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Adjoining inverses to noncommutative Banach algebras and extensions of operators

by

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Abstract. We exhibit an example of a Banach algebra A with an element $u \in A$ such that u is left invertible in an extension $B \supset A$, u is right invertible in another extension $B' \supset A$ and u is invertible in no extension $C \supset A$. This solves some problems of W. Żelazko ([6], Problems 2.8 and 2.9) and shows that Arens' characterization of permanently singular elements is not true in noncommutative Banach algebras. Further, two problems of B. Bollobás [2] are solved and the following is proved: If T is a bounded operator on a Banach space X then there exists a Banach space $Y \supset X$ and $S \in B(Y)$ such that $S|X = T$ and $\sigma(S) = \{\lambda: \inf\{\|(T-\lambda)x\|: x \in X, \|x\| = 1\} = 0\}$.

Let A, B be Banach algebras (all Banach algebras in this paper will be complex and unital). We say that B is an *extension* of A (we write shortly $B \supset A$) if there is an isometric, unit preserving homomorphism $f: A \to B$.

Let u be an element of a Banach algebra A. We say that u is a *left topological divisor of zero* if $\inf\{\|uz\|: z \in A, \|z\| = 1\} = 0$. If $u \in A$ is a left topological divisor of zero then u is left invertible in no extension $B \supset A$. Indeed, suppose on the contrary $bu = 1_B$ for some $b \in B$, $B \supset A$. Then

$$1 = \inf\{\|buz\|: z \in A, \|z\| = 1\} \leqslant \|b\| \inf\{\|uz\|: z \in A, \|z\| = 1\} = 0,$$

a contradiction.

For commutative Banach algebras the following characterization (Arens [1]) holds:

$u \in A$ is invertible in some (commutative) extension $B \supset A$ if and only if u is not a topological divisor of zero.

The analogous statement for one-sided inverses in noncommutative Banach algebras is an open problem.

PROBLEM. Let A be a Banach algebra and suppose $u \in A$ is not a left topological divisor of zero. Does there exist an extension $B \supset A$ such that u is left invertible in B?

In this paper we show that the analogous statement for two-sided inverses is not true. It is possible that $u \in A$ is neither a left nor right topological divisor of zero and still u is invertible in no extension $B \supset A$

([6], Problem 2.8). In fact, we construct an example giving also an answer to Problem 2.9 of [6]:

1. Theorem. *There exist a Banach algebra A, an element $u \in A$, and two extensions $B \supset A$, $B' \supset A$ such that:*
1. *u is left invertible in B.*
2. *u is right invertible in B'.*
3. *u is invertible in no extension $C \supset A$.*

2. Lemma. *There exists a semigroup P with unit 1_P and elements $a_1, \ldots, a_{11}$, $u \in P$ such that:*
1. $a_8 \neq a_9$.
2. $ua_5 = a_1$, $ua_6 = a_{10}$, $a_7 u = a_{11}$, $a_4 u = a_1$, $a_2 a_6 = a_8$, $a_7 a_3 = a_9$, $a_4 a_3 = a_{10}$, $a_2 a_5 = a_{11}$.
3. *If $p_1, p_2 \in P$, $p_1 \neq p_2$, then $up_1 \neq up_2$ and $p_1 u \neq p_2 u$.*

Proof. Let $P = \{z\} \cup \{u^m \colon m \geqslant 0\} \cup \{u^k a_i u^l \colon (k, i, l) \in S\}$ where

$$S = \{(k, i, l)\colon k \geqslant 0,\ l \geqslant 0,\ i = 1, 2, 3, 8, 9, 10, 11\} \cup \{(0, i, l)\colon l \geqslant 0,\ i = 5, 6\} \cup \{(k, i, 0)\colon k \geqslant 0,\ i = 4, 7\}$$

(we consider z, u^m, $u^k a_i u^l$ as formal mutually distinct symbols).

