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A proof of Pełczyński's conjecture for the Haar system

by

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Abstract. Let H be a real or complex Hilbert space with norm $|\cdot|$. Let $1 < p < \infty$ and $p^* = \max\{p, p/(p-1)\}$. Suppose that f and g belong to the Lebesgue–Bochner space $L_H^p[0, 1)$ and $(h_n)_{n \geq 0}$ is the sequence of Haar functions on $[0, 1)$. Let

$$f = \sum_{k=0}^{\infty} a_k h_k, \quad g = \sum_{k=0}^{\infty} b_k h_k$$

where $a_k, b_k \in H$ and the two series converge in $L_H^p[0, 1)$. The main result of the paper is: If $|b_k| \leq |a_k|$ for all $k \geq 0$, then

$$\|g\|_p \leq (p^*-1)\|f\|_p$$

and the constant p^*-1 is best possible. Strict inequality holds if $p \neq 2$ and $\|f\|_p > 0$.

This result yields Pełczyński's conjecture: The classical inequality of Paley and Marcinkiewicz for the Haar system holds with the same constant if the multiplier sequence of signs ± 1 is replaced by a sequence of unimodular complex numbers.

1. Introduction. We begin with an inequality for Haar polynomials with coefficients in a real or complex Hilbert space H. If n is a positive integer, let L_n be the left half and R_n the right half of the nth interval I_n in the sequence $[0, 1)$, $[0, 1/2)$, $[1/2, 1)$, $[0, 1/4)$, $[1/4, 1/2)$, ... So, for example, $L_1 = I_2$ and $R_1 = I_3$. Here it is convenient to define the nth Haar function as follows: $h_n = 1$ on L_n, $h_n = -1$ on R_n, and $h_n = 0$ on $I_n^c = [0, 1) \setminus I_n$. Furthermore, $h_0 \equiv 1$.

We shall denote the norm of $x \in H$ by $|x|$ and the maximum of p and $p/(p-1)$ by p^*. Note that $p^*-1 = \max\{p-1, 1/(p-1)\}$.

THEOREM 1. *Let $1 < p < \infty$. If $a_k, b_k \in H$ and $|b_k| \leq |a_k|$, then, for all $n \geq 0$,*

$$\Big\|\sum_{k=0}^{n} b_k h_k\Big\|_p \leq (p^*-1)\Big\|\sum_{k=0}^{n} a_k h_k\Big\|_p \tag{1}$$

and the constant p^*-1 is best possible. Strict inequality holds if and only if (i) *$p \neq 2$ and $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$ or* (ii) *$p = 2$ and $\sum_{k=0}^{n} |b_k|^2 < \sum_{k=0}^{n} |a_k|^2$.*

Paley [8] and Marcinkiewicz [7] assumed that $H = \mathbf{R}$. Their proof did not yield the best constant. We gave a proof in [3] that did yield the best constant for the case $H = \boldsymbol{R}$ and a shorter such proof in [4]. Pełczyński [9] observed that with the aid of the general theory of the complexification of operators our result implies that p^*-1 is also the best constant for the case $a_k \in \boldsymbol{C}$ and $b_k = \varepsilon_k a_k$ where $\varepsilon_k \in \{1, -1\}$. In addition to implying these results, Theorem 1 gives Pełczyński's conjecture: keeping the same constant, the ε_k's can be replaced by unimodular complex numbers.

Suppose that B is a Banach space that is not isomorphic to a Hilbert space. If H is replaced by B in the above theorem, then (1) does not hold for any constant. This rests on a result of Kwapień [6]; see Section 5 of [1]. Nevertheless, for the special case $b_k = \varepsilon_k a_k$, there is a large class of Banach spaces such that, for some constant, the inequality (1) does hold; see [1] and, for a more recent discussion, [5].

