mathcal{T}}(\sigma)^{\perp\perp} = L^{\mathcal{T}}(\sigma)$. However, that is precisely what we shall do here. The conditions on $\mathcal{T}$ explain clearly the counterexample that was found in [5]. Furthermore, we obtain immediately that $L^{PM}(\sigma)^{\perp\perp} = L^{PM}(\sigma)$ for all σ, which was the main unanswered question in [5].

Our result depends on sufficient conditions recently found by A. Louveau [3, Chap. IX] that ensure that $\mathcal{C}^{\perp\perp} = \mathcal{C}$ for a general class $\mathcal{C}$. Recall that $\mathcal{C}$ is said to be a *band* if $\nu \ll \mu \in \mathcal{C} \Rightarrow \nu \in \mathcal{C}$. Since every class of the form $\mathcal{E}^{\perp}$ is evidently a band, we may as well assume $\mathcal{C}$ to be a band. With this assumption, we may in fact restrict ourselves to subprobability measures, i.e., measures μ such that $\mu \geq 0$ and $\|\mu\|_{M(X)} \leq 1$. That is, if $X^{\#}$ denotes the space of subprobability measures, then for bands $\mathcal{C}$,

$$\mathcal{C} = \mathcal{C}^{\perp\perp} \Leftrightarrow \mathcal{C} \cap X^{\#} = (\mathcal{C} \cap X^{\#})^{\perp\perp} \cap X^{\#}.$$

We shall therefore abuse notation and let $\mathcal{E}^{\perp}$ be understood as a subclass of $X^{\#}$ when discussing measures in $X^{\#}$. Similarly, we shall call a class $\mathcal{C} \subset X^{\#}$ a band if $\mu \in \mathcal{C} \Rightarrow L(\mu) \cap X^{\#} \subset \mathcal{C}$. For convenience, we write $L^{\mathcal{T}}_{\#}(\sigma) = L^{\mathcal{T}}(\sigma) \cap X^{\#}$.

Let $M(X)$ have the weak* topology. If Λ is a (weak*) Borel probability measure *on* $M(X)$ with compact support, then $\nu \in M(X)$ is said to be its *barycenter* if for all $f \in \mathcal{C}(X)$,

$$\langle f, \nu\rangle = \int_{M(X)} \langle f, \mu\rangle \, d\Lambda(\mu), \tag{1}$$

where

$$\langle f, \mu\rangle \overset{\text{def}}{=} \int_X f\, d\mu.$$

It is clear that every Λ has exactly one barycenter. A class $\mathcal{C}$ is called *measure convex* if it contains the barycenter of every Λ carried by $\mathcal{C}$. Now if (1) holds for all $f \in C(X)$, then (1) holds also for every $f \in \mathcal{B}(X)$, the class of bounded Borel-measurable functions on X, since the smallest class containing $C(X)$ and closed under bounded pointwise limits is $\mathcal{B}(X)$. Thus it is evident that every class of the form $\mathcal{E}^{\perp}$ is measure convex. It is also evident that $\mathcal{E}^{\perp}$ is norm-closed. It is remarkable that these conditions are almost sufficient as well.

THEOREM 1 [3, Chap. IX]. *Let $\mathcal{C} \subset X^{\#}$ be a norm-closed measure convex band. If $\mathcal{C}$ is weak* analytic, then $\mathcal{C} = \mathcal{C}^{\perp\perp}$.*

We shall give a short proof of this based on work of G. Mokobodzki.

Proof. Let $\mu \in \mathcal{C}^{\perp\perp}$. Because $\mathcal{C}$ is a norm-closed convex band, we may decompose μ as $\mu = \mu_1 + \mu_2$, with $\mu_1 \in \mathcal{C}$ and $\mu_2 \perp \mathcal{C}$ (let $\mu_1 = \mu|_E$, where E is a Borel set such that $\mu(E) = \sup\{\mu(F)\colon F \text{ Borel}, \mu|_F \in \mathcal{C}\}$). Because $\mathcal{C}$ is analytic and measure convex, there is [2, p. 191, Remark 37] a Borel set $E \subset X$ such that E carries every $\nu \in \mathcal{C}$ and $\mu_2(E) = 0$. Thus $E^c \in \mathcal{C}^{\perp}$, whence $\mu(E^c) = 0$, and so $\mu_2(E^c) = 0$. Therefore $\mu_2 = 0$ and $\mu \in \mathcal{C}$. ∎

It remains to be seen under what conditions $L^{\mathcal{T}}_{\#}(\sigma)$ satisfies the hypotheses of Theorem 1. As we have supposed $\mathcal{T}$ to be weaker than the usual norm topology, it is automatic that $L^{\mathcal{T}}(\sigma)$ is a norm-closed band. All the topologies considered in [5] have the following form: $\mathcal{T}$ is a linear topology with base at zero consisting of the sets

$$\{\mu \in M(X)\colon \forall f \in F\ f(\mu) < \varepsilon\} \quad (F \in \mathcal{F},\ \varepsilon > 0), \tag{2}$$

where $\mathcal{F}$ is a collection of sets F and each $f \in F$ is a nonnegative function on $M(X)$. For example, if $\mathcal{T}$ is the PM topology on T, we may take $\mathcal{F}$ to consist of the single set $F = \{\mu \mapsto |\hat{\mu}(n)|\colon n \in \mathbf{Z}\}$; if $\mathcal{T}$ is the weak* topology, we may take

$$\mathcal{F} = \{F\colon \exists n \geq 1\ \exists f_1, \ldots, f_n \in C(X)\ F = \{\mu \mapsto |\textstyle\int f_i\, d\mu|\colon 1 \leq i \leq n\}\}.$$

