$$(2) \qquad \bigcap_{j=0}^{n} f^{-j}(B_1) \times \bigvee_{j=0}^{n} \tau^{-j}(\alpha)$$

of the partition $\bigvee_{j=0}^{n} T^{-j}\beta_1$. The set $\bigcap_{j=0}^{n} f^{-j}(B_1)$ is nonempty and open. Hence $f^k(x) \in \bigcap_{j=0}^{n} f^{-j}(B_1)$ for some k by the density of the orbit of x. The points $(f^k(x), \tau^k(y))$ and $(f^k(x), \tau^k(y_1))$ belong to different atoms of the set (2) because $d(\tau^k(y), \tau^k(y_1)) > \varepsilon > \delta_n$. Hence (x, y) and (x, y_1) belong to different elements of the partition

$$T^{-k}\Big(\bigvee_{j=0}^{n} T^{-j}\beta_1\Big) \leqslant \bigvee_{j=0}^{n+k} T^{-j}\beta_1 < \beta_1^-.$$

This is a contradiction and hence $y = y_1$, which finishes the proof.

COROLLARY. *The partition* $\beta = \{B \times B, B \times B^c, B^c \times S\}$ *where* $B = \{e^{2\pi i\alpha}: 0 < \alpha < \frac{1}{2}\}$ *is a generator for the transformation* $T(z, w) = (z^2, aw)$ *of the torus* S^2 *where* a *is not a root of unity.*

Remark. The inequalities in Theorem 1 establish the best estimate of the minimal cardinality of a generator.

The example given in the corollary belongs to the following class of transformations: Let T be the 1-sided $(1/k, \ldots, 1/k)$ Bernoulli shift and let τ be any distal automorphism. Then the generator β_1 constructed for $T \times \tau$ has $k+1$ elements and the other generators have at least $k+1$ elements.

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INSTYTUT MATEMATYKI POLITECHNIKI WROCŁAWSKIEJ
INSTITUTE OF MATHEMATICS, WROCŁAW TECHNICAL UNIVERSITY
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

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On the strong Cesàro summability of double orthogonal series

by

I. SZALAY* (Szeged)

Abstract. In a recent paper [3], Móricz gave a coefficient test for the strong summability of double orthogonal series in the case of the parameters α and β greater than 1/2 and the index λ equal to 2. Using the definition of convergence in Pringsheim's sense with a bound, the present author extends the definition of strong summability to the case of λ positive and α and β nonnegative. The case $\alpha = \beta = 0$ is the so-called strong convergence. This note contains coefficient conditions for nine cases of parameters and indices.

1. Introduction. First of all we mention that for a double sequence $\{\omega_{m,n}\}_{m,n=0}^{\infty}$ the "little o"

$$\omega_{m,n} = o(1) \quad \text{as } \min(m, n) \to \infty$$

$$(\text{or } \max(m, n) \to \infty, \text{ or } m \to \infty, \text{ or } n \to \infty)$$

means that $\omega_{m,n} \to 0$ as $\min(m, n) \to \infty$ (or $\max(m, n) \to \infty$, or $m \to \infty$, or $n \to \infty$) and in addition there exists a constant K such that $|\omega_{m,n}| \leqslant K$ for $m, n = 0, 1, \ldots$ The case

$$\omega_{m,n} = o(1) \quad \text{as } \min(m, n) \to \infty$$

may be called *convergence in Pringsheim's sense with a bound*. Our next definitions are understood in this sense.

We say that a series

$$(1) \qquad \sum_{i,k=0} c_{i,k}$$

is *Cesàro summable with parameters* $\alpha, \beta > -1$ — *or* $(C, (\alpha, \beta))$ *summable* — to s if

$$\sigma_{m,n}^{(\alpha,\beta)} - s = o(1) \quad \text{as } \min(m, n) \to \infty,$$

and it is said to be *strongly Cesàro summable with parameters* $\alpha, \beta > 0$ *and index* $\lambda > 0$ — *or* $[C, (\alpha, \beta)]_\lambda$ *summable* — to s if

* This work was done while the author was visiting the Steklov Mathematical Institute in Moscow, U.S.S.R.

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}|\sigma_{i,k}^{(\alpha-1,\beta-1)}-s|^{\lambda}=o(1) \quad \text{as } \min(m,n)\to\infty,$$

where

$$\sigma_{m,n}^{(\alpha,\beta)}=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha)}A_{n-k}^{(\beta)}c_{i,k} \tag{2}$$

is the (m, n)th rectangular $(C, (\alpha, \beta))$ mean of the partial sums

$$s_{m,n}=\sum_{i=0}^{m}\sum_{k=0}^{n}c_{i,k}$$

with the Cesàro numbers

$$A_0^{(\alpha)}=1,\qquad A_n^{(\alpha)}=\frac{(1+\alpha)(2+\alpha)\dots(n+\alpha)}{n!},\qquad n=1,2,\dots$$

(see e.g. [6], Vol. 1, pp. 76–77). Obviously, $s_{m,n}=\sigma_{m,n}^{(0,0)}$, so $(C, (0, 0))$ summability means the convergence of the series (1) in Pringsheim's sense with a bound.

In the limit cases, when α or β or both are zero, we say that the series (1) is $[C, (0, \beta)]_\lambda$, $\beta > 0$, $[C, (\alpha, 0)]_\lambda$, $\alpha > 0$, and $[C, (0, 0)]_\lambda$ *summable* to s if

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}\left|\Delta_{11}\left((i+1)(k+1)\sigma_{i,k}^{(0,\beta)}\right)-s\right|^{\lambda}=o(1),$$

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}\left|\Delta_{11}\left((i+1)(k+1)\sigma_{i,k}^{(\alpha,0)}\right)-s\right|^{\lambda}=o(1),$$

and

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}\left|\Delta_{11}\left((i+1)(k+1)s_{i,k}\right)-s\right|^{\lambda}=o(1)$$

respectively as $\min(m, n)\to\infty$, where $\Delta_{11}\omega_{i,k}=\omega_{i,k}-\omega_{i-1,k}-\omega_{i,k-1}+\omega_{i-1,k-1}$ for i, $k = 0, 1, 2, \dots$, with $\omega_{i,k}=0$ if i or k or both are -1.

