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Minimal generators for ergodic endomorphisms

by

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Abstract. Every ergodic endomorphism f of a nonatomic Lebesgue space which possesses a finite 1-sided generator has a 1-sided generator α such that $k_f \leqslant \operatorname{card}\alpha \leqslant k_f+1$. This is the best estimate for the minimal cardinality of a 1-sided generator. If f belongs to the class of transformations described in the second part of the paper then the generator of minimal cardinality (equal to k_f+1) is presented in an explicit form.

Let f be an ergodic endomorphism of a nonatomic Lebesgue space $(X, \mathscr{B}, \mu)$. Let $f^{-1}\varepsilon$ denote the partition $\{f^{-1}(x)\colon x\in X\}$ and let $\{m_{f^{-1}(x)}\}_{x\in X}$ be the canonical system of measures. Denote by $h(f)$ the entropy of f. If $h(\varepsilon, f) = h(f) < \infty$, then the canonical measures are purely atomic. In this case we can define a number k_f in the following way:

$$k_f = \min \{k\colon \operatorname{card}\{y\colon y\in f^{-1}(x) \text{ and } m_{f^{-1}(x)}(y) > 0\} \leqslant k \text{ a.e.}\}.$$

The equality $h(\varepsilon, f) = h(f) < \infty$ implies that f admits a 1-sided generator of finite entropy. Therefore f is measure-theoretically conjugate to a positive nonsingular endomorphism $\tilde{f}$ (see [4], p. 107) and we obtain a simple description of k_f:

$$k_f = \min\{k\colon \operatorname{card}\tilde{f}^{-1}(x) \leqslant k \text{ a.e.}\}.$$

The number k_f is connected with the existence of a finite 1-sided generator.

Theorem A [2]. *An ergodic endomorphism f has a finite 1-sided generator iff $h(\varepsilon, f) = h(f) < \infty$ and $k_f < \infty$.*

Analysing the proof of Theorem A, it is not difficult to see that there exists a 1-sided generator α such that

$$k_f \leqslant \operatorname{card}\alpha \leqslant (e^{h(f)}+1)\,k_f. \tag{1}$$

It is obvious by $\alpha \vee f^{-1}\varepsilon = \varepsilon$ that for every 1-sided generator α the left side of the above inequality holds. By (1), the problem arises of finding the exact estimate of the minimal cardinality of a 1-sided generator for f. This problem is solved by the following theorem:

THEOREM 1. *If an ergodic endomorphism f admits a finite 1-sided generator, then there exists a 1-sided generator α such that $k_f \leqslant \operatorname{card}\alpha \leqslant k_f+1$.*

Before proving the theorem we shall prove two lemmas.

LEMMA 1. *There exists a partition $\beta = \{B_1, \ldots, B_{k_f}\}$ such that $\beta \vee f^{-1}\varepsilon = \varepsilon$ and $h(\beta, f) = h(f)$.*

Proof. Using the reasoning of Rokhlin (see [5], p. 41) we get the partition $\beta = \{B_1, \ldots, B_{k_f}\}$ such that $B_1 \cap f^{-1}(x)$ consists of an atom of the greatest $m_{f^{-1}(x)}$ measure, next $B_2 \cap f^{-1}(x)$ consists of an atom of the greatest $m_{f^{-1}(x)}$ measure in $f^{-1}(x)-B_1$, etc. This partition satisfies the following conditions: $\beta \vee f^{-1}\varepsilon = \varepsilon$ and $f^{-1}\beta^- \leqslant \beta^-$ where $\beta^- = \bigvee_{i=0}^{\infty} f^{-1}\beta$. According to Theorems 1 and 2 in [3], f is represented by the skew product $f(z, y) = (f_{\beta^-}(z), \sigma_z(y))$ where $(z, y) \in X_{\beta^-} \times Y \simeq X$ and f_{β^-} denotes the factor endomorphism of f. We also have the equality $J_f(z, y) = J_{\beta^-}(z) J_{\sigma_z}(y)$ where J denotes the Jacobian of an endomorphism. In fact, $J_f(z, y) = J_{\beta^-}(z)$ because σ_z is an automorphism a.e.

Hence we get the following equalities:

$$h(f) = h(\varepsilon, f) = H(\varepsilon \mid f^{-1}\varepsilon) = \int \log J_f\, dm = \int_{X_{\beta^-}} \log J_{\beta^-}\, dm_{\beta^-}$$
$$= h(\beta^-, f_{\beta^-}) = h(\beta, f),$$

which finishes the proof.