We shall say that $f\colon M(X) \to \mathbf{R}$ is *measure convex* if f is weak* universally measurable and if whenever ν is a barycenter of Λ, we have

$$f(\nu) \leq \int_{M(X)} f(\mu)\, d\Lambda(\mu).$$

LEMMA 2. *If every $F \in \mathcal{F}$ is equicontinuous in the norm topology, each $f \in F$ is measure convex, each basic open set (2) is weak* analytic, $\sigma \in M(X)$, and $L^{\mathcal{T}}_{\#}(\sigma)$ is weak* analytic, then $L^{\mathcal{T}}_{\#}(\sigma)$ is measure convex and $L^{\mathcal{T}}(\sigma) = L^{\mathcal{T}}(\sigma)^{\perp\perp}$.*

Proof. Since F is equicontinuous, every set (2) contains a (norm) ball about the origin; therefore $\mathcal{T}$ is weaker than the norm topology. Let ν be a barycenter of any (Borel) measure Λ carried by $L^{\mathcal{T}}_{\#}(\sigma)$. In order to show that $\nu \in L^{\mathcal{T}}_{\#}(\sigma)$, we must find, for each U as in (2), a measure $\omega \in L^1(\sigma)$ such that $\omega - \nu \in U$. Now $(\varrho, \mu) \mapsto \varrho - \mu$ is continuous as a map $L^1_{\#}(\sigma) \times X^{\#} \to M(X)$ (where $M(X)$ has the weak* topology, which $L^1_{\#}(\sigma)$ and $X^{\#}$ inherit as well). Since U is (weak*) analytic, it follows [1, p. 43, Theorem 11] that $\{(\varrho, \mu) \in L^1_{\#}(\sigma) \times X^{\#}\colon \varrho - \mu \in U\}$ is an analytic subset of $L^1_{\#}(\sigma) \times X^{\#}$. Hence [1, p. 160] there is a selection map $h\colon X^{\#} \to L^1_{\#}(\sigma)$, measurable from the σ-algebra generated by the analytic subsets of $X^{\#}$ to the Borel subsets of

$L^1_{\#}(\sigma)$, such that $h(\mu)-\mu\in U$ for all $\mu\in X^{\#}$ for which there is a measure $\varrho\in L^1_{\#}(\sigma)$ with $\varrho-\mu\in U$ —in particular, for all $\mu\in L^{\mathcal{T}}_{\#}(\sigma)$. Thus, h is universally measurable, so that we may define

$$\omega=\int_{X^{\#}} h(\mu)\,d\Lambda(\mu).$$

Since $L^1_{\#}(\sigma)$ is measure convex, we have $\omega\in L^1_{\#}(\sigma)$. Now for $f\in F$, we have, by measure convexity,

$$f(\omega-\nu)=f\Big(\int_{X^{\#}}[h(\mu)-\mu]\,d\Lambda(\mu)\Big)\leqslant\int_{X^{\#}} f\big(h(\mu)-\mu\big)\,d\Lambda(\mu)<\int_{X^{\#}}\varepsilon\,d\Lambda(\mu)=\varepsilon,$$

where we have used the definition of U. Since this is true for all $f\in F$, it follows that $\omega-\nu\in U$, as desired.

The last part of the lemma follows from Theorem 1. ∎

We say that a nonnegative function f is *coanalytic* if the set $\{f<a\}$ is analytic for $a\geqslant 0$ [1, p. 74].

THEOREM 3. *Let $\mathcal{T}$ be a linear topology with base at zero given by sets (2), where $\mathcal{F}$ is countable, each $F\in\mathcal{F}$ is countable and equicontinuous in the norm topology, and each $f\in F$ is weak* coanalytic and measure convex. Then for all $\sigma\in M(X)$, $L^{\mathcal{T}}(\sigma)=L^{\mathcal{T}}(\sigma)^{\perp\perp}$.*

Proof. It is clear that each basic open set (2) is weak* analytic [1, p. 42, Theorem 8]. If S denotes any countable norm-dense subset of $L^1(\sigma)$, then

$$L^{\mathcal{T}}(\sigma)=\bigcap_{n\geqslant 1}\ \bigcap_{F\in\mathcal{F}}\ \bigcup_{\omega\in S}\ \bigcap_{f\in F}\{\mu\in M(X)\colon f(\mu-\omega)<1/n\}.$$

Therefore $L^{\mathcal{T}}(\sigma)$ is also weak* analytic. Lemma 2 completes the proof. ∎

This theorem applies to the usual norm topology by taking $\mathcal{F}=\{F\}$, $F=\{\mu\mapsto|\int f\,d\mu|\colon f\in S\}$, where S is a countable dense subset of the unit ball of $C(X)$; to the weak* topology by a similar artifice; and to the PM and WN topologies on $\boldsymbol{T}$ in the obvious ways. This explains the positive results of [5]. Furthermore, it is easily seen that the example given in [5] of a topology $\mathcal{T}$ with $L^{\mathcal{T}}(\sigma)\neq L^{\mathcal{T}}(\sigma)^{\perp\perp}$ works because $L^{\mathcal{T}}(\sigma)$ is not measure convex; indeed, while the basic open sets are of the form (2), the elements $f\in F$ are not measure convex.

Two defects of this approach are the following: the results in [5] were obtained by first identifying $L^{\mathcal{T}}(\sigma)$ explicitly. It would still be interesting to identify $L^{\mathrm{PM}}(\sigma)$, for example, or even $L^{\mathrm{PM}}(\sigma)^{\perp}$ (cf. [4]). Furthermore, it is not clear how to extend these methods to (locally) compact (abelian) groups which are not Polish spaces, whereas we know, for example, that $L^{\mathrm{PM}}(\lambda)=L^{\mathrm{PM}}(\lambda)^{\perp\perp}$ for λ the Haar measure on any compact abelian group [4].

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