We use the following notation:

$$z_{m,n}^{(\alpha,\beta)}=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha-1)}A_{n-k}^{(\beta)}ic_{i,k}, \tag{3}$$

$$t_{m,n}^{(\alpha,\beta)}=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha)}A_{n-k}^{(\beta-1)}kc_{i,k}, \tag{4}$$

$$\tau_{m,n}^{(\alpha,\beta)}=\frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha-1)}A_{n-k}^{(\beta-1)}ikc_{i,k}, \tag{5}$$

for the series (1), where the last expression is the (m, n)th rectangular $(C, (\alpha, \beta))$ mean of the sequence $\{mnc_{m,n}\}_{m,n=0}^{\infty}$.

In [4] and [5] we proved the following theorems.

THEOREM A. *If the series* (1) *is* $[C, (\alpha, \beta)]_\lambda$ *summable to* s, $\alpha, \beta \geqslant 0$, $\lambda > 0$, *then it is also* $[C, (\alpha, \beta)]_\mu$ *summable to* s *for every* μ *such that* $0 < \mu < \lambda$.

THEOREM B. *Let* $\lambda > 1$, $\alpha, \beta > 1/\lambda$ *and* $\delta, \gamma < 1-1/\lambda$. *If the series* (1) *is* $[C, (\alpha, \beta)]_\lambda$ *summable to* s *then it is also* $(C, (\alpha-\delta, \beta-\gamma))$ *summable to* s.

THEOREM C. *If the series* (1) *is* $[C, (\alpha, \beta)]_1$ *summable to* s, $\alpha, \beta \geqslant 0$, *then it is also* $(C, (\alpha, \beta))$ *summable to* s.

THEOREM D. *Let* $\alpha, \beta \geqslant 0$ *and* $\lambda \geqslant 1$. *A necessary and sufficient condition for the series* (1) *to be* $[C, (\alpha, \beta)]_\lambda$ *summable is that it be* $(C, (\alpha, \beta))$ *summable and that*

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}|z_{i,k}^{(\alpha,\beta)}|^{\lambda}=o(1), \tag{6}$$

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}|t_{i,k}^{(\alpha,\beta)}|^{\lambda}=o(1), \tag{7}$$

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}=o(1) \tag{8}$$

as $\min(m, n)\to\infty$.

THEOREM E. *Let* $\alpha, \beta \geqslant 0$ *and* $\lambda \geqslant 1$. *The series* (1) *is* $[C, (\alpha, \beta)]_\lambda$ *summable to* s *if and only if*

$$\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{k=0}^{n}\left|\Delta_{11}\left((i+1)(k+1)\sigma_{i,k}^{(\alpha,\beta)}\right)-s\right|^{\lambda}=o(1)$$

as $\min(m, n)\to\infty$.

THEOREM F. *Let* $\alpha, \beta, \gamma, \delta \geqslant 0$ *and* $\lambda \geqslant 1$. *If the series* (1) *is* $[C, (\alpha, \beta)]_\lambda$ *summable to* s *then it is also* $[C, (\alpha+\delta, \beta+\gamma)]_\lambda$ *summable to* s.

THEOREM G. *If the series* (1) *is* $[C, (\alpha, \beta)]_\lambda$ *summable to* s, $\alpha, \beta \geqslant 0$ *and* $\lambda \geqslant 1$, *then it is also* $[C, (\alpha+\delta, \beta+\gamma)]_\mu$ *summable to* s *for* $\gamma, \delta \geqslant 0$ *and* $0 < \mu \leqslant \lambda$.

THEOREM H. *Suppose that either*

(i) $\alpha, \beta \geqslant 0$, $\mu > \lambda > 1$ *and* $\delta, \gamma \geqslant 1/\lambda - 1/\mu$, *or*

(ii) $\alpha, \beta > 0$, $\mu > \lambda = 1$ *and* $\delta, \gamma > 1-1/\mu$.

Then if the series (1) *is* $[C, (\alpha, \beta)]_\lambda$ *summable to* s, *it is* $[C, (\alpha+\delta, \beta+\gamma)]_\mu$ *summable to* s.

Turning to the double orthogonal series, let $(X, \mathscr{F}, \varrho)$ be a given arbitrary positive measure space and $\{\varphi_{i,k}(x)\}_{i,k=0}^{\infty}$ an orthonormal system on X. We consider the double orthogonal series

$$\sum_{i,k=0}^{\infty} a_{i,k}\,\varphi_{i,k}(x). \tag{9}$$

For the series (9) we use $\sigma_{m,n}^{(\alpha,\beta)}(x)$, $z_{m,n}^{(\alpha,\beta)}(x)$, $t_{m,n}^{(\alpha,\beta)}(x)$ and $\tau_{m,n}^{(\alpha,\beta)}(x)$ defined in (2)–(5) with $c_{i,k} = a_{i,k}\,\varphi_{i,k}(x)$. For a double sequence $\{f_{m,n}(x)\}_{m,n=0}^{\infty}$ of functions in $L^2 = L^2(X, \mathscr{F}, \varrho)$, the symbol

$$f_{m,n}(x) = o_x(1) \quad \text{a.e. as } \min(m, n) \to \infty,$$
$$(\text{or } \max(m, n) \to \infty, \text{ or } m \to \infty, \text{ or } n \to \infty)$$

means that $f_{m,n}(x) \to 0$ as $\min(m, n) \to \infty$ (or $\max(m, n) \to \infty$, or $m \to \infty$, or $n \to \infty$) and in addition there exists a function $F \in L^2$ such that $\sup_{m,n\geq 0}|f_{m,n}(x)| \leqslant F(x)$ a.e. on X.

By the well-known Riesz–Fischer theorem, if

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 < \infty$$

then there exists a function $f \in L^2$ such that the rectangular partial sums

$$s_{m,n}(x) = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{i,k}\,\varphi_{i,k}(x)$$

of the series (9) converge to f in the L^2-norm, i.e.

$$\int (s_{m,n}(x) - f(x))^2\, d\varrho(x) \to 0 \quad \text{as } \min(m, n) \to \infty.$$

Here and in the sequel, the integrals are taken over the entire space X. We call f the *L^2-sum* of the series (9).

The following coefficient tests for $(C, (\alpha, \beta))$ summability of double orthogonal series are known.