Let $\bar{f}$ denote the natural extension of f to an automorphism. The transformation $\bar{f}$ is an automorphism of the measurable space $(\bar{X}, \bar{\mathscr{B}}, \bar{m})$ where $\mathscr{B}$ is an exhaustive σ-algebra of $\bar{\mathscr{B}}$. The following generalization of Proposition 28.2 in [1] holds:

LEMMA 2. *Let $\beta = \{B_1, \ldots, B_s\}$ be a partition such that $\bar{m}(B_1) > 0$, $B_i \in \mathscr{B}$, $i = 1, \ldots, s$. Let*

$$\eta = [\bar{m}(B_1)]^{-1} [h(\bar{f}) - h(\beta, \bar{f})].$$

If $n \in N$, $\log n > \eta$, then there exists a partition $\{A_1, \ldots, A_n\}$ of B_1 such that $A_i \in \mathscr{B}$, $i = 1, \ldots, n$, and $\beta_1 = \{A_1, \ldots, A_n, B_2, \ldots, B_s\}$ is a generator for $\bar{f}$.

Proof. Except for the symbol Ψ, which we replace by $\bar{f}$, we will use the same notation as in [1]. Proposition 28.2 is a consequence of the proof of Theorem 28.1 [1] where a generator is constructed by adjoining different blocks to certain subsets of a set H_i for $i = 1, 2, \ldots$ The set H_i is the sum of some levels of a suitable Rokhlin tower. We obtain the assertion of the lemma if we use a Rokhlin tower such that the levels contained in H_i are elements of $\mathscr{B}$. Therefore we take a sequence of partitions γ_i, $i = 1, 2, \ldots$, such that $\gamma_i \subseteq \mathscr{B}$ and $\bigvee_{i \in N} \sum_f \gamma_i = \bar{\mathscr{B}}$.

Assume that $H_{i-1} \in \mathscr{B}$. In the next step of the induction proof we find a set $F_i \subseteq S_i \cap H_{i-1}$ such that $F_i \in \mathscr{B}$ and F_i is an $(\bar{f}, q_i, \xi_i)$-Rokhlin set and $\bar{F}_i = \bar{f}_{H_{i-1}}^{-n_i}(F_i)$ is an $(\bar{f}_{H_{i-1}}, n_i, \xi_i)$-Rokhlin set. Analysing the proof of Theorem 26.4 [1] gives easily the existence of such an F_i. By

$$H_i = \bigcup_{j=2l+l(a)}^{2l+l(a)+k_i} f_{H_{i-1}}^{j}(\bar{F}_i) \quad \text{for } j = k_i - c + j, \ldots, n_{i-1}$$

and by using the levels $f_{H_{i-1}}^{j}(\bar{F}_j)$ for $j = k_i - c + j, \ldots, n_{i-1}$ and some atoms of the partition $\bigvee_{j=0}^{q_i-1} f^{-j}(\gamma_i)$ in coding we get the assertion.

Proof of Theorem 1. Let $\beta = \{B_1, \ldots, B_{k_f}\}$ be the partition given by Lemma 1. By $h(\bar{f}) = h(f) = h(\beta, f)$ we get $\eta = 0$. Therefore by applying Lemma 2 to β, where $n = 2$, we get the generator $\alpha = \{A_1, A_2, B_2, \ldots, B_{k_f}\}$ for $\bar{f}$ such that $\alpha \subseteq \mathscr{B}$ and $\alpha \vee f^{-1}\varepsilon = \varepsilon$. We get the assertion by using the same reasoning as in the proof of Rokhlin's theorem 10.11 in [6].

In the second part of the paper we describe a simple construction of a 1-sided generator for a class of simple products of measure-preserving transformations. Let (X, f, μ) and (Y, τ, p) be dynamical systems where X, Y are compact metric spaces, f is a continuous transformation, τ is a homeomorphism and μ, p are Borel invariant measures. Let $T(x, y) = (f(x), \tau(y))$ for $(x, y) \in X \times Y$. T is a continuous endomorphism of the space $X \times Y$ with the product measure $\mu \times p$. Assume that f has a k_0-element 1-sided generator $\beta = \{B_1, \ldots, B_{k_0}\}$ such that B_i, $i = 1, \ldots, k_0$, is an open set and τ has a 2-element 1-sided generator $\alpha = \{A, A^c\}$.

DEFINITION. A homeomorphism τ is *distal* if $\forall x \neq y\ \exists \varepsilon > 0$ $d(\tau^n(x), \tau^n(y)) > \varepsilon$ for $n = 0, \pm 1, \ldots$

Here d denotes the metric in Y.

THEOREM 2. *If for all $k \geqslant 0$, $\bigcap_{n=0}^{k} f^{-n}(B_1) \neq \emptyset$, τ is distal and the diameter δ_n of the partition $\alpha_n = \bigvee_{i=0}^{n} \tau^{-i}\alpha$ tends to 0 and the orbit of μ-a.e. $x \in X$ is dense in X, then $\beta_1 = \{B_1 \times A, B_1 \times A^c, B_2, \ldots, B_{k_0}\}$ is a 1-sided generator for T.*

Proof. Denote by X' (Y') the subset of X (Y) coded by the generator β (α). Let X'' be the set of all points $x \in X'$ such that the positive orbit of x is dense in X. Then we have $\mu \times p(X'' \times Y') = 1$. Now, we prove that the set $X'' \times Y'$ is coded by β_1, i.e. every atom of $\beta_1^- = \bigvee_{i=0}^{\infty} T^{-i}\beta_1$ has at most a 1-point intersection with $X'' \times Y'$.