The multiplication on P is defined by:

$$zp = pz = z \quad (p \in P), \quad \text{i.e. } z \text{ is a zero element in } P,$$

$$u^0 p = pu^0 = p \quad (p \in P), \quad \text{i.e. } u^0 = 1_P,$$

$$u^m u^n = u^{m+n} \quad (m, n \geqslant 0),$$

$$u^m (u^k a_i u^l) = u^{m+k} a_i u^l \quad (k, l \geqslant 0,\ m \geqslant 1,\ i \neq 5, 6),$$

$$u^m (a_5 a^l) = u^{m-1} a_1 a^l, \quad u^m (a_6 u^l) = u^{m-1} a_{10} u^l \quad (l \geqslant 0,\ m \geqslant 1),$$

$$(u^k a_i u^l) u^m = u^k a_i u^{l+m} \quad (k, l \geqslant 0,\ m \geqslant 1,\ i \neq 4, 7),$$

$$(u^k a_4) u^m = u^k a_1 u^{m-1}, \quad (u^k a_7) u^m = u^k a_{11} u^{m-1} \quad (k \geqslant 0,\ m \geqslant 1),$$

$$(u^k a_i u^l)(u^{k'} a_{i'} u^{l'}) = z \quad (l \geqslant 1 \text{ or } k' \geqslant 1 \text{ or } (i, i') \notin \{(2, 6), (7, 3), (4, 3), (2, 5)\}),$$

$$\left.\begin{aligned} (u^k a_2)(a_6 u^{l'}) &= u^k a_8 u^{l'} \\ (u^k a_7)(a_3 u^{l'}) &= u^k a_9 u^{l'} \\ (u^k a_4)(a_3 u^{l'}) &= u^k a_{10} u^{l'} \\ (u^k a_2)(a_5 u^{l'}) &= u^k a_{11} u^{l'} \end{aligned}\right\} \quad (k, l' \geqslant 0).$$

We write a_i, $u^k a_i$, $a_i u^l$, u instead of $u^0 a_i u^0$; $u^k a_i u^0$, $u^0 a_i u^l$, u^1 respectively.

It is easy to check (although rather tedious) that the multiplication defined in this way is associative. Also conditions 1–3 of Lemma 2 are satisfied.

3. Lemma. (i) *Let P be a semigroup with unit 1_P and let $u \in P$ satisfy $up_1 \neq up_2$ whenever $p_1, p_2 \in P$, $p_1 \neq p_2$. Then there exists a semigroup Q with unit $1_Q = 1_P$ containing P as a subsemigroup such that u is left invertible in Q.*

(ii) *If $p_1, p_2 \in P$, $p_1 \neq p_2$, implies $p_1 u \neq p_2 u$ then there exists a semigroup $Q' \supset P$ with unit $1_{Q'} = 1_P$ such that u is right invertible in Q'.*

Proof. Lemma 3 is a well-known fact from the theory of semigroups (see e.g. [4], X.1). For the sake of convenience we give an outline of the proof here. Let Q be the semigroup of all mappings $f\colon P \to P$. We may identify an element $p \in P$ with a mapping $L_p \in Q$ defined by $L_p(p') = pp'$ $(p' \in P)$. In this way P becomes a subsemigroup of Q. As L_u is a one-to-one mapping, it is left invertible in Q. Part 2 can be proved analogously.

Proof of Theorem 1.

1. Let P, $Q \supset P$ be the semigroups constructed in Lemmas 2, 3 and let $b \in Q$ be the left inverse of the element $u \in P$, $bu = 1_Q$.

Denote by A the l^1 algebra over P, i.e. A consists of all formal series $a = \sum_{p \in P} \lambda_p p$ with complex coefficients λ_p $(p \in P)$ such that $\|a\|_A = \sum_{p \in P} |\lambda_p| < \infty$. The algebraic operations on A are defined by

$$\Big(\sum_{p \in P} \lambda_p p\Big) + \Big(\sum_{p \in P} \mu_p p\Big) = \sum_{p \in P} (\lambda_p + \mu_p) p,$$

$$\alpha\Big(\sum_{p \in P} \lambda_p p\Big) = \sum_{p \in P} (\alpha \lambda_p) p \quad (\alpha \in \mathbf{C}),$$

$$\Big(\sum_{p \in P} \lambda_p p\Big)\Big(\sum_{p \in P} \mu_p p\Big) = \sum_{p \in P} \Big(\sum_{p_1 p_2 = p} \lambda_{p_1} \mu_{p_2}\Big) p.$$

Similarly, let B be the l^1 algebra over the semigroup Q. Clearly A, B are unital Banach algebras, $A \subset B$. Furthermore, the element $u \in A \subset B$ is left invertible in B.