THEOREM I. *If*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)]^2 [\log(k+2)]^2 < \infty \tag{10}$$

then the series (9) *is a.e.* $(C, (0, 0))$ *summable to its L^2-sum.* (See [1], Theorem 8.1.)

THEOREM J. *Suppose that either*

(i) $\alpha > 0$, $\beta = 0$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log\log(i+4)]^2 [\log(k+2)]^2 < \infty, \tag{11}$$

or

(ii) $\alpha = 0$, $\beta > 0$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)]^2 [\log\log(k+4)]^2 < \infty, \tag{12}$$

or

(iii) $\alpha > 0$, $\beta > 0$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log\log(i+4)]^2 [\log\log(k+4)]^2 < \infty. \tag{13}$$

Then the series (9) *is a.e.* $(C, (\alpha, \beta))$ *summable to its L^2-sum.* (See [3], Theorems 1 and 2.)

For $[C, (\alpha, \beta)]_2$ summability, $\alpha, \beta > \frac{1}{2}$, Móricz ([3], Theorem 6) proved

THEOREM K. *If $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ and the condition* (13) *is satisfied then the series* (9) *is a.e.* $[C, (\alpha, \beta)]_2$ *summable to its L^2-sum.*

Remark. Taking $\lambda = 2$, by Theorem B, we can easily deduce case (iii) of Theorem J from Theorem K.

The aim of this note to prove the following

THEOREM. *Let $0 < \nu \leqslant \mu$, $\mu \geqslant 2$, and let one of the following conditions be satisfied:*

(i) $\alpha, \beta > 1 - 1/\mu$ *and* (13) *holds.*

(ii) $\alpha = 1 - 1/\mu$, $\beta > 1 - 1/\mu$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)] [\log\log(k+4)]^2 < \infty. \tag{14}$$

(iii) $\frac{1}{2} - 1/\mu \leqslant \alpha < 1 - 1/\mu$, $\beta > 1 - 1/\mu$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{2(1-1/\mu-\alpha)} [\log\log(k+4)]^2 < \infty. \tag{15}$$

(iv) $\alpha = \beta = 1 - 1/\mu$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)] [\log(k+2)] < \infty. \tag{16}$$

(v) $\frac{1}{2} - 1/\mu \leqslant \alpha < 1 - 1/\mu$, $\beta = 1 - 1/\mu$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{2(1-1/\mu-\alpha)} [\log(k+2)] < \infty. \tag{17}$$

(vi) $\frac{1}{2} - 1/\mu \leqslant \alpha < 1 - 1/\mu$, $\frac{1}{2} - 1/\mu \leqslant \beta < 1 - 1/\mu$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{2(1-1/\mu-\alpha)} (k+1)^{2(1-1/\mu-\beta)} < \infty. \tag{18}$$

(vii) $\alpha > 1-1/\mu$, $\beta = 1-1/\mu$ and

(19) $$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log\log(i+4)]^2 [\log(k+2)] < \infty .$$

(viii) $\alpha > 1-1/\mu$, $\frac{1}{2}-1/\mu \leqslant \beta < 1-1/\mu$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log\log(i+4)]^2 (k+1)^{2(1-1/\mu-\beta)} < \infty .$$

(ix) $\alpha = 1-1/\mu$, $\frac{1}{2}-1/\mu \leqslant \beta < 1-1/\mu$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)] (k+1)^{2(1-1/\mu-\beta)} < \infty .$$

Then (9) *is a.e.* $[C, (\alpha, \beta)]_\nu$ *summable to its* L^2*-sum.*

Remark. For $\mu = 2$ in case (i) we recover Theorem K.

COROLLARY. *Let* $\mu > 0$. *Under the condition* (13),

$$\frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} |s_{i,k}(x) - f(x)|^\mu = o_x(1) \quad \text{a.e. as } \min(m, n) \to \infty,$$

where f *is the* L^2*-sum of the series* (9).

Proof. If $\mu \geqslant 2$ then for $\alpha = \beta = 1$ we immediately get the corollary from case (i) of the Theorem. If $0 < \mu < 2$ then the $[C, (1, 1)]_2$ summability is obtained by case (i) of the Theorem. Furthermore, taking $\lambda = 2$, we may apply Theorem A to get the a.e. $[C, (1, 1)]_\mu$ summability of the series (9) to its L^2-sum.

2. Auxiliary results. Using the method of Móricz, we prove the following two lemmas.

LEMMA 1. *In the cases* (i) $\alpha > \frac{1}{2}$, (ii) $\alpha = \frac{1}{2}$, (iii) $-1 < \alpha < \frac{1}{2}$, *the conditions*

(20) $$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(k+2)]^2 < \infty,$$

(21) $$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)] [\log(k+2)]^2 < \infty,$$

and

(22) $$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{1-2\alpha} [\log(k+2)]^2 < \infty$$

respectively are sufficient in order that

(23) $$2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2 = o_x(1) \quad \text{a.e. as } p \to \infty,$$

uniformly in n.

Proof. First, assume that $n = 2^q$, $q = 0, 1, \ldots$ By (3) and Cauchy's inequality we get

$$2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,2^q}^{(\alpha,0)}(x)|^2$$
$$= 2^{-p} \sum_{m=2^p+1}^{2^{p+1}} \Big|(A_m^{(\alpha)})^{-1} \sum_{i=0}^{m} \sum_{k=0}^{2^q} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|^2$$
$$= 2^{-p} \sum_{m=2^p+1}^{2^{p+1}} \Big| \sum_{t=-2}^{q-1} \Big((t+3)$$
$$\times (A_m^{(\alpha)})^{-1} \sum_{i=0}^{m} \sum_{k=2^t+1}^{2^{t+1}} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big)(t+3)^{-1}\Big|^2$$
$$\leqslant 2^{-p} \sum_{m=2^p+1}^{2^{p+1}} \Bigg(\sum_{t=-2}^{q-1} \left(\frac{t+3}{A_m^{(\alpha)}}\right)^2 \Big| \sum_{i=0}^{m} \sum_{k=2^t+1}^{2^{t+1}} A_{m-i}^{(\alpha-1)}$$
$$\times i a_{i,k} \varphi_{i,k}(x)\Big|^2 \Bigg) \sum_{t=-2}^{q-1} (t+3)^{-2}$$
$$\leqslant \tfrac{1}{6}\pi^2 \cdot 2^{-p} \sum_{t=-2}^{\infty} (t+3)^2 \sum_{m=2^p+1}^{2^{p+1}} (A_m^{(\alpha)})^{-2}$$
$$\times \Big| \sum_{i=0}^{m} \sum_{k=2^t+1}^{2^{t+1}} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|^2,$$

where the convention

(24) $$2^t = \begin{cases} -1 & \text{for } t = -2 \\ 0 & \text{for } t = -1 \end{cases}$$

has been used.

