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BMO estimates for biharmonic multiple layer potentials

by

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Abstract. In this paper BMO estimates are obtained for the trace of the biharmonic multiple layer potential on C^1 domains in the plane. The methods extend to singular integral operators acting on compatible triples which satisfy the appropriate L^2 boundedness, reproducing properties and kernel estimates. In particular, the estimates extend to homogeneous fourth order real constant coefficient elliptic partial differential equations in the plane.

Agmon introduced multiple layer potentials in [1] in order to study elliptic boundary value problems. These potentials are of interest not only because of their widespread applicability (see Agmon [1]) but also because they give integral representations for solutions of elliptic p.d.e.'s. From the analyst's point of view the explicit form of the kernels of these integrals permits the extension of boundary estimates to minimally smooth domains (see Fabes, Jodeit and Rivière [9], Verchota [12], and Cohen and Gosselin [3]) via the singular integral estimates of Calderón [2] and Coifman, McIntosh and Meyer [6]. Multiple layer potentials also provide a nice way to characterize harmonic functions with BMO data (see Fabes and Kenig [8], p. 11) and suggest a way to construct a Hardy space theory connected with adjoint boundary value problems (see [8], [4] and [5]).

In this paper we obtain BMO estimates for the gradient of the biharmonic multiple layer potential on the boundary of a C^1 domain. A norm $\|\cdot\|_*$, analogous to the BMO norm of John and Nirenberg, is defined in § 2.1 on triples of L^2 boundary data. We introduce a space BMO2 consisting of those triples $\dot{f} = (f, g, h)$ for which $\|\dot{f}\|_* < \infty$ and the compatibility condition $f_s = gx_s + hy_s$ holds. (As usual, $(x(s), y(s))$ is an arc length parametrization of the boundary and the subscript s denotes differentiation with respect to arc length). Although it is easily seen that for $\dot{f} \in \mathrm{BMO2}$, $\|\dot{f}\|_* \approx \|g\|_{\mathrm{BMO}} + \|h\|_{\mathrm{BMO}}$, it is convenient to work with the space BMO2 since the L^2 estimates for the trace of the biharmonic multiple layer potential are given in terms of maps from triples to triples (see § 3 of [3] for details).

This paper's main theorem is a biharmonic analog of a result of Fabes and Kenig (Theorem 1.1 of [9]) in which they showed that the trace of the classical double layer potential preserved the space BMO. Using the atomic Hardy space theory of Coifman and Weiss [7] and h^1-BMO duality, they constructed an H^1 space for a C^1 domain from harmonic functions whose normal derivatives are in atomic h^1. This paper is the first step in developing an analogous H^1 theory connected with the solutions of the adjoint boundary value problems studied by Cohen and Gosselin in [4] and [5].

In Theorem (2.1.6) we show that the singular integral $\dot{\mathcal{K}}$, which characterizes the trace of the multiple layer potential and its gradient along the boundary of a C^1 domain in the plane, is bounded from BMO2 to itself. Our proof follows the same general outline as the theorem of Fabes and Kenig [8] but is considerably more complicated. In particular, the elements of BMO2 are compatible triples rather than scalar-valued functions. The BMO2 norm, as defined in § 2.1, is obtained by taking the best fit $\dot{\omega}(\dot{f})$ for a triple $\dot{f}$ in the space of polynomials of degree at most one and their gradients in an arc γ, averaging the difference $|\dot{f}-\dot{\omega}(\dot{f})|$ along γ and taking the supremum over all arcs γ. A pointwise estimate is obtained for the difference between the best fit for a triple $\dot{f}$ over an arc and the best fit for $\dot{f}$ over the double of the arc in terms of a sharp function of $\dot{f}$. A Poincaré type estimate is obtained bounding the $L^2(\gamma)$ norm of $f-\omega(\dot{f})$ in terms of the $L^2\times L^2\times L^2(\gamma)$ norm of $\dot{f}-\dot{\omega}(\dot{f})$ where $\dot{\omega}(\dot{f})$ is the best fit for $\dot{f}$ on the arc γ and $f-\omega(\dot{f})$ is the first component of the triple $\dot{f}-\dot{\omega}(\dot{f})$.

One technical aspect of our work is worth noting here. To analyze the behaviour of the matrix kernel τ of the operator $\dot{\mathcal{K}}$ far from the singularity, we develop in § 2.3 a kind of matrix-valued Taylor series approximation for τ which differs from it by what we call the "remainder matrix". This somewhat technical section has the advantage of suggesting ways to handle estimates of this type for any multiple layer potential.

It is important to note that the estimates obtained in the main theorem make no use of the fact that the components of the matrix kernel are biharmonic. The proof shows that an integral operator defined on a space of compatible triples and satisfying appropriate L^2 boundedness, reproducing properties and kernel estimates extends to a bounded operator in the BMO metric defined in this paper. This fact is of sufficient interest that we include it as a separate result (Theorem (2.1.7)).

Integral operators on spaces of compatible triples are not artificial. Multiple layer potentials as defined by Agmon [1] are defined on exactly this type of compatibility space and a study of solutions of elliptic equations via such potentials involves the type of singular integral estimates obtained in this paper.

In particular, Theorem (2.1.7) immediately applies to all constant real coefficient elliptic equations of homogeneous fourth order on C^2 domains in $\boldsymbol{R}^2$. The general fourth order results extend to C^1 domains if L^2 boundedness can be shown for the appropriate multiple layer potentials. The methods of the theorem seem adaptable to obtaining BMO estimates for the $(m-1)$th derivatives of the solutions of elliptic equations of order $2m$.

Finally, I would like to acknowledge the contributions of John Gosselin with whom I have studied the properties of biharmonic layer potentials, and Garth Baker and Henry Simpson, for several useful suggestions about error estimates for best fit approximations for compatible triples.

§ 1. **Preliminaries.** Before stating the main results of this paper we review some of the relevant definitions, notations, and estimates for biharmonic potentials on C^1 domains. Throughout the paper we assume that Ω is a bounded simply connected C^1 domain in $\boldsymbol{R}^2$. By C^1 we will mean that for any $P\in\partial\Omega$, the boundary of Ω, there exists a ball $B(P,\delta)$ of positive radius δ, centered at P, and a coordinate system (z,w) of $\boldsymbol{R}^2$ with origin at P such that with respect to this coordinate system, $\Omega\cap B(P,\delta) = \{(z,w)\colon z\in\boldsymbol{R},\ w>\varphi(x)\}\cap B(P,\delta)$ where $\varphi\in C_0^1(\boldsymbol{R})$ and $\varphi'(0)=\varphi(0)=0$.