2. The proof is quite analogous. We use the semigroup Q' constructed in Lemma 3(ii) instead of Q.

3. Suppose on the contrary that there exists a Banach algebra $C \supset A$ such that $u \in A$ is (two-sided) invertible in C. Put $c = u^{-1}$. Then

$$\begin{aligned} 0 &= a_2 c a_4 (1 - uc) a_3 - a_2 (1 - cu) a_5 c a_3 + a_2 (1 - cu) a_6 - a_7 (1 - uc) a_3 \\ &= a_2 c a_4 a_3 - a_2 c a_4 u c a_3 - a_2 a_5 c a_3 + a_2 c u a_5 c a_3 \\ &\qquad + a_2 a_6 - a_2 c u a_6 - a_7 a_3 + a_7 u c a_3 \\ &= a_2 c a_{10} - a_2 c a_1 c a_3 - a_{11} c a_3 + a_2 c a_1 c a_3 + a_8 \\ &\qquad - a_2 c a_{10} - a_9 + a_{11} c a_3 = a_8 - a_9 \neq 0, \end{aligned}$$

a contradiction.

In [2], B. Bollobás asked the following two questions: Let T be a bounded linear operator on a Banach space X, i.e. $T\in B(X)$.

QUESTION 1. Does there exist a Banach space $Y\supset X$ and an operator $S\in B(Y)$ such that $S|X=T$ and $\sigma(S)$ is the essential spectrum of T?

QUESTION 2. Can one find a Banach space $Y\supset X$ and an isometrical algebra homomorphism $\varphi\colon B(X)\to B(Y)$ such that $\varphi(S)|X=S$ $(S\in B(X))$ and $\sigma(\varphi(T))$ is the essential spectrum of T?

(By the *essential spectrum* of T is meant the set $\{\lambda\in C\colon \inf\{\|(T-\lambda)x\|\colon x\in X,\ \|x\|=1\}=0\}$). We show that Question 1 has an affirmative answer while the answer to Question 2 is negative.

Let A be a Banach algebra and $a\in A$. Then $\tau_l^A(a)$ ($\tau_r^A(a)$) denotes the left (right) approximate point spectrum of $a\in A$, i.e. the set of all complex λ for which $a-\lambda$ is a left (right) topological divisor of zero in A. If $A=B(X)$ is the algebra of all bounded linear operators on a Banach space X and $T\in B(X)$ then

$$\tau_l^{B(X)}(T)=\{\lambda\colon \inf\{\|(T-\lambda)x\|\colon x\in X,\ \|x\|=1\}=0\},$$

$$\tau_r^{B(X)}(T)=\{\lambda\colon (T-\lambda)X\neq X\},$$

hence $\tau_l^{B(X)}(T)\cup\tau_r^{B(X)}(T)=\sigma^{B(X)}(T)$ (see [3]).

Let $\varphi\colon B(X)\to C$ be an isometrical algebra homomorphism from $B(X)$ to a Banach algebra C. Then $\tau_l^{B(X)}(T)\cup\tau_r^{B(X)}\subset\sigma^C(\varphi(T))\subset\sigma^{B(X)}(T)$, hence $\sigma^C(\varphi(T))=\sigma^{B(X)}(T)$ (the condition $\varphi(S)|X=S$, $S\in B(X)$, was not used). We have proved:

4. PROPOSITION. *Let φ be an isometrical algebra homomorphism $\varphi\colon B(X)\to C$, where X is a Banach space and C a Banach algebra. Then $\sigma(\varphi(T))=\sigma(T)$ for every $T\in B(X)$.*

Question 1 has an affirmative answer:

5. THEOREM. *Let T be a bounded linear operator acting on a Banach space X. Then there exists a Banach space $Y\supset X$ and an operator $S\in B(Y)$ such that $S|X=T$ and $\sigma(S)=\{\lambda\in C\colon \inf\{\|(T-\lambda)x\|\colon x\in X,\ \|x\|=1\}=0\}$.*

Proof. Let $\mathscr{A}\subset B(X)$ be the closed algebra generated by I, T and $(T-\lambda)^{-1}$ $(\lambda\notin\sigma(T))$. Then $\mathscr{A}$ is a commutative Banach algebra and $\sigma^{\mathscr{A}}(T)=\sigma^{B(X)}(T)$.