Let us introduce the function

$$F^2(x) = \sum_{p=0}^{\infty} \sum_{t=-2}^{\infty} \frac{(t+3)^2}{2^p} \sum_{m=2^p+1}^{2^{p+1}} (A_m^{(\alpha)})^{-2} \Big| \sum_{i=0}^{m} \sum_{k=2^t+1}^{2^{t+1}} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|^2$$

and integrate it over X. Then we can write

$$\int F^2(x)\, d\varrho(x)$$
$$= \sum_{p=0}^{\infty} \sum_{t=-2}^{\infty} \frac{(t+3)^2}{2^p} \sum_{m=2^p+1}^{2^{p+1}} (A_m^{(\alpha)})^{-2} \sum_{i=0}^{m} \sum_{k=2^t+1}^{2^{t+1}} (A_{m-i}^{(\alpha-1)})^2 i^2 a_{i,k}^2$$
$$\leqslant K_\alpha \sum_{p=0}^{\infty} \sum_{t=-2}^{\infty} \sum_{k=2^t+1}^{2^{t+1}} [\log(8k+2)]^2$$
$$\times 2^{-p} \sum_{m=2^p+1}^{2^{p+1}} (m+1)^{-2\alpha} \sum_{i=0}^{m} (m-i+1)^{2\alpha-2} i^2 a_{i,k}^2$$

$$\leqslant K_\alpha \sum_{k=0}^{\infty} [\log(8k+2)]^2 \sum_{m=0}^{\infty} (m+1)^{-2\alpha-1} \sum_{i=0}^{m} (m-i+1)^{2\alpha-2} i^2 a_{i,k}^2$$

$$= K_\alpha \sum_{k=0}^{\infty} [\log(8k+2)]^2 \sum_{i=0}^{\infty} i^2 a_{i,k}^2 \sum_{m=i}^{\infty} (m-i+1)^{2\alpha-2}(m+1)^{-2\alpha-1}.$$

A usual computation shows that

$$\leqslant \begin{cases} K_\alpha (i+1)^{-2} & \text{if } \alpha > \frac{1}{2}, \\ K_\alpha (i+1)^{-2}\log(i+1) & \text{if } \alpha = \frac{1}{2}, \\ K_\alpha (i+1)^{-2\alpha-1} & \text{if } -1 < \alpha < \frac{1}{2}. \end{cases}$$

Hence by (20)–(22) Levi's theorem yields $F \in L^2$ in each case, which proves that

$$(26) \qquad 2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,2^q}^{(\alpha,0)}(x)|^2 = o_x(1) \quad \text{a.e. as } p \to \infty,$$

uniformly in q.

Now, consider $2^q < n \leqslant 2^{q+1}$, $q = 0, 1, \ldots$ Using (3) again, by Minkowski's inequality we obtain

$$\Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2\Big\}^{1/2}$$

$$\leqslant \Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} \Big|(A_m^{(\alpha)})^{-1} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|^2\Big\}^{1/2}$$

$$\leqslant \Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,2^q}^{(\alpha,0)}(x)|^2\Big\}^{1/2}$$

$$+ \Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} \Big|(A_m^{(\alpha)})^{-1} \sum_{i=0}^{m} \sum_{k=2^q+1}^{n} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|^2\Big\}^{1/2}.$$

Setting

$$M_{p,q}^{(\alpha,0)}(x) = \Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} (A_m^{(\alpha)})^{-2} \times \max_{2^q < n \leqslant 2^{q+1}} \Big|\sum_{i=0}^{m} \sum_{k=2^q+1}^{n} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|^2\Big\}^{1/2},$$

we get

$$(27) \qquad \max_{2^q < n \leqslant 2^{q+1}} \Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2\Big\}^{1/2} \leqslant \Big\{2^{-p} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,2^q}^{(\alpha,0)}(x)|^2\Big\}^{1/2} + M_{p,q}^{(\alpha,0)}(x).$$

Applying the Men'shov–Rademacher lemma (see e.g. [2], p. 79) gives

$$\int\Big(\max_{2^q < n \leqslant 2^{q+1}} \Big|\sum_{i=0}^{m} \sum_{k=2^q+1}^{n} A_{m-i}^{(\alpha-1)} i a_{i,k} \varphi_{i,k}(x)\Big|\Big)^2 d\varrho(x) \leqslant K(\log^2 2^{q+1}) \sum_{i=0}^{m} \sum_{k=2^q+1}^{2^{q+1}} (A_{m-i}^{(\alpha-1)})^2 i^2 a_{i,k}^2,$$

and so we have

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \int |M_{p,q}^{(\alpha,0)}(x)|^2 d\varrho(x) \leqslant K \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(q+1)^2}{2^p} \sum_{m=2^p+1}^{2^{p+1}} (A_m^{(\alpha)})^{-2} \sum_{i=0}^{m} \sum_{k=2^q+1}^{2^{q+1}} (A_{m-i}^{(\alpha-1)})^2 i^2 a_{i,k}^2.$$

Hence repeating the previous arguments we see that for each case of the parameter α, the series

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \int |M_{p,q}^{(\alpha,0)}(x)|^2 d\varrho(x)$$

converges. By Levi's theorem,

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |M_{p,q}^{(\alpha,0)}(x)|^2$$

is convergent almost everywhere and its sum is in L^2. This yields that

$$(28) \qquad M_{p,q}^{(\alpha,0)}(x) = o_x(1) \quad \text{a.e. as } \max(p, q) \to \infty.$$

Combining (26), (27) and (28) completes the proof.

LEMMA 2. *Suppose that $\beta > 0$ and that either*

(i) $\alpha > \frac{1}{2}$ *and*

$$(29) \qquad \sum_{i,k=0}^{\infty} a_{i,k}^2 [\log\log(k+4)]^2 < \infty,$$

or

(ii) $\alpha = \frac{1}{2}$ *and* (14) *holds,*

or

(iii) $-1 < \alpha < \frac{1}{2}$ *and*

$$(30) \qquad \sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{1-2\alpha} [\log\log(k+4)]^2 < \infty.$$

Then

$$(31) \qquad 2^{-p-q} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^q+1}^{2^{q+1}} |z_{m,n}^{(\alpha,\beta)}(x)|^2 = o_x(1) \quad \text{a.e. as } p \to \infty,$$

uniformly in q.