We let $H=(L^2\times L^2\times L^2)(\partial\Omega)$ where we assume that the components of elements in H are real-valued. We define the inner product of two elements $\vec{f}_1,\vec{f}_2\in H$ by $(\vec{f}_1,\vec{f}_2)=\int_{\partial\Omega}\vec{f}_1\vec{f}_2^t\,ds$ where $\vec{f}_2^t$ denotes the column vector which is the transpose of the row vector $\vec{f}_2$ and the product $\vec{f}_1\vec{f}_2^t$ is the scalar function obtained by matrix multiplication. H is clearly a Hilbert space with norm arising from the inner product.

We next define an important subspace.

Definition (1.1.) $\mathcal{B}_2=\{\dot{f}=(f,g,h)\colon \dot{f}\in(L_1^2\times L^2\times L^2)(\partial\Omega)$ and $f_s=gx_s+hy_s$ a.e.$\}$. (Here $(x(s),y(s))$ is an arc length parametrization of $\partial\Omega$, the subscript s in f_s, x_s, and y_s denotes differentiation with respect to arc length and $f\in L_1^2(\partial\Omega)$ means that $\int_{\partial\Omega}(|f|^2+|f_s|^2)\,ds<\infty$). It is important to note that because of the compatibility condition the metrics $\{\int_{\partial\Omega}(|f|^2+|g|^2+|h|^2)\,ds\}^{1/2}$ and $\{\int(|f|^2+|f_s|^2+|g|^2+|h|^2)\,ds\}^{1/2}$ are equivalent for elements in $\mathcal{B}_2$. Generally, the arrow notation, $\vec{f}$, will denote elements of H whereas the dot notation, $\dot{f}$, will denote compatible triples. The symbol $\|\dot{f}\|$ or $\|\vec{f}\|$ will denote the norm arising from the inner product in H.

For $X=(x,y)$, a point in $\boldsymbol{R}^2$ with Cartesian coordinates (x,y), the function

$$F(X)=\frac{-1}{4\pi}\{(x^2+y^2)\log(x^2+y^2)^{1/2}+y^2\}$$

is the choice of fundamental solution for the biharmonic operator used by Agmon in [1].

Definition (1.2.). For the point $Q\in\partial\Omega$ fixed and $X\in\boldsymbol{R}^2$ we define the differential operator $\vec{K}$ by

$$\vec{K}v=(K_1v,\ K_2v,\ K_3v) \tag{1.3}$$

where

$$K_1 v(X) = \langle \vec{\nabla}\Delta v(X),\ \vec{N}_Q\rangle + 2\langle \vec{\nabla} v_{xy}(X),\ \vec{T}_Q\rangle,$$

$$K_2 v(X) = \big(v_{xx}(X) - v_{yy}(X)\big)\, y_s(Q),$$

$$K_3 v(X) = \big(v_{xx}(X) - v_{yy}(X)\big)\, y_s(Q) + 4 v_{xy}(X)\, y_s(Q).$$

The vectors $\vec{N}_Q$ and $\vec{T}_Q$ are the unit inner normal and unit tangent at Q respectively.

Definition (1.4). For $X \notin \partial\Omega$, the *multiple layer potential* with density $\dot{f} \in \mathscr{B}_2$ is defined by

$$u(\dot{f};\, X) = \int_{\partial\Omega} \dot{f}(Q)\, \vec{K}^Q F(X-Q)^t\, ds(Q) \tag{1.5}$$

where the superscript Q indicates that the differential operator $\vec{K}$ is acting at the point Q.

Definition (1.6). For $X \notin \partial\Omega$, we let $\dot{u}(X) = (u(X),\, u_x(X),\, u_y(X))$ and note that since we may interchange the order of integration and differentiation in computing u_x and u_y we get the integral representation

$$\dot{u}(\dot{f};\, X) = \int_{\partial\Omega} \dot{f}(Q)\, \tau(X, Q)\, ds(Q) \tag{1.7}$$

where $\tau(X, Q)$ is the 3×3 matrix given by

$$\tau(X, Q) = \begin{bmatrix} K_1^Q F(X-Q) & K_1^Q \partial_x^X F(X-Q) & K_1^Q \partial_y^X F(X-Q) \\ K_2^Q F(X-Q) & K_2^Q \partial_x^X F(X-Q) & K_2^Q \partial_y^X F(X-Q) \\ K_3^Q F(X-Q) & K_3^Q \partial_x^X F(X-Q) & K_3^Q \partial_y^X F(X-Q) \end{bmatrix}. \tag{1.8}$$

The superscripts denote the variables on which the differential operator is acting.

Definition (1.9). For $X = (x, y)$ and S a subset of $\boldsymbol{R}^2$ we define the space of polynomials $\mathscr{I}(S) = \{w\colon\ w(X) = \alpha x + \beta x + \gamma(x^2+y^2) + \delta\}$. We next define the space of triples $\dot{\mathscr{I}}(S) = \{\dot{w} = (w,\, w_x,\, w_y)\colon\ w \in \mathscr{I}(S)\}$. We will often use this notation for $\dot{\mathscr{I}}(\partial\Omega)$ and if it is clear from the context we will simply denote the space by $\dot{\mathscr{I}}$.

The trace of $\dot{u}$ on $\partial\Omega$ cannot be obtained at a point $P \in \partial\Omega$ by substituting P for X in (1.7). However, for $P \neq Q$, the matrix $\tau(P, Q)$ can be defined as in (1.8) and we may define

$$\mathscr{K}_\varepsilon \dot{f}(P) = \int_{|P-Q|>\varepsilon} \dot{f}(Q)\, \tau(P, Q)\, ds(Q). \tag{1.9}$$

The main results of Cohen and Gosselin in [3] and [4] which we will need here can be summarized as follows:

Theorem (1.10). (i) *$\mathscr{K}\dot{f}(P) = \lim_{\varepsilon\to 0} \mathscr{K}_\varepsilon \dot{f}(P)$ exists a.e. and $\mathscr{K}$ is bounded and compact from $\mathscr{B}_2$ to itself.*