Let (x, y), $(x_1, y_1) \in X'' \times Y'$ belong to the same atom of β_1^-. Since $\beta^- \times Y \leqslant \beta_1^-$ we get $x = x_1$. If $y \neq y_1$ then by the distality of τ there exists $\varepsilon > 0$ such that $d(\tau^n(y), \tau^n(y_1)) > \varepsilon$ for $n = 0, 1, \ldots$ Since $\lim_{n\to\infty} \delta_n = 0$ there is n such that $\delta_n < \varepsilon$. Consider the following family of atoms

(2) $$\bigcap_{j=0}^{n} f^{-j}(B_1) \times \bigvee_{j=0}^{n} \tau^{-j}(\alpha)$$

of the partition $\bigvee_{j=0}^{n} T^{-j}\beta_1$. The set $\bigcap_{j=0}^{n} f^{-j}(B_1)$ is nonempty and open. Hence $f^k(x) \in \bigcap_{j=0}^{n} f^{-j}(B_1)$ for some k by the density of the orbit of x. The points $(f^k(x), \tau^k(y))$ and $(f^k(x), \tau^k(y_1))$ belong to different atoms of the set (2) because $d(\tau^k(y), \tau^k(y_1)) > \varepsilon > \delta_n$. Hence (x, y) and (x, y_1) belong to different elements of the partition

$$T^{-k}\Big(\bigvee_{j=0}^{n} T^{-j}\beta_1\Big) \leqslant \bigvee_{j=0}^{n+k} T^{-j}\beta_1 < \beta_1^{-}.$$

This is a contradiction and hence $y = y_1$, which finishes the proof.

COROLLARY. *The partition* $\beta = \{B \times B, B \times B^c, B^c \times S\}$ *where* $B = \{e^{2\pi i \alpha}: 0 < \alpha < \frac{1}{2}\}$ *is a generator for the transformation* $T(z, w) = (z^2, aw)$ *of the torus* S^2 *where* a *is not a root of unity.*

Remark. The inequalities in Theorem 1 establish the best estimate of the minimal cardinality of a generator.

The example given in the corollary belongs to the following class of transformations: Let T be the 1-sided $(1/k, \ldots, 1/k)$ Bernoulli shift and let τ be any distal automorphism. Then the generator β_1 constructed for $T \times \tau$ has $k+1$ elements and the other generators have at least $k+1$ elements.

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On the strong Cesàro summability of double orthogonal series

by

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Abstract. In a recent paper [3], Móricz gave a coefficient test for the strong summability of double orthogonal series in the case of the parameters α and β greater than 1/2 and the index λ equal to 2. Using the definition of convergence in Pringsheim's sense with a bound, the present author extends the definition of strong summability to the case of λ positive and α and β nonnegative. The case $\alpha = \beta = 0$ is the so-called strong convergence. This note contains coefficient conditions for nine cases of parameters and indices.

1. Introduction. First of all we mention that for a double sequence $\{\omega_{m,n}\}_{m,n=0}^{\infty}$ the "little o"

$$\omega_{m,n} = o(1) \quad \text{as } \min(m, n) \to \infty$$

$$(\text{or } \max(m, n) \to \infty, \text{ or } m \to \infty, \text{ or } n \to \infty)$$

means that $\omega_{m,n} \to 0$ as $\min(m, n) \to \infty$ (or $\max(m, n) \to \infty$, or $m \to \infty$, or $n \to \infty$) and in addition there exists a constant K such that $|\omega_{m,n}| \leqslant K$ for $m, n = 0, 1, \ldots$ The case

$$\omega_{m,n} = o(1) \quad \text{as } \min(m, n) \to \infty$$

may be called *convergence in Pringsheim's sense with a bound*. Our next definitions are understood in this sense.

We say that a series

(1) $$\sum_{i,k=0} c_{i,k}$$

is *Cesàro summable with parameters* $\alpha, \beta > -1$ – or $(C, (\alpha, \beta))$ *summable* – to s if

$$\sigma_{m,n}^{(\alpha,\beta)} - s = o(1) \quad \text{as } \min(m, n) \to \infty,$$

and it is said to be *strongly Cesàro summable with parameters* $\alpha, \beta > 0$ *and index* $\lambda > 0$ – or $[C, (\alpha, \beta)]_\lambda$ *summable* – to s if

* This work was done while the author was visiting the Steklov Mathematical Institute in Moscow, U.S.S.R.