2. Proof of Theorem 1. Define $v\colon H \times H \to \boldsymbol{R}$ by

$$v(x, y) = |y|^p - (p^*-1)^p |x|^p \tag{2}$$

and assume there is a function u on $H \times H$ such that if $a, b, x, y \in H$ and $|b| \leqslant |a|$, then

$$v(x, y) \leqslant u(x, y), \tag{3}$$

$$u(x, y) = u(-x, -y), \tag{4}$$

$$u(0, 0) = 0, \tag{5}$$

$$u(x+a, y+b) + u(x-a, y-b) \leqslant 2u(x, y). \tag{6}$$

Let $f_n = \sum_{k=0}^{n} a_k h_k$ and $g_n = \sum_{k=0}^{n} b_k h_k$. Then, by (2) and (3), the integral of $v(f_n, g_n)$ on $[0, 1)$ satisfies

$$\|g_n\|_p^p - (p^*-1)^p \|f_n\|_p^p = \int v(f_n, g_n) \leqslant \int u(f_n, g_n).$$

We show now that

$$\int u(f_n, g_n) \leqslant \int u(f_{n-1}, g_{n-1}) \leqslant \ldots \leqslant \int u(f_0, g_0) \leqslant 0, \tag{7}$$

which gives (1). The inequality on the right follows from (4)–(6):

$$\int u(f_0, g_0) = u(a_0, b_0) = [u(a_0, b_0) + u(-a_0, -b_0)]/2 \leqslant u(0, 0) = 0.$$

The inequality on the left follows from (6) and the fact that f_{n-1} and g_{n-1} are constant on I_n, the support of h_n:

$$\int_{I_n^c} u(f_n, g_n) = \int_{I_n^c} u(f_{n-1}, g_{n-1}),$$

$$\begin{aligned}\int_{I_n} u(f_n, g_n) &= \int_{L_n} u(f_{n-1}+a_n, g_{n-1}+b_n) + \int_{R_n} u(f_{n-1}-a_n, g_{n-1}-b_n)\\ &= \int_{I_n} [u(f_{n-1}+a_n, g_{n-1}+b_n) + u(f_{n-1}-a_n, g_{n-1}-b_n)]/2\\ &\leqslant \int_{I_n} u(f_{n-1}, g_{n-1}).\end{aligned}$$

To complete the proof of (1) we must find a function u on $H \times H$ satisfying (3)–(6), or show that one exists. This is the key step. Here is such a function:

$$u(x, y) = \alpha_p (|x|+|y|)^{p-1} (|y| - (p^*-1)|x|) \tag{8}$$

where $\alpha_p = p(1-1/p^*)^{p-1}$. Clearly, (4) and (5) are satisfied. If $|x|+|y| = 0$, then (3) is satisfied. Therefore, to prove (3) we may assume that $|x|+|y| > 0$ and, by homogeneity, that $|x|+|y| = 1$. Letting $s = |x|$, we see that the proof of (3) reduces to showing that

$$F(s) = \alpha_p (1-p^* s) - (1-s)^p + (p^*-1)^p s^p$$

is nonnegative for $0 \leqslant s \leqslant 1$. If $p = 2$, then $F \equiv 0$. If $p > 2$, then there is a number s_0 satisfying $0 < s_0 < 1/p^*$ such that F is concave on $[0, s_0]$ and is convex on $[s_0, 1]$. Therefore, it follows from $F(1/p^*) = F'(1/p^*) = 0$ that F is nonnegative on $[s_0, 1]$. Since $F(0) \geqslant 0$, concavity implies that the function F is also nonnegative on $(0, s_0)$. If $1 < p < 2$, the argument is similar: F is convex on an interval containing $[0, 1/p^*]$, F is concave on its complement, and $F(1) \geqslant 0$.

To prove (6), we may assume, by continuity, that x and a are linearly independent over $\boldsymbol{R}$ and the same for y and b. The case $H = \boldsymbol{R}$ can be handled by thinking of $\boldsymbol{R}$ as a subspace of $\boldsymbol{C}$. Then $G\colon \boldsymbol{R} \to \boldsymbol{R}$ defined by

$$\alpha_p G(t) = u(x+at, y+bt)$$

is infinitely differentiable and, as we shall show, G'' is nonpositive. Therefore, $G(1) + G(-1) \leqslant 2G(0)$, which proves (6). By translation, it is enough to prove that $G''(0) \leqslant 0$. Letting $x' = x/|x|$ and $y' = y/|y|$, we have, for $2 \leqslant p < \infty$,