(ii) *$\mathscr{K}\dot{w} = \dot{w}$ for all $\dot{w} \in \dot{\mathscr{I}}$.*

(iii) *The nontangential $\lim_{X\to P\in\partial\Omega} \dot{u}(\dot{f};\, X)$ equals*

$$\begin{cases} (\dot{I} + \mathscr{K})\dot{f}(P) \ \text{a.e.}, & X \in \Omega, \\ (-\dot{I} + \mathscr{K})\dot{f}(P) \ \text{a.e.}, & X \notin \bar{\Omega}. \end{cases}$$

(iv) *$(\dot{I} + \mathscr{K})^{-1}$ exists on $\mathscr{B}_2$, $(-\dot{I} + \mathscr{K})^{-1}$ exists on $(-\dot{I} + \mathscr{K})\mathscr{B}_2$.*

Corollary (1.11). *The interior Dirichlet problem $\Delta^2 u(X) = 0$ in Ω, $\dot{u} = \dot{f} \in \mathscr{B}_2$ on $\partial\Omega$, is solvable by $u = u\big((\dot{I}+\mathscr{K})^{-1}\dot{f};\, X\big)$. The exterior Dirichlet problem is solvable by $u = u\big((-\dot{I}+\mathscr{K})^{-1}\dot{f}_0;\, X\big) + w_0$ where $\dot{f} = \dot{f}_0 + \dot{w}_0$, $\dot{f}_0 \in (-\dot{I}+\mathscr{K})\mathscr{B}_2$ and $\dot{w}_0 \in \dot{\mathscr{I}}$.*

§ 2. **The BMO2 space.** In this section we define a space BMO2 which is analogous to the space of bounded mean oscillation introduced by John and Nirenberg [10], and adapted here to the space of compatible triples. For $\dot{f} \in \mathscr{B}_2$ we define a sharp function of $\dot{f}$, a corresponding BMO2 norm and prove that the operator $\mathscr{K}$ defined in Theorem (1.10) is bounded in the BMO2 norm.

§ **2.1. The basic definitions.** We start with some notation. For $P_0 \in \partial\Omega$ we let $B_r(P_0)$ denote the set $\{P \in \partial\Omega\colon\ |P - P_0| < r\}$. When the "center" P_0 is clear from the context we will simply write B_r and $|B_r|$ will denote the arc length of B_r. Because the domain is C^1 and the boundary is compact there are constants c_1 and c_2 such that $c_1 r \leqslant |B_r| \leqslant c_2 r$ and this will hold for all arcs B_r as long as $r \leqslant |\partial\Omega|$.

For $S \subset \boldsymbol{R}^2$ we let $\mathscr{P}_1(S) = \{f(X) = \alpha x + \beta y + \delta\colon\ X = (x, y),\ X \in S\}$ and $\dot{\mathscr{P}}_1(S) = \{\dot{f} = (f, f_x, f_y)\colon\ f \in \mathscr{P}_1(S)\}$. When the set S can be inferred from the context we write these sets as $\mathscr{P}_1$ and $\dot{\mathscr{P}}_1$.

Definition (2.1.1). For $\vec{f} \in H$ we define the *sharp function* of $\vec{f}$,

$$\vec{f}^{\#}(P_0) = \sup_{r>0} \Big\{ \inf_{\dot{w}\in\dot{\mathscr{P}}_1} |B_r|^{-1} \int_{B_r} |\vec{f} - \dot{w}|^2\, ds \Big\}^{1/2} \tag{2.1.2}$$

and the BMO2 *norm*,

$$\|\vec{f}\|_* = \|\vec{f}^{\#}\|_\infty. \tag{2.1.3}$$

Definition (2.1.4). $\mathrm{BMO2} = \{\dot{f} \in \mathscr{B}_2\colon\ \|\dot{f}\|_* < \infty\}$.

Remark (2.1.5). From the definition of the sharp function it is clear that if $\vec{f}_0, \vec{f}_1 \in H$ and $\vec{f}_0 - \vec{f}_1 \in \dot{\mathscr{P}}_1$, then $\|\vec{f}_0 - \vec{f}_1\|_* = 0$. This means that BMO2 is not a Banach space. However, the relation $\vec{f}_0 \sim \vec{f}_1 \Leftrightarrow \vec{f}_0 - \vec{f}_1 \in \dot{\mathscr{P}}_1$ is an equivalence relation. Thus by choosing equivalence classes in BMO2 or introducing the metric $\|\vec{f}\|_* + \|\vec{f}\|$ we can turn BMO2 into a Banach space. For the purposes of this paper that is unnecessary.

We can now state our main theorem:

THEOREM (2.1.6). *For $\mathscr{K}$ defined as in (1.10)(i), there is a constant c, depending only on the shape of the domain, such that* $\|\mathscr{K}\dot{f}\|_* \leqslant c\|\dot{f}\|_*$.

The proof requires several lemmas concerning approximation of elements in $\mathscr{B}_2$ by elements in $\dot{\mathscr{P}}_1$ along arcs B_r. While the machinery developed for the proof is complicated it is useful because it is immediately applicable to the following more general theorem.

THEOREM (2.1.7). *Let $k(P, Q)$ be a 3×3 matrix of functions defined on $\partial\Omega\times\partial\Omega$ for $P\neq Q$ and assume*

$$\mathscr{K}\dot{f}(P) = \lim_{\varepsilon\to 0}\int_{|P-Q|>\varepsilon}\dot{f}(Q)\,k(P,Q)\,ds(Q)$$

exists for almost every $P\in\partial\Omega$. Then if

(2.1.8) $$\|\mathscr{K}\dot{f}\|_{(L^2\times L^2\times L^2)(\partial\Omega)} \leqslant c\,\|\dot{f}\|_{(L^2\times L^2\times L^2)(\partial\Omega)},$$

(2.1.9) $\mathscr{K}\dot{\omega}(P) = \dot{\omega}(P)$ *for every $\dot{\omega}\in\dot{\mathscr{P}}_1$ and for almost every $P\in\partial\Omega$,*

(2.1.10) *the components k_{ij} of k satisfy the pointwise estimates*

$$|D^2 k_{11}(P, Q)| \leqslant c/|P-Q|^3,$$
$$|D^2 k_{i1}(P, Q)| \leqslant c/|P-Q|^2, \quad i = 2, 3,$$
$$|Dk_{1j}(P, Q)| \leqslant c/|P-Q|^3, \quad j = 2, 3,$$
$$|Dk_{ij}(P, Q)| \leqslant c/|P-Q|^2, \quad i, j = 2, 3,$$

where D denotes one derivative taken with respect to the variable P and D^2 denotes a second derivative with respect to the variable P, then

(2.1.11) $$\|\mathscr{K}\dot{f}\|_* \leqslant c\,\|\dot{f}\|_*.$$

§ 2.2. The best fit polynomial. In this section we consider the restriction of elements in $\mathscr{B}_2$ to arcs $B_r(P_0)$. We define a triple $\dot{\omega}_r(\dot{f})$ which is the best fit for $\dot{f}$ in the space $\dot{\mathscr{P}}_1$ along the arc B_r. The best fit $\dot{\omega}_r(\dot{f})$ will play the role in the BMO2 theory that the average value of a function on an interval plays in the BMO theory introduced by John and Nirenberg [10].