Proof. By (3) and Minkowski's inequality we get

$$\Big\{2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\alpha,\beta)}(x)|^2\Big\}^{1/2}$$
$$=\Big\{2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\Big|(A_m^{(\alpha)}A_n^{(\beta)})^{-1}\times\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{(\alpha-1)}A_{n-k}^{(\beta)}\,ia_{i,k}\,\varphi_{i,k}(x)\Big|^2\Big\}^{1/2}$$
$$\leqslant\Big\{2^{-p}\sum_{m=2^p+1}^{2^{p+1}}\Big|\sum_{i=0}^{m}\sum_{k=0}^{2^q}\frac{A_{m-i}^{(\alpha-1)}}{A_m^{(\alpha)}}\,ia_{i,k}\,\varphi_{i,k}(x)\Big|^2\Big\}^{1/2}$$
$$+\Big\{2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\Big|\sum_{i=0}^{m}\sum_{k=2^q+1}^{n}\frac{A_{m-i}^{(\alpha-1)}}{A_m^{(\alpha)}}\,ia_{i,k}\,\varphi_{i,k}(x)\Big|^2\Big\}^{1/2}$$
$$+\Big\{2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\Big|\sum_{i=0}^{m}\sum_{k=0}^{n}\frac{A_{m-i}^{(\alpha-1)}}{A_m^{(\alpha)}}\Big(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}}-1\Big)ia_{i,k}\,\varphi_{i,k}(x)\Big|^2\Big\}^{1/2}$$
$$\equiv T_{p,q}^{(1)}(x)+T_{p,q}^{(2)}(x)+T_{p,q}^{(3)}(x).$$

First, we estimate $T_{p,q}^{(1)}(x)$. Using the convention (24) in the summation, consider the sequence

$$a_{i,t}^{*}=\Big\{\sum_{k=2^t+1}^{2^{t+1}}a_{i,k}^2\Big\}^{1/2}\qquad (i=0,1,2,\ldots,\ t=-2,-1,0,1,\ldots)$$

and the system

$$\varphi_{i,t}^{*}(x)=\begin{cases}(a_{i,t}^{*})^{-1}\displaystyle\sum_{k=2^t+1}^{2^{t+1}}a_{i,k}\,\varphi_{i,k}(x) & \text{if } a_{i,t}^{*}\neq 0,\\ \varphi_{i,2^t+1}(x) & \text{if } a_{i,t}^{*}=0,\end{cases}$$

and observe that

$$(T_{p,q}^{(1)}(x))^2=2^{-p}\sum_{m=2^p+1}^{2^{p+1}}\Big|(A_m^{(\alpha)})^{-1}\sum_{i=0}^{m}\sum_{t=-2}^{q-1}A_{m-i}^{(\alpha-1)}\,ia_{i,t}^{*}\,\varphi_{i,t}^{*}(x)\Big|^2.$$

Then (29), (14) and (30) imply (20)–(22) for $a_{i,t}^{*}$.

Now we may apply Lemma 1 to the orthogonal series

$$\sum_{i=0}^{\infty}\sum_{t=-2}^{\infty}a_{i,t}^{*}\,\varphi_{i,t}^{*}(x).$$

By (3), putting $n=q-1$ in (23) gives

$$T_{p,q}^{(1)}(x)=o_x(1)\quad\text{a.e. as } p\to\infty, \tag{32}$$

uniformly in q.

Next, we estimate $T_{p,q}^{(2)}(x)$:

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int|T_{p,q}^{(2)}(x)|^2\,d\varrho(x)$$
$$\leqslant K_\alpha\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\sum_{i=0}^{m}\sum_{k=2^q+1}^{n}(m-i+1)^{2\alpha-2}\times(m+1)^{-2\alpha}i^2a_{i,k}^2$$
$$\leqslant K_\alpha\sum_{m=2}^{\infty}\sum_{q=0}^{\infty}\sum_{i=0}^{m}\sum_{k=2^q+1}^{2^{q+1}}(m-i+1)^{2\alpha-2}(m+1)^{-2\alpha-1}i^2a_{i,k}^2$$
$$\leqslant K_\alpha\sum_{k=2}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{m}(m-i+1)^{2\alpha-2}(m+1)^{-2\alpha-1}i^2a_{i,k}^2$$
$$=K_\alpha\sum_{k=2}^{\infty}\sum_{i=0}^{\infty}i^2a_{i,k}^2\sum_{m=i}^{\infty}(m-i+1)^{2\alpha-2}(m+1)^{-2\alpha-1}.$$

Now (25) shows that

$$\sum_{p,q=0}^{\infty}\int|T_{p,q}^{(2)}(x)|^2\,d\varrho(x)\leqslant\begin{cases}K_\alpha\displaystyle\sum_{i,k=0}^{\infty}a_{i,k}^2 & \text{if } \alpha>\tfrac12,\\ K_\alpha\displaystyle\sum_{i,k=0}^{\infty}a_{i,k}^2[\log(i+2)] & \text{if } \alpha=\tfrac12,\\ K_\alpha\displaystyle\sum_{i,k=0}^{\infty}a_{i,k}^2(i+1)^{1-2\alpha} & \text{if } -1<\alpha<\tfrac12.\end{cases}$$

Hence, by (29), (14) and (30) we may apply Levi's theorem to get

$$T_{p,q}^{(2)}(x)=o_x(1)\quad\text{a.e. as } \max(p,q)\to\infty. \tag{33}$$

Finally, we estimate $T_{p,q}^{(3)}(x)$:

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int|T_{p,q}^{(3)}(x)|^2\,d\varrho(x)$$
$$\leqslant K_\alpha\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\sum_{i=0}^{m}\sum_{k=0}^{n}(m-i+1)^{2\alpha-2}\times(m+1)^{-2\alpha}\Big(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}}-1\Big)^2i^2a_{i,k}^2$$
$$\leqslant K_\alpha\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(n+1)^{-1}\sum_{i=0}^{m}\sum_{k=0}^{n}(m-i+1)^{2\alpha-2}$$

$$\times (m+1)^{-2\alpha-1}\left(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}}-1\right)^2 i^2 a_{i,k}^2$$

$$\leqslant K_\alpha \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} i^2 a_{i,k}^2 \Big(\sum_{m=i}^{\infty}(m-i+1)^{2\alpha-2}(m+1)^{-2\alpha-1}\Big)$$

$$\times\left(\sum_{n=k}^{\infty}(n+1)^{-1}\left(\frac{A_{n-k}^{(\beta)}}{A_n^{(\beta)}}-1\right)^2\right).$$

Since for any positive β,

$$\sum_{n=k}^{\infty}(n+1)^{-1}\left(\frac{A_{m-k}^{(\beta)}}{A_n^{(\beta)}}-1\right)^2 \leqslant K_\beta \qquad (k=0,1,2,\ldots)$$

(see [3], (4.9)), (25) shows that

$$\sum_{p,q=0}^{\infty}\int |T_{p,q}^{(3)}(x)|^2\,d\varrho(x) = \begin{cases} K_{\alpha,\beta}\sum_{i,k=0}^{\infty} a_{i,k}^2 & \text{if } \alpha>\frac{1}{2},\\ K_{\alpha,\beta}\sum_{i,k=0}^{\infty} a_{i,k}^2[\log(i+2)] & \text{if } \alpha=\frac{1}{2},\\ K_{\alpha,\beta}\sum_{i,k=0}^{\infty} a_{i,k}^2(i+1)^{1-2\alpha} & \text{if } -1<\alpha<\frac{1}{2},\end{cases}$$

and again (29), (14), (30) and Levi's theorem give

(34) $$T_{p,q}^{(3)}(x)=o_x(1) \quad \text{a.e. as } \max(p,q)\to\infty.$$

Finally, by (32), (33) and (34) we have (31).

The following two lemmas are analogous to Lemmas 1 and 2, so we omit their proofs.

LEMMA 3. *In the cases* (i) $\beta>\frac{1}{2}$, (ii) $\beta=\frac{1}{2}$, (iii) $-1<\beta<\frac{1}{2}$, *the conditions*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2[\log(i+2)]^2<\infty,$$

$$\sum_{i,k=0}^{\infty} a_{i,k}^2[\log(i+2)]^2[\log(k+2)]<\infty,$$

$$\sum_{i,k=0}^{\infty} a_{i,k}^2[\log(i+2)]^2(k+1)^{1-2\beta}<\infty$$

respectively are sufficient in order that

(35) $$2^{-q}\sum_{n=2^q+1}^{2^{q+1}}|t_{m,n}^{(0,\beta)}(x)|^2=o_x(1) \quad \text{a.e. as } q\to\infty,$$

uniformly in m.

LEMMA 4. *Suppose that $\alpha>0$ and that either*
(i) $\beta>\frac{1}{2}$ *and*

(36) $$\sum_{i,k=0}^{\infty} a_{i,k}^2[\log\log(i+4)]^2<\infty,$$

or
(ii) $\beta=\frac{1}{2}$ *and* (19) *holds,*
or
(iii) $-1<\beta<\frac{1}{2}$ *and*

(37) $$\sum_{i,k=0}^{\infty} a_{i,k}^2[\log\log(i+4)]^2(k+1)^{1-2\beta}<\infty.$$

Then

(38) $$2^{-p-q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1.}^{2^{q+1}}|t_{m,n}^{(\alpha,\beta)}(x)|^2=o_x(1) \quad \text{a.e. as } q\to\infty,$$

uniformly in p.

Finally, we prove

LEMMA 5. *Let one of the following conditions be satisfied:*
(i) $\alpha,\beta>\frac{1}{2}$ *and*

(39) $$\sum_{i,k=0}^{\infty} a_{i,k}^2<\infty.$$

(ii) $\alpha=\frac{1}{2}$, $\beta>\frac{1}{2}$ *and*

(40) $$\sum_{i,k=0}^{\infty} a_{i,k}^2[\log(i+2)]<\infty.$$

(iii) $-1<\alpha<\frac{1}{2}$, $\beta>\frac{1}{2}$ *and*

(41) $$\sum_{i,k=0}^{\infty} a_{i,k}^2(i+1)^{1-2\alpha}<\infty.$$

(iv) $\alpha=\beta=\frac{1}{2}$ *and* (16) *holds.*
(v) $-1<\alpha<\frac{1}{2}$, $\beta=\frac{1}{2}$ *and*

(42) $$\sum_{i,k=0}^{\infty} a_{i,k}^2(i+1)^{1-2\alpha}[\log(k+2)]<\infty.$$

(vi) $-1<\alpha<\frac{1}{2}$, $-1<\beta<\frac{1}{2}$ *and*

(43) $$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{1-2\alpha}(k+1)^{1-2\beta} < \infty.$$

(vii) $\alpha>\frac{1}{2}$, $\beta=\frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(k+2)] < \infty.$$

(viii) $\alpha>\frac{1}{2}$, $-1<\beta<\frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (k+1)^{1-2\beta} < \infty.$$

(ix) $\alpha=\frac{1}{2}$, $-1<\beta<\frac{1}{2}$ *and*

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)](k+1)^{1-2\beta} < \infty.$$

Then the series

(44) $$\sum_{m,n=0}^{\infty} \frac{1}{(m+1)(n+1)} |\tau_{m,n}^{(\alpha,\beta)}(x)|^2$$

converges a.e.

Proof. By (5) we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(m+1)(n+1)} \int |\tau_{m,n}^{(\alpha,\beta)}(x)|^2 d\varrho(x)$$

$$\leqslant K_{\alpha,\beta} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(m+1)^{1+2\alpha}(n+1)^{1+2\beta}} \sum_{i=0}^{m}\sum_{k=0}^{n} (m-i+1)^{2\alpha-2} \times (n-k+1)^{2\beta-2} i^2 k^2 a_{i,k}^2$$

$$= K_{\alpha,\beta} \sum_{i=0}^{m}\sum_{k=0}^{n} i^2 k^2 a_{i,k}^2 \Big(\sum_{m=i}^{\infty} (m-i+1)^{2\alpha-2}(m+1)^{-2\alpha-1}\Big) \times \Big(\sum_{n=k}^{\infty} (n-k+1)^{2\beta-2}(n+1)^{-2\beta-1}\Big).$$

Hence using (25) and applying Levi's theorem yield the assertion.