$$\begin{aligned}G''(0) = &-p(p-1)(|a|^2 - |b|^2)(|x|+|y|)^{p-2}\\ &-p(p-2)[|b|^2 - (y', b^2)]\,|y|^{-1}(|x|+|y|)^{p-1}\\ &-p(p-1)(p-2)[(x', a)+(y', b)]^2\,|x|(|x|+|y|)^{p-3}\end{aligned} \tag{9}$$

where (x', a) denotes the real part of the inner product of x' and a. Using the assumption that $|b| \leqslant |a|$ and the Cauchy–Schwarz inequality, we see that $G''(0) \leqslant 0$. If $1 < p < 2$, a similar expression for $(p-1)G''(0)$ can be obtained from (9) by interchanging x and y, a and b, and then multiplying the right-hand side by -1. This follows from

$$(p-1)u(x, y) = -\alpha_p (|y|+|x|)^{p-1} (|x| - (p-1)|y|)$$

and completes the proof of (1).

The constant p^*-1 is best possible since it is best possible for $H = \boldsymbol{R}$; see [2] or [3].

To prove the strictness of the inequality (1) in the nontrivial case (i), let

m be the least integer k satisfying $0 \leqslant k \leqslant n$ and $a_k \neq 0$. Then $|g_m(t)| \leqslant |f_m(t)|$, $t \in [0, 1)$, and $\|f_m\|_p > 0$. Therefore, by (8),

$$u(f_m, g_m) \leqslant \alpha_p |f_m|^{p-1} \left(|f_m| - (p^* - 1)|f_m|\right)$$

and, by (7),

$$\int u(f_n, g_n) \leqslant \int u(f_m, g_m) \leqslant \alpha_p (2 - p^*) \|f_m\|_p^p < 0.$$

So

$$\|g_n\|_p^p \leqslant (p^* - 1)^p \|f_n\|_p^p + \alpha_p (2 - p^*) \|f_m\|_p^p \tag{10}$$

and this implies the strictness of the inequality (1).

3. Remarks. Under the conditions of Theorem 1, if the series $\sum_{k=0}^{\infty} a_k h_k$ converges in $L_H^p[0, 1)$, then, by (1), the sequence of partial sums of the series $\sum_{k=0}^{\infty} b_k h_k$ is a Cauchy sequence so the latter series must also converge in $L_H^p[0, 1)$. In this case, by (1),

$$\Big\| \sum_{k=0}^{\infty} b_k h_k \Big\|_p \leqslant (p^* - 1) \Big\| \sum_{k=0}^{\infty} a_k h_k \Big\|_p.$$

If $p \neq 2$ and $(a_0, a_1, \ldots) \neq (0, 0, \ldots)$, then, by (10), strict inequality holds. Now, if B is a Banach space, every $f \in L_B^p[0, 1)$ is the limit in $L_B^p[0, 1)$ of a series of the form $\sum_{k=0}^{\infty} a_k h_k$ where the coefficients $a_k \in B$. (The original proof of Schauder [10] for the case $B = \mathbf{R}$ carries over.) Thus, we have the following theorem.

THEOREM 2. *Let $1 < p < \infty$ and $f, g \in L_H^p[0, 1)$. Suppose that*

$$f = \sum_{k=0}^{\infty} a_k h_k, \quad g = \sum_{k=0}^{\infty} b_k h_k$$

where $a_k, b_k \in H$ and the two series converge in $L_H^p[0, 1)$. If $|b_k| \leqslant |a_k|$ for all $k \geqslant 0$, then

$$\|g\|_p \leqslant (p^* - 1) \|f\|_p$$

and the constant $p^* - 1$ is best possible. Strict inequality holds if and only if (i) *$p \neq 2$ and $\|f\|_p > 0$ or* (ii) *$p = 2$ and $\sum_{k=0}^{\infty} |b_k|^2 < \sum_{k=0}^{\infty} |a_k|^2$. In fact, if* (i) *holds,*

$$\sup_g \|g\|_p < (p^* - 1) \|f\|_p$$

where the supremum is taken over all such g as above.

The discovery of the function u also makes possible the proof of sharp inequalities for differentially subordinate martingales and stochastic integrals taking values in a Hilbert space. These inequalities and their proofs will appear in Astérisque (Colloque Paul Lévy).

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