Let $B_r(P_0) = \{P\in\partial\Omega\colon |P-P_0|<r\}$. H_r will denote the Hilbert space $(L^2\times L^2\times L^2)(B_r)$ with norm arising from the inner product

$$(\vec{f}_1, \vec{f}_2)_r = \int_{B_r}(f_1 f_2 + g_1 g_2 + h_1 h_2)\,ds$$

where $\vec{f}_1 = (f_1, g_1, h_1)$, $\vec{f}_2 = (f_2, g_2, h_2)$ and $\vec{f}_1, \vec{f}_2\in H_r$. We let $\mathscr{B}_2(r)$ denote the subspace of H_r consisting of those triples $(f, g, h)\in H_r$ which satisfy the compatibility condition $f_s = gx_s + hy_s$ almost everywhere.

Recall that $\dot{\mathscr{P}}_1(B_r)$ is the restriction of the triples $\dot{f} = (f, f_x, f_y)$ to the set B_r where $f(x, y) = \alpha x + \beta y + \delta$. We will need an orthonormal basis for $\dot{\mathscr{P}}_1(B_r)$. To find such a basis we start with the polynomials $d_1^r(P) = 1$, $d_2^r(P) = x(P) - x(P_0)$ and $d_3^r(P) = y(P) - y(P_0)$. We form the following corresponding triples:

(2.2.1) $$\begin{aligned}\dot{d}_1^r(P) &= (1, 0, 0),\\ \dot{d}_2^r(P) &= (x(P)-x(P_0), 1, 0),\\ \dot{d}_3^r(P) &= (y(P)-y(P_0), 0, 1).\end{aligned}$$

Clearly the set $\{\dot{d}_1^r, \dot{d}_2^r, \dot{d}_3^r\}$ forms a basis for $\dot{\mathscr{P}}_2(B_r)$. Applying the Gram–Schmidt process with the inner product in H_r we arrive at an orthonormal basis $\{\dot{e}_1^r, \dot{e}_2^r, \dot{e}_3^r\}$. If e_j^r, $j = 1, 2, 3$, denotes the first component of $\dot{e}_j^r$, then $(e_j^r)_x$ and $(e_j^r)_y$, the partial derivatives of e_j^r, are the second and third components.

DEFINITION (2.2.2). For $\vec{f}\in H$, the *best fit* for $\vec{f}$ in $\dot{\mathscr{P}}_1(B_r)$ is defined to be

(2.2.3) $$\dot{\omega}_r = \dot{\omega}_r(\vec{f}; P) = \sum_{j=1}^{3}(\vec{f}, \dot{e}_j^r)_r\,\dot{e}_j^r(P).$$

We let ω_r denote the first component of $\dot{\omega}_r$. Clearly ω_r is a polynomial and $\dot{\omega}_r = (\omega_r, (\omega_r)_x, (\omega_r)_y)$. Furthermore, the polynomial ω_r extends in an obvious way to all of $\boldsymbol{R}^2$. When we have occasion to refer to the polynomial ω_r its domain will be inferred from the context.

We begin by establishing an identity relating the best fit $\dot{\omega}_r$ and its "double", $\dot{\omega}_{2r}$.

LEMMA (2.2.4). *For* $\vec{f}\in H$,

$$\omega_r(\vec{f}) - \omega_{2r}(\vec{f}) = \omega_r(\vec{f} - \dot{\omega}_{2r}(\vec{f})), \quad \dot{\omega}_r(\vec{f}) - \dot{\omega}_{2r}(\vec{f}) = \dot{\omega}_r(\vec{f} - \dot{\omega}_{2r}(\vec{f})).$$

Proof. Expand the $\dot{e}_j^{2r}$'s in terms of the $\dot{e}_j^r$'s and combine the coefficients of the $\dot{e}_j^r$'s.

LEMMA (2.2.5). *Let B_r denote $B_r(P_0)$ and $\{\dot{e}_1^r, \dot{e}_2^r, \dot{e}_3^r\}$ the orthonormal basis for $\dot{\mathscr{P}}_1(B_r)$ obtained from the $\dot{d}_j^r$'s via the Gram–Schmidt process. Then for $j = 1, 2, 3$, the polynomials e_j^r, $(e_j^r)_x$ and $(e_j^r)_y$ satisfy:*

(2.2.6) $$|e_1^r(P)| \leqslant C/r^{1/2}, \quad |e_j^r(P)| \leqslant C\{r + |P-P_0|\}/r^{1/2} \quad \text{for } j = 2, 3,$$

(2.2.7) $$\begin{aligned}|(e_j^r)_x(P)| &\leqslant C/r^{1/2} \quad \text{for } j = 2, 3, \quad (e_1^r)_x = 0,\\ |(e_j^r)_y(P)| &\leqslant C/r^{1/2} \quad \text{for } j = 2, 3, \quad (e_1^r)_y = 0.\end{aligned}$$

Proof. Adapt the Gram–Schmidt procedure used on p. 120 of Taibleson and Weiss [11] to the sets B_r via the use of local coordinates. The calculations are tedious but elementary.

COROLLARY (2.2.8). *For* $\dot f \in \mathscr{B}_2$, $|(\dot f, \dot e_j^r)_r \dot e_j^r| \leqslant c|B_r|^{-1}\int_{B_r}|\dot f|\,ds$ *where the constant* c *is independent of* r, P_0, *and* $\dot f$.