3. Proof of the Theorem. Our proof is divided into two parts.

Part 1. In this part we prove the Theorem for $\mu=2$ only. The argu ment is based on Theorem D, so we have to show that the series (9) is a.($(C,(\alpha,\beta))$ summable to its L^2-sum (see also Theorems A and C) and tha

(45) $$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M}\sum_{n=0}^{N} |z_{m,n}^{(\alpha,\beta)}(x)|^2 = o_x(1),$$

(46) $$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M}\sum_{n=0}^{N} |t_{m,n}^{(\alpha,\beta)}(x)|^2 = o_x(1),$$

(47) $$\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M}\sum_{n=0}^{N} |\tau_{m,n}^{(\alpha,\beta)}(x)|^2 = o_x(1)$$

a.e. as $\min(M,N)\to\infty$.

Take the integer w such that $2^{w-1}<M\leqslant 2^w$. Using the convention (24) for w and p, we can write for $M=0,1,2,\ldots$ and $n=0,1,2,\ldots$

$$\frac{1}{M+1}\sum_{m=0}^{M} |z_{m,n}^{(\alpha,0)}(x)|^2 \leqslant \frac{2}{2^w+1}\sum_{m=0}^{2^w} |z_{m,n}^{(\alpha,0)}(x)|^2$$

$$= \frac{2}{2^w+1}\sum_{p=-2}^{w-1} 2^{p+2}\frac{1}{2^{p+2}} \sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2.$$

The estimate (23) means that there is $F_1 \in L^2$ such that for $p=-2$, $-1,0,\ldots$ and $n=0,1,2,\ldots$

$$2^{-p-2}\sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2 \leqslant F_1(x) \quad \text{a.e.},$$

and furthermore, for any positive ε, there is a number $\varkappa=\varkappa_x(\varepsilon)$ such that if $p>\varkappa$ then for $n=0,1,2,\ldots$

$$2^{-p-2}\sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2 < \varepsilon \quad \text{a.e.}$$

Hence we see that for $M=0,1,2,\ldots$ and $n=0,1,2,\ldots$

$$(M+1)^{-1}\sum_{m=0}^{M} |z_{m,n}^{(\alpha,0)}(x)|^2 \leqslant 4F_1(x) \quad \text{a.e.}$$

Furthermore, assuming that $M>2^{\varkappa+1}$ we have

$$(M+1)^{-1}\sum_{m=0}^{M} |z_{m,n}^{(\alpha,0)}(x)|^2 \leqslant \frac{2}{2^w+1}\Big(\sum_{p=-2}^{\varkappa}+\sum_{p=\varkappa+1}^{w-1}\Big) 2^{p+2}\frac{1}{2^{p+2}}\sum_{m=2^p+1}^{2^{p+1}} |z_{m,n}^{(\alpha,0)}(x)|^2,$$

so the right-hand side is a.e. small uniformly in n if M is large enough.

Thus in the case $\alpha>-1$, $\beta=0$ we have shown (45) under the assumptions of Lemma 1.

Now take the integers w and v such that $2^{w-1} < M \leqslant 2^w$ and $2^{v-1} < N \leqslant 2^v$. Using the convention (24) for w, v, p and q, we can write for $M = 0, 1, \ldots$ and $N = 0, 1, \ldots$

$$\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}|z_{m,n}^{(\alpha,\beta)}(x)|^2 \leqslant \frac{4}{(2^w+1)(2^v+1)}\sum_{m=0}^{2^w}\sum_{n=0}^{2^v}|z_{m,n}^{(\alpha,\beta)}(x)|^2$$

$$= \frac{4}{(2^w+1)(2^v+1)}\sum_{p=-2}^{w-1}\sum_{q=2}^{v-1}2^{p+q+4}\frac{1}{2^{p+2}2^{q+2}}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\alpha,\beta)}(x)|^2.$$

The estimate (31) means that there is $F_2 \in L^2$ such that for $p = -2, -1, 0, \ldots$ and $q = -2, -1, 0, \ldots$

$$2^{-p-2}2^{-q-2}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\alpha,\beta)}(x)|^2 \leqslant F_2(x) \quad \text{a.e.},$$

and furthermore, for any positive ε, there is a number $\varkappa = \varkappa_x(\varepsilon)$ such that if $p > \varkappa$ then for $q = -2, -1, 0, \ldots$

$$2^{-p-2}2^{-q-2}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}|z_{m,n}^{(\alpha,\beta)}(x)|^2 < \varepsilon \quad \text{a.e.}$$

Hence we conclude that for $M = 0, 1, 2, \ldots$ and $N = 0, 1, 2, \ldots$

$$\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}|z_{m,n}^{(\alpha,\beta)}(x)|^2 \leqslant 64F_2(x) \quad \text{a.e.}$$

Furthermore, assuming that $M > 2^{\varkappa+1}$, we have

$$\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}|z_{m,n}^{(\alpha,\beta)}(x)|^2 \leqslant \frac{4F_2(x)}{(2^w+1)(2^v+1)}\sum_{p=-2}^{\varkappa}\sum_{q=-2}^{v-1}2^{p+q+4}$$

$$+\frac{4\varepsilon}{(2^w+1)(2^v+1)}\sum_{p=\varkappa+1}^{w-1}\sum_{q=-2}^{v-1}2^{p+q+4} < \frac{8F_2(x)2^{\varkappa+1}}{M}+4\varepsilon \quad (N = 0, 1, 2, \ldots).$$

Thus (45) is satisfied in the case $\alpha > -1$, $\beta > 0$ under the assumptions of Lemma 2.

Replacing (23) and (31) by (35) and (38), respectively, we can prove that (46) is satisfied in the case $\alpha = 0$, $\beta > -1$ under the assumptions of Lemma 3, and in the case $\alpha > 0$, $\beta > -1$ under the assumptions of Lemma 4.