Let $\tilde{\mathscr{A}}=\mathscr{A}\oplus X$. Define a norm on $\tilde{\mathscr{A}}$ by

$$\|A\oplus x\|=\|A\|+\|x\|\qquad (A\in\mathscr{A},\ x\in X)$$

and a multiplication by

$$(A\oplus x)(A'\oplus x')=AA'\oplus(Ax'+A'x)\qquad (A,\ A'\in\mathscr{A},\ x,\ x'\in X).$$

Then $\tilde{\mathscr{A}}$ is a commutative Banach algebra.

The mapping $f_1\colon \mathscr{A}\to\tilde{\mathscr{A}}$ defined by $f_1(A)=A\oplus 0$ $(A\in\mathscr{A})$ is an isometrical algebra homomorphism so

$$\sigma^{\tilde{\mathscr{A}}}(T\oplus 0)\subset\sigma^{\mathscr{A}}(T)=\sigma^{B(X)}(T).$$

Let $\lambda\in C$. Define $d(T-\lambda)=\inf\{\|(T-\lambda)x\|\colon x\in X,\ \|x\|=1\}$. Clearly,

$$\inf\{\|((T-\lambda)\oplus 0)(A\oplus x)\|\colon A\in\mathscr{A},\ x\in X,\ \|A\oplus x\|=1\}$$
$$=\inf\{\|(T-\lambda)A\|+\|(T-\lambda)x\|\colon A\in\mathscr{A},\ x\in X,\ \|A\|+\|x\|=1\}\leqslant d(T-\lambda).$$

On the other hand, $\|(T-\lambda)A\|+\|(T-\lambda)x\|\geqslant d(T-\lambda)\|A\|+d(T-\lambda)\|x\|$ so

$$\inf\{\|((T-\lambda)\oplus 0)(A\oplus x)\|\colon A\in\mathscr{A},\ x\in X,\ \|A\oplus x\|=1\}=d(T-\lambda)$$

and $\tau^{\tilde{\mathscr{A}}}(T\oplus 0)=\{\lambda\colon d(T-\lambda)=0\}$.

By [5] there exists a Banach algebra $\mathscr{C}\supset\tilde{\mathscr{A}}$ such that $\sigma^{\mathscr{C}}(T\oplus 0)=\tau^{\tilde{\mathscr{A}}}(T\oplus 0)=\{\lambda\colon d(T-\lambda)=0\}$. Consider the operator $S\colon \mathscr{C}\to\mathscr{C}$ defined by $Sc=(T\oplus 0)c$ $(c\in\mathscr{C})$. Then clearly

$$\sigma^{B(\mathscr{C})}(S)\subset\sigma^{\mathscr{C}}(T\oplus 0)=\{\lambda\colon d(T-\lambda)=0\}.$$

Let $x\in X$, i.e. $0\oplus x\in\tilde{\mathscr{A}}\subset\mathscr{C}$. Then

$$S(0\oplus x)=(T\oplus 0)(0\oplus x)=0\oplus Tx.$$

If we identify $x\in X$ with $0\oplus x\in\tilde{\mathscr{A}}\subset\mathscr{C}$ then $S|X=T$ and $\sigma(S)=\{\lambda\colon d(T-\lambda)=0\}$.

References

[1] R. Arens, *Extensions of Banach algebras*, Pacific J. Math. 10 (1960), 1–16.

[2] B. Bollobás, *To what extent can the spectrum of an operator be diminished under an extension*, in: Linear and Complex Analysis Problem Book, V. P. Havin, S. V. Hruščev and N. K. Nikol'skij (eds.), Lecture Notes in Math. 1043, Springer, Berlin 1984, 210.

[3] R. Harte, *Spectral mapping theorems*, Proc. Roy. Irish Acad. Sect. A 72 (1972), 89–107.

[4] E. S. Lyapin, *Semigroups*, Gos. Izdat. Fiz.-Mat. Literat., Moscow 1960 (in Russian).

[5] C. J. Read, *Inverse producing extensions of a Banach algebra which eliminates the residual spectrum of one element*, Trans. Amer. Math. Soc. 286 (1986), 715–725.

[6] W. Żelazko, *On Ideal Theory in Banach and Topological Algebras*, Monograf. Inst. Mat. 15, Univ. Nac. Autónoma Mexico, 1984.

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