Proof. Using the pointwise estimates from Lemma (2.2.5) we have

$$(2.2.9)\qquad |(\dot f, \dot e_j^r)_r \dot e_j^r(P)| \leqslant c\big(r^{-1/2}\int |\dot f|\,ds\big)|\dot e_j^r(P)| \leqslant c|B_r|^{-1}\int_{B_r}|\dot f|\,ds.$$

COROLLARY (2.2.10). *For* $\dot f \in \mathscr{B}_2$, $P_0 \in \partial\Omega$, $B_r = B_r(P_0)$ *and* $\dot\omega_r$, $\dot\omega_{2r}$ *the best fits for* $\dot f$ *over the arcs* B_r *and* B_{2r} *respectively,*

$$|\dot\omega_r(\dot f) - \dot\omega_{2r}(\dot f)| \leqslant c\dot f^{\#}(P_0).$$

Proof. From Lemma (2.2.4), the projection estimates and Schwarz' inequality,

$$(2.2.11)\qquad |\dot\omega_r(\dot f) - \dot\omega_{2r}(\dot f)|^2 = |\dot\omega_r(\dot f - \dot\omega_{2r}(\dot f))|^2 \leqslant c|B_r|^{-1}\int_{B_r}|\dot f - \dot\omega_{2r}(\dot f)|^2 \leqslant c\dot f^{\#}(P_0)^2.$$

The next lemma is a kind of Poincaré estimate which bounds the L^2 norm of the first component $f-\omega_r(\dot f)$ in terms of the L^2 norms of its x and y derivatives.

LEMMA (2.2.12). *For* $\dot f = (f, g, h) \in \mathscr{B}_2$,

$$\Big\{\int_{B_r}|f-\omega_r(\dot f)|^2\,ds\Big\}^{1/2} \leqslant c|B_r|\Big\{\int_{B_r}\big(|g-(\omega_r(\dot f))_x|^2 + |h-(\omega_r(\dot f))_y|^2\big)\,ds\Big\}^{1/2}.$$

Proof. Since $\dot\omega_r(\dot f)$ is the best fit for $\dot f$ in the space H_r, there is a point $\tilde P \in B_r$ for which $f(\tilde P) - \omega_r(\dot f, \tilde P) = 0$. We may then write

$$(2.2.13)\qquad f(P) - \omega_r(\dot f)(P) = \int_{\tilde P}^{P}\big[\big(g-(\omega_r(\dot f))_x\big)x_s + \big(h-(\omega_r(\dot f))_y\big)y_s\big]\,ds.$$

Squaring both sides, applying Schwarz' inequality and integrating both sides over B_r gives

$$(2.2.14)\qquad \int_{B_r}|f-\omega_r(\dot f)|^2\,ds \leqslant |B_r|^2\int_{B_r}\big(|g-(\omega_r(\dot f))_x|^2 + |h-(\omega_r(\dot f))_y|^2\big)\,ds.$$

Take square roots and obtain the estimate in this lemma.

COROLLARY (2.2.15). *For* $\dot f = (f, g, h) \in \mathscr{B}_2$,

$$\Big\{|B_r|^{-1}\int_{B_r}|f-\omega_r(\dot f)|^2\,ds\Big\}^{1/2} \leqslant |B_r|\dot f^{\#}(P_0).$$

Proof. This estimate follows by taking the square root of both sides of (2.2.14), dividing by $|B_r|^{1/2}$ and applying the definition of $\dot f^{\#}(P_0)$.

COROLLARY (2.2.16). *For* $\dot f \in \mathscr{B}_2$, $P_0 \in \partial\Omega$, $B_r = B_r(P_0)$ *and* $\dot\omega_r$ *and* $\dot\omega_{2r}$ *the best fits for* $\dot f$ *over the arcs* B_r *and* B_{2r} *respectively, the first components* ω_r *and* ω_{2r} *satisfy for all* $P \in \partial\Omega$

$$(2.2.17)\qquad |\omega_r(\dot f)(P) - \omega_{2r}(\dot f)(P)| \leqslant c(|P-P_0|+r)\dot f^{\#}(P_0).$$

Proof. For $j=1$ use $|e_1^r(P)| \leqslant c/r^{1/2}$, $(e_1^r)_x = (e_1^r)_y = 0$, Schwarz' inequality and Corollary (2.215) to obtain

$$|(\dot f - \dot\omega_{2r}(\dot f), \dot e_1^r)_r e_1^r(P)| \leqslant \Big\{\int_{B_{2r}}|f-\omega_{2r}(\dot f)|^2\,ds\Big\}^{1/2}|e_1^r(P)| \leqslant cr\dot f^{\#}(P_0).$$

For $j = 2, 3$, use (2.2.6) and Schwarz' inequality to obtain

$$|(\dot f - \dot\omega_{2r}(\dot f), \dot e_j^r)_r e_j^r(P)| \leqslant c\Big\{\int_{B_{2r}}|\dot f - \dot\omega_{2r}(\dot f)|^2\,ds\Big\}^{1/2}\{r+|P-P_0|\}/r^{1/2} \leqslant c\{r+|P-P_0|\}\dot f^{\#}(P_0).$$

§ 2.3. The osculating polynomial and the remainder matrix. Our goal in Chapter 2 is to show that $\mathscr{K}$ is bounded in the BMO2 norm. In this section we introduce a polynomial which approximates the operator away from the singularity. The approximation will be by a kind of boundary defined Taylor series which Agmon [1] called the osculating polynomial.

In our case we will actually be looking at a certain matrix of polynomials which will be made up of polynomials which resemble Taylor series remainders. To do this we must develop some notation which is a little cumbersome.

DEFINITION (2.3.1). Let $P, Q \in \partial\Omega$ and $\dot f \in \mathscr{B}_2$. The *osculating polynomial of order two* is

$$(2.3.2)\qquad T_2(\dot f; P, Q) = f(Q) + g(Q)\big(x(P)-x(Q)\big) + h(Q)\big(y(P)-y(Q)\big).$$

We define $T_1(\dot f; P, Q)$ to be $f(Q)$ and call it the *osculating polynomial of order one*.

The function $T_2(\dot f; P, Q)$ is a polynomial in the variable $(x(P), y(P))$ so we can extend it to be a polynomial in the entire plane. We can then define an element $\dot T_2 \in \mathscr{B}_2$ by

$$\dot T_2(\dot f; P, Q) = \big(T_2(\dot f; P, Q), \partial_x^P T_2(\dot f; P, Q), \partial_y^P T_2(\dot f; P, Q)\big) = \big(T_2(\dot f; P, Q), T_1(g; P, Q), T_1(h; P, Q)\big).$$

For three elements $\dot f_1, \dot f_2, \dot f_3 \in \mathscr{B}_2$ we can let $\vec f = (f_1, f_2, f_3)$ and define the matrix

$$(2.3.3)\qquad (\vec f^t)^{\cdot} = \begin{bmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{bmatrix}$$

where t denotes the transpose of $\vec f$.