The identity

$$\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}(m+1)(n+1)b_{m,n} = \frac{1}{(M+1)(N+1)}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{m}\sum_{k=0}^{n}b_{ik}\Big)$$

$$-\frac{1}{M+1}\sum_{m=0}^{M-1}\Big(\sum_{i=0}^{m}\sum_{k=0}^{N}b_{i,k}\Big)-\frac{1}{N+1}\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{M}\sum_{k=0}^{n}b_{i,k}\Big)+\sum_{i=0}^{M}\sum_{k=0}^{N}b_{i,k},$$

with

$$b_{m,n} = \frac{1}{(m+1)(n+1)}|\tau_{m,n}^{(\alpha,\beta)}(x)|^2,$$

shows that the convergence of the series (44) implies (47), i.e. (47) is satisfied under the assumptions of Lemma 5.

It remains to show that a suitable test for $(C, (\alpha, \beta))$ summability and Lemmas 1–5 are applicable in cases (i)–(ix) of the Theorem. Since (vii), (viii) and (ix) are analogous to (ii), (iii) and (v), respectively, we investigate (i)–(vi) only.

In case (i), $\alpha > \frac{1}{2}$ and $\beta > \frac{1}{2}$. By (13), we may apply Theorem J(iii), and so the series (9) is a.e. $(C, (\alpha, \beta))$ summable to its L^2-sum.

On the other hand, (13) implies (29), (36) and (39), which yields the assertion of the Theorem.

In case (ii), $\alpha = \frac{1}{2}$ and $\beta > \frac{1}{2}$. Then (14) implies (13), so the $(C, (\alpha, \beta))$ summability is obtained as in case (i). Furthermore, applying Lemma 2(ii), we have (45). The conditions (36) and (40) follow from (14), which completes the proof in this case.

In case (iii), $0 \leqslant \alpha < \frac{1}{2}$ and $\beta > \frac{1}{2}$. Then (15) implies (12) and (13), so the $(C, (\alpha, \beta))$ summability is obtained from Theorem J(ii), (iii). Here $\mu = 2$, so (15) gives (30), and applying Lemma 2(iii), we have (45). By (15), we may apply Lemmas 3(i) and 4(i) to get (46). Finally, (41) follows from (15) and we have the assertion.

In case (iv), $\alpha = \beta = \frac{1}{2}$. Then (16) implies (13), so the $(C, (\alpha, \beta))$ summability is obtained as in case (i); furthermore, applying Lemma 5(iv), we have (47). The conditions (14) (see Lemma 2(ii)) and (19) (see Lemma 4(ii)) follow from (16), which yields the assertion.

In case (v), $0 \leqslant \alpha < \frac{1}{2}$ and $\beta = \frac{1}{2}$. Then (17) implies (12) and (13), so the $(C, (\alpha, \beta))$ summability is obtained as in case (iii). Furthermore, for $\mu = 2$, (17) implies (42), and applying Lemma 5(v), we have (47). For $\mu = 2$, (17) implies (30), so we have (45). Since by (17) we may apply Lemmas 3(ii) and 4(ii), we get (46) and the assertion of the Theorem.

In case (vi), $0 \leqslant \alpha < \frac{1}{2}$ and $0 \leqslant \beta < \frac{1}{2}$. Then (18) implies (10)–(13), so Theorems I and J yield the a.e. $(C, (\alpha, \beta))$ summability of the series (9) to its L^2-sum. For $\mu = 2$, (18) gives (43), and applying Lemma 5(vi), we have (47). The conditions (22) and (30) follow from (18), so we get (45). By using Lemmas 3 and 4, we get (46) in a similar way, which finishes the proof in the case $\mu = 2$.

Part 2. Assume that $\mu > 2$. Our proof is based on Theorem H(i) with $\lambda = 2$.

In part 1 we have shown that the series (9) is a.e. $[C, (\tilde{\alpha}, \tilde{\beta})]_2$ summable to its L^2-sum in each of the following cases:

(I) $\bar{\alpha}, \bar{\beta} > \frac{1}{2}$ and (13) holds.
(II) $\bar{\alpha} = \frac{1}{2}$, $\bar{\beta} > \frac{1}{2}$ and (14) holds.
(III) $0 \leqslant \bar{\alpha} < \frac{1}{2}$, $\bar{\beta} > \frac{1}{2}$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{1-2\bar{\alpha}} [\log\log(k+4)]^2 < \infty .$$

(IV) $\bar{\alpha} = \bar{\beta} = \frac{1}{2}$ and (16) holds.
(V) $0 \leqslant \bar{\alpha} < \frac{1}{2}$, $\bar{\beta} = \frac{1}{2}$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{1-2\bar{\alpha}} [\log(k+2)] < \infty .$$

(VI) $0 \leqslant \bar{\alpha} < \frac{1}{2}$, $0 \leqslant \bar{\beta} < \frac{1}{2}$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 (i+1)^{1-2\bar{\alpha}} (k+1)^{1-2\bar{\beta}} < \infty .$$

(VII) $\bar{\alpha} > \frac{1}{2}$, $\bar{\beta} = \frac{1}{2}$ and (19) holds.
(VIII) $\bar{\alpha} > \frac{1}{2}$, $0 \leqslant \bar{\beta} < \frac{1}{2}$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log\log(i+4)]^2 (k+1)^{1-2\bar{\beta}} < \infty .$$

(IX) $\bar{\alpha} = \frac{1}{2}$, $0 \leqslant \bar{\beta} < \frac{1}{2}$ and

$$\sum_{i,k=0}^{\infty} a_{i,k}^2 [\log(i+2)] (k+1)^{1-2\bar{\beta}} < \infty .$$

Writing

$$\alpha = \bar{\alpha} + \tfrac{1}{2} - 1/\mu, \qquad \beta = \bar{\beta} + \tfrac{1}{2} - 1/\mu,$$

we observe that cases (i)–(ix) of the Theorem give (I)–(IX), respectively. Hence we see that under the assumptions of the Theorem, the series (9) is a.e. $[C, (\bar{\alpha}, \bar{\beta})]_2$ summable to its L^2-sum. Applying Theorem H(i), we get the a.e. $[C, (\alpha, \beta)]_\mu$ summability of (9) to the same sum.

Finally, replacing the pair λ, μ by μ, ν and applying Theorem A complete the proof.

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BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
Aradi vertanuk tere 1
6720 Szeged, Hungary

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ERRATUM

Line 4 on p. 96 is missing. It should read:

$$(25) \qquad \sum_{m=i}^{\infty} (m-i+1)^{2\alpha-2} (m+1)^{-2\alpha-1}$$

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