DEFINITION (2.3.4). The *osculating polynomial* for the matrix $(\vec f^t)^{\cdot}$ is defined by

$$(2.3.5)\qquad T_2\big((\vec f^t)^{\cdot}; P, Q\big) = [T_2(\dot f_1; P, Q), T_2(\dot f_2; P, Q), T_2(\dot f_3; P, Q)]^t$$

and then we define a matrix-valued version

$$(2.3.6)\qquad \dot T_2\big((\vec f^t)^{\cdot}; P, Q\big) = \begin{bmatrix} T_2(\dot f_1; P, Q) & T_1(g_1; P, Q) & T_1(h_1; P, Q) \\ T_2(\dot f_2; P, Q) & T_1(g_2; P, Q) & T_1(h_2; P, Q) \\ T_2(\dot f_3; P, Q) & T_1(g_3; P, Q) & T_1(h_3; P, Q) \end{bmatrix}.$$

We now turn our attention to the corresponding remainders.

DEFINITION (2.3.7). For $\dot f \in \mathscr{B}_2$, $P, Q \in \partial\Omega$ we define

$$(2.3.8)\qquad \begin{aligned} R_2(\dot f; P, Q) &= f(P) - T_2(\dot f; P, Q), \\ R_1(f; P, Q) &= f(P) - f(Q). \end{aligned}$$

The corresponding elements in $\mathscr{B}_2$ are given by

$$(2.3.9)\qquad \begin{aligned} \dot R_2(\dot f; P, Q) &= \dot f(P) - \dot T_2(\dot f; P, Q) \\ &= \big(R_2(\dot f; P, Q), R_1(g; P, Q); R_1(h; P, Q)\big). \end{aligned}$$

For the matrix $(\vec f^t)^{\cdot}$ we define

$$(2.3.10)\qquad \dot R_2\big((\vec f^t)^{\cdot}; P, Q\big) = \begin{bmatrix} R_2(\dot f_1; P, Q) & R_1(g_1; P, Q) & R_1(h_1; P, Q) \\ R_2(\dot f_2; P, Q) & R_1(g_2; P, Q) & R_1(h_2; P, Q) \\ R_2(\dot f_3; P, Q) & R_1(g_3; P, Q) & R_1(h_3; P, Q) \end{bmatrix}.$$

We have introduced the osculating polynomials and the various remainders to apply to a particular compatible triple which will arise in the next section. In this section we will obtain the estimates which will be used to show that $\dot{\mathscr{K}}$ of Theorem (1.10) is bounded in the BMO2 norm.

Let $P_0 \in \partial\Omega$ and choose some $r > 0$. Let $\eta(X) \in C_0^\infty(\boldsymbol{R}^2)$ with $\eta(X) = 1$ for $|X - P_0| \leqslant 2r$, $\eta(X) \equiv 0$ for $|X - P_0| \geqslant 4r$ and $0 \leqslant \eta(X) \leqslant 1$, $\forall X \in \boldsymbol{R}_2$.

Let $P, Q \in \partial\Omega$ with $|P_0 - P| \leqslant r$ and $|P_0 - Q| \geqslant 2r$. Then for $\dot f \in \mathscr{B}_2$, the vector-valued integral $\dot{\mathscr{K}}\big((1-\eta)\dot f\big)(P) = \int (1-\eta(Q))\dot f(Q)\tau(P, Q)\,ds(Q)$ is well defined. Since $|P - Q| \geqslant r$, $(1-\eta(Q))\tau(P, Q)$ is C^∞ on $\partial\Omega$, so we can justify an interchange of differentiation and integration yielding

$$(2.3.11)\qquad \begin{aligned} \dot{\mathscr{K}}\big((1-\eta)\dot f(P)\big) &= \int_{\partial\Omega} (1-\eta(Q))\dot f(Q)\tau(P, Q)\,ds(Q) \\ &= (I, \partial_x^P, \partial_y^P)\int_{\partial\Omega} (1-\eta(Q))\dot f(Q)\vec K^Q F(P-Q)^t\,ds(Q). \end{aligned}$$

So obviously $\dot{\mathscr{K}}\big((1-\eta)\dot f\big) \in \mathscr{B}_2$ and we can define the osculating polynomial $T_2\big(\dot{\mathscr{K}}((1-\eta)\dot f); P, P_0\big)$ and the related remainder:

$$(2.3.12)\qquad \begin{aligned} \dot R_2\big(\dot{\mathscr{K}}((1-\eta)\dot f); P, P_0\big) &= \dot{\mathscr{K}}\big((1-\eta)\dot f\big)(P) - \dot T_2\big(\dot{\mathscr{K}}((1-\eta)\dot f); P, P_0\big) \\ &= \int_{\partial\Omega} (1-\eta(Q))\dot f(Q)\varrho(P_0, P, Q)\,ds(Q) \end{aligned}$$

where

$$(2.3.13)\qquad \varrho(P_0, P, Q) = \dot R_2\big((\vec K^Q F(\cdot - Q)^t)^{\cdot}; P, P_0\big) =$$

$$\begin{bmatrix} R_2(K_1^Q F(\cdot - Q); P, P_0) & R_1(K_1^Q F_x(\cdot - Q); P, P_0) & R_1(K_1^Q F_y(\cdot - Q); P, P_0) \\ R_2(K_2^Q F(\cdot - Q); P, P_0) & R_1(K_2^Q F_x(-Q); P, P_0) & R_1(K_2^Q F_y(\cdot - Q); P, P_0) \\ R_2(K_3^Q F(\cdot - Q); P, P_0) & R_1(K_3^Q F_x(\cdot - Q); P, P_0) & R_1(K_3^Q F_y(\cdot - Q); P, P_0) \end{bmatrix}.$$

We will refer to $\varrho(P_0, P, Q)$ as the *remainder matrix* and note that from direct calculation we get the following estimates:

LEMMA (2.3.14). *For $P, Q, P_0 \in \partial\Omega$, $|P - P_0| \leqslant r$ and $|P_0 - Q| \geqslant 2r$, the components of the remainder matrix satisfy*:

$$(2.3.15)\qquad \begin{aligned} &|\varrho_{11}(P_0, P, Q)| \leqslant cr^2/|P_0 - Q|^3, \\ &|\varrho_{i1}(P_0, P, Q)| \leqslant cr^2/|P_0 - Q|^2, \quad i = 2, 3, \\ &|\varrho_{1j}(P_0, P, Q)| \leqslant cr/|P_0 - Q|^3, \quad j = 2, 3, \\ &|\varrho_{ij}(P_0, P, Q)| \leqslant cr/|P_0 - Q|^2, \quad i = 2, 3,\ j = 2, 3. \end{aligned}$$

Proof. The estimates follow from the fact that for the points P_0, P, Q described here $F(\cdot - Q)$ is a smooth function in $\boldsymbol{R}^2$ and the remainders are simply two-dimensional Taylor series remainders restricted to the boundary. The estimates in (2.3.15) follow from computing the appropriate derivatives of $F(\cdot - Q)$.

§ **2.4. The basic estimates for $\dot{\mathscr{K}}$.** In this section we show that $\dot{\mathscr{K}}$ is bounded from BMO2 to itself in the BMO2 norm. The argument follows a strategy similar to the proof of Theorem 1.1 of Fabes and Kenig [8]. However, the norm is more complicated here and the estimates require the use of both the best fit polynomial for $\dot f$ and the remainder operator.

Proof of Theorem (2.1.6). Let $P_0 \in \partial\Omega$ and choose $r > 0$. Choose $\theta \in C_0^\infty(\boldsymbol{R}^2)$ such that $\theta(X) \equiv 1$ for $|X| \leqslant 2$, $\theta(X) = 0$ for $|X| \geqslant 4$ and

$0 \leqslant \theta(X) \leqslant 1$, $\forall X \in \mathbf{R}^2$. Let $\eta(Q) = \theta(P_0 - Q/r)$. Then $\eta(Q) = 1$ for $|P_0 - Q| \leqslant 2r$, $\eta(Q) = 0$ for $|P_0 - Q| \geqslant 4r$, $0 \leqslant \eta(Q) \leqslant 1$ and $|\vec{\nabla}\eta(Q)| \leqslant M/r$ where $M = \max\{|\vec{\nabla}\theta(X)|:\ X \in \mathbf{R}^2\}$.

For $\dot f \in \mathrm{BMO2}$ let $\dot\omega_{4r} = \dot\omega_{4r,P_0}(\dot f)(Q)$ denote the best fit for $\dot f$ in the set $B_{4r}(P_0)$. Let $T = T_2\big(\mathscr{K}((1-\eta)(\dot f - \dot\omega_{4r})); P, P_0\big)$ be the osculating polynomial for $\mathscr{K}((1-\eta)(\dot f - \dot\omega_{4r}))$ based at P_0 and let $\dot T$ denote its corresponding element in $\mathscr{B}_2 \cap \mathscr{P}_1$ obtained by taking the gradient of T with respect to the variable P. We can then write

$$\dot{\mathscr{K}}\dot f(P) - \dot\omega_{4r}(P) - \dot T(P) = \dot{\mathscr{K}}\big((\eta(f-\omega_{4r}))^{\cdot}\big)(P) - \int (f(Q) - \omega_{4r}(Q))(0, \eta_x(Q), \eta_y(Q))\,\tau(P, Q)\,ds(Q) + \int_{\partial\Omega} (1-\eta(Q))(\dot f(Q) - \dot\omega_{4r}(Q))\,\varrho(P_0, P, Q)\,ds(Q) \tag{2.4.1}$$

where $\varrho(P_0, P, Q)$ is the remainder matrix defined in (2.3.13).

To estimate the first term on the right-hand side of (2.4.1) we use the boundedness of $\dot{\mathscr{K}}$ from $\mathscr{B}_2$ to $\mathscr{B}_2$. This yields

$$\begin{aligned}\int_{B_r} \big|\dot{\mathscr{K}}\big((\eta(f-\omega_{4r}))^{\cdot}\big)(P)\big|^2 ds(P) &\leqslant \int_{\partial\Omega} \big|\dot{\mathscr{K}}\big((\eta(f-\omega_{4r}))^{\cdot}\big)(P)\big|^2 ds(P)\\ &\leqslant C\int_{\partial\Omega} |(\eta(f-\omega_{4r}))^{\cdot}(P)|^2 ds(P) \leqslant C\int_{\partial\Omega} |\eta(P)|^2\,|\dot f(P) - \dot\omega_{4r}(P)|^2 ds(P)\\ &\quad + C\int_{\partial\Omega} |\vec{\nabla}\eta(P)|^2\,|f(P) - \omega_{4r}(P)|^2 ds(P)\\ &\leqslant Cr\,\Big\{|B_r|^{-1}\int_{B_{4r}} |\dot f(P) - \dot\omega_{4r}(P)|^2 ds(P)\Big\}\\ &\quad + CrM^2 \dot f^{\#}(P_0)^2 \leqslant Cr\dot f^{\#}(P_0)^2.\end{aligned} \tag{2.4.3}$$

To estimate the second term in (2.4.1) we employ Lemma (2.2.12) and straightforward estimates for the components of the matrix $\tau(P, Q)$ defined in (1.8). We get

$$\begin{aligned}\Big|\int_{\partial\Omega} (f(Q) - \omega_{4r}(Q))(0, \eta_x(Q), \eta_y(Q))\,\tau(P, Q)\,ds(Q)\Big| &\leqslant C\frac{M}{r}\int_{B_{4r}} |f(Q) - \omega_{4r}(Q)|\\ &\qquad \times(|\tau_{21}| + |\tau_{31}| + |\tau_{22}| + |\tau_{23}| + |\tau_{32}| + |\tau_{33}|)(P, Q)\,ds(Q)\\ &\leqslant C\frac{M}{r}\Big\{r^{-1}\int_{B_{4r}} |f(Q) - \omega_{4r}(Q)|^2 ds(Q)\Big\}^{1/2}\\ &\leqslant C\frac{M}{r}\{[r\dot f^{\#}(P_0)]^2\}^{1/2} = CM\dot f^{\#}(P_0).\end{aligned} \tag{2.4.4}$$

For the third integral in (2.4.1) we begin by noting that it suffices to estimate the integrals

$$\mathrm{I} = \int_{\partial\Omega} (1-\eta(Q))(f(Q) - \omega_{4r}(Q))\,\varrho_{11}(P_0, P, Q)\,ds(Q),$$

$$\mathrm{II} = \int_{\partial\Omega} (1-\eta(Q))(f(Q) - \omega_{4r}(Q))\,\varrho_{12}(P_0, P, Q)\,ds(Q).$$

The remaining terms are similar and if anything easier.

To simplify the notation we let $B_j = B_{2^j r}(P_0)$, $\omega_j = \omega_{2^j r}(\dot f)$ and let N denote the largest integer for which $B_{j+1}\setminus B_j$ is nonempty. Since $1 - \eta(Q) = 0$ for $Q \in B_r$ we may estimate

$$\begin{aligned}|\mathrm{I}| &\leqslant \sum_{j=1}^{N}\int_{B_{j+1}\setminus B_j} |f(Q) - \omega_2(Q)|\,|\varrho_{11}(P_0, P, Q)|\,ds(Q)\\ &\leqslant \sum_{j=1}^{N}\int_{B_{j+1}\setminus B_j} \Big\{|f(Q) - \omega_{j+1}(Q)| + \sum_{i=2}^{j+1} |\omega_{i+1}(Q) - \omega_i(Q)|\Big\}\\ &\qquad \times|\varrho_{11}(P_0, P, Q)|\,ds(Q).\end{aligned} \tag{2.4.5}$$

To estimate the "$f - \omega_{j+1}$" term we use the estimates for the remainder matrix in (2.3.15) to get

$$\begin{aligned}&\int_{B_{j+1}\setminus B_j} |f(Q) - \omega_{j+1}(Q)|\,|\varrho_{11}(P_0, P, Q)|\,ds(Q)\\ &\quad\leqslant C\int_{B_{j+1}/B_j} |f(Q) - \omega_{j+1}(Q)|\,r^2\,|P_0 - Q|^{-3}\,ds(Q)\\ &\quad\leqslant Cr^2(2^j r)^{-3}\int_{B_{j+1}} |f(Q) - \omega_{j+1}(Q)|\,ds(Q) \leqslant c2^{-2j}\dot f^{\#}(P_0).\end{aligned} \tag{2.4.6}$$

To estimate the "$\sum_{i=2}^{j+1}$" term we once again use the remainder estimates in (2.3.15) along with Corollary (2.2.10) to get

$$\begin{aligned}&\sum_{i=1}^{j+1}\int_{B_{j+1}\setminus B_j} |\omega_{i+1}(Q) - \omega_i(Q)|\,|\varrho_{11}(P_0, P, Q)|\,ds(Q)\\ &\quad\leqslant \sum_{i=2}^{j+1} c\dot f^{\#}(P_0)\int_{B_{j+1}\setminus B_j} (2^i r + |P - Q|)\,r^2\,|P_0 - Q|^{-3}\,ds(Q)\\ &\quad\leqslant cj2^{-j}\dot f^{\#}(P_0).\end{aligned} \tag{2.4.7}$$

Summing in j we get the estimate

$$|\mathrm{I}| \leqslant c\dot f^{\#}(P_0). \tag{2.4.8}$$

To estimate II we begin as in estimating I and we are led first to estimate the integral

$$(2.4.9)\quad \int_{B_{j+1}\setminus B_j} |f(Q)-\omega_{j+1}(Q)|\,|\varrho_{12}(P_0, P, Q)|\,ds(Q)$$
$$\leqslant c \int_{B_{j+1}\setminus B_j} |f(Q)-\omega_{j+1}(Q)|\,r\,|P_0-Q|^{-3}\,ds(Q)$$
$$\leqslant c2^{-3j} r^{-2} \int_{B_{j+1}} |f(Q)-\omega_{j+1}(Q)|\,ds(Q)$$
$$\leqslant c2^{-2j} r^{-1} \{|B_{j+1}|^{-1} \int_{B_{j+1}} |f(Q)-\omega_{j+1}(Q)|^2\,ds(Q)\}^{1/2}$$
$$\leqslant c2^{-j} \dot{f}^{\#}(P_0) \quad \text{by Corollary (2.2.15).}$$

As in the estimate for I we must estimate the integral

$$(2.4.10)\quad \int_{B_{j+1}\setminus B_j} |\omega_{i+1}(Q)-\omega_i(Q)|\,|\varrho_{12}(P_0, P, Q)|\,ds(Q)$$
$$\leqslant c\dot{f}^{\#}(P_0) \int_{B_{j+1}\setminus B_j} (2^i r+|P_0-Q|)\,r\,|P_0-Q|^{-3}\,ds(Q)$$
$$\leqslant c\dot{f}^{\#}(P_0)(2^{i-2j}+2^{-j}).$$

Summing in i we get

$$(2.4.11)\quad \sum_{i=2}^{j+1} \int_{B_{j+1}\setminus B_j} |\omega_{i+1}(Q)-\omega_i(Q)|\,|\varrho_{12}(P_0, P, Q)|\,ds(Q)$$
$$\leqslant cj2^{-j}\dot{f}^{\#}(P_0).$$

Combining estimates (2.4.9) with (2.4.11) and summing in j we get

$$(2.4.12)\quad |\mathrm{II}| \leqslant c\dot{f}^{\#}(P_0) \sum_{j=1}^{\infty} j2^{-j} \leqslant c\dot{f}^{\#}(P_0).$$

Hence we get the BMO2 estimate:

$$(2.4.13)\quad \{|B_r|^{-1} \int_{B_r} |\mathrm{I}|^2\,ds\}^{1/2} + \{|B_r|^{-1} \int_{B_r} |\mathrm{II}|^2\,ds\}^{1/2} \leqslant c\dot{f}^{\#}(P_0).$$

The proof of (2.1.7) now proceeds in a manner similar to the proof of Theorem (2.1.6). One can still define the polynomial $T = T_2\big(\mathscr{K}((1-\eta)(\dot{f}-\dot{\omega}_{4r})); P, P_0\big)$ and let $\dot{T}$ denote the corresponding element in $\mathscr{B}_2 \cap \dot{\mathscr{P}}_1$. This replaces the matrix kernel ϱ on the right-hand side of (2.4.1) by a matrix of Taylor series remainders for the functions $k_{ij}(\cdot, Q)$. In particular, ϱ_{11} is replaced by $R_2\big(k_{11}(\cdot, Q); P, P_0\big)$ and ϱ_{12} is replaced by $R_1\big(k_{12}(\cdot, Q); P, P_0\big)$. The pointwise estimates for $k_{ij}(P, Q)$ have been chosen to guarantee that the replacements for the ϱ_{ij} will satisfy pointwise estimates sufficient to allow the proof of (2.1.6) to go through for this case.

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