[7] S. J. L. van Eijndhoven and P. Kruszyński, *Spectral trajectories and duality*, EUT-Report, Eindhoven University of Technology, 1986.
[8] J. de Graaf, *A theory of generalized functions based on holomorphic semigroups*, Nederl. Akad. Wetensch. Proc. Ser. A 86 (4) (1983), 407–420; 87 (2) (1984), 173–187.
[9] P. Kruszyński, *Orthogonally scattered measures on relatively orthocomplemented lattices*, preprint, Univ. of Toronto, 1985.
[10] P. Masani, *Orthogonally scattered measures*, Adv. in Math. 2 (1968), 61–177.
[11] H. Schaefer, *Topological Vector Spaces*, Springer, New York 1971.

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCES
EINDHOVEN UNIVERSITY OF TECHNOLOGY
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

and

DEPARTMENT OF MATHEMATICS AND INFORMATICS
DELFT UNIVERSITY OF TECHNOLOGY
P.O. Box 356, 2600 AJ Delft, The Netherlands

Received December 30, 1986 (2265)

Added in proof (March 1988). In a recent paper by A. F. M. ter Elst, *On the connection between a symmetry condition and several nice properties of the space $S_{\Phi(A)}$ and $T_{\Phi(A)}$*, preprint, Eindhoven University of Technology, 1987, it is proved that Assumption III (3.2) is equivalent to a lot of topological properties of the spaces $T_{\Phi(A)}$ and $S_{\Phi(A)}$ constructed in [4].

STUDIA MATHEMATICA, T. XCI. (1988)

Geometrical properties of Banach spaces and the distribution of the norm for a stable measure

by

MICHAŁ RYZNAR (Wrocław)

Abstract. Let μ be a symmetric p-stable measure, $0 < p < 1$, on a locally convex separable metric linear space E and let q be a lower semicontinuous seminorm on E which is finite μ-a.s. We prove that the density of $F(t) = \mu\{q < t\}$ is bounded. If $1 \leqslant p < 2$ and (E, q) is a Banach space containing l_p^n's uniformly, then for every $\eta > 1$ we find a symmetric p-stable measure on E and a norm $\tilde{q}$ which is η-equivalent to the norm q such that the density of $F(t) = \mu\{\tilde{q} < t\}$ is unbounded.

1. Let μ be a symmetric p-stable measure, $0 < p \leqslant 2$, on a locally convex separable metric linear space E, with a measurable seminorm q. Then the distribution function $F(t) = \mu\{q < t\}$ is absolutely continuous apart from a possible jump (for $p = 2$, i.e., for the Gaussian case see [3], and for $0 < p < 2$, see [2]).

In this note we examine whether the density of $F(t)$ is bounded. This information is very essential to estimate the rate of convergence in CLT. It is well known that if E is a Hilbert space and q is the standard Hilbertian norm then, in the Gaussian case, the density is bounded [6]. However, as was shown by Rhee and Talagrand [14], a small change of the Hilbertian norm may spoil the boundedness. This result was recently generalized to all separable Banach spaces by Rhee [13]. She proved that for any infinite-dimensional Banach space (E, q) and every $\eta > 1$ there exists a new norm $\tilde{q}$ which is η-equivalent to q and a Gaussian measure μ such that the density of the μ-distribution of $\tilde{q}$ is unbounded.

In the first part of this note we consider the case of symmetric p-stable measures, $0 < p < 1$. Applying the explicit formula for the density proved in [8] we show that it is bounded whenever q is lower semicontinuous.

For $1 \leqslant p < 2$ we constructed in [15] some examples of p-stable measures μ on c_0 such that the density of $F(t) = \mu\{\|\cdot\|_\infty < t\}$ is unbounded. In this note we provide such examples in Banach spaces in which l^p is finitely representable. If (E, q) is a Banach space of this type, then for every function $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ with $\lim_{t\to 0^+} f(t) = \infty$, we are able to find an equivalent norm $\tilde{q}$, and a symmetric p-stable measure μ such that the density $F'(t)$ of $F(t)$

$= \mu\{\tilde{q} < t\}$ is unbounded, and moreover $F'(t_j) \geq f(t_j)$ for some sequence $t_j \to 0^+$. Also, it turns out that the last property characterizes Banach spaces in which l^p is finitely representable. To carry out our construction we modify some arguments of Rhee [13].

2. Let E be a locally convex separable linear metric space. By q we denote a measurable seminorm on E, i.e. a measurable function $q: E \to [0, \infty]$ such that $q(x+y) \leq q(x)+q(y)$ and $q(\alpha x) = |\alpha| q(x)$ for all $x, y \in E$ and all $\alpha \in \boldsymbol{R}$.

A probability measure μ on E is called *p-stable*, $0 < p < 2$, if for any independent random vectors X, Y with distribution μ and for all $\alpha, \beta > 0$ with $\alpha^p + \beta^p = 1$, the distribution of $\alpha X + \beta Y$, after a suitable translation, is identical with μ. If μ is a symmetric measure then there exists a symmetric σ-finite measure ν on E with $\nu(U^c) < \infty$ for every neighbourhood U of the origin, such that $\mu = \lim \exp(\nu|_{U_n^c})$ for $U_n \searrow \{0\}$. The measure ν is called the *Lévy measure* of μ. Suppose that q is a lower semicontinuous seminorm which is finite μ-a.s. Then $\sigma = p\nu\{q \geq 1\} < \infty$ and for every Borel subset A of $\boldsymbol{R}^+$ and every $\varepsilon > 0$ we have [2]

$$\nu|_{q>\varepsilon}\{q \in A\} = \sigma \int_{\varepsilon}^{\infty} \mathbf{1}_A t^{-(1+p)} dt. \tag{1}$$

Now, suppose that (E, q) is a Banach space. We say that it is of *Rademacher type r*, $1 \leq r \leq 2$, if there exists a positive constant K such that for all $x_1, \ldots, x_n \in E$

$$Eq^r\Big(\sum_{i=1}^{n} x_i r_i\Big) \leq K \sum_{i=1}^{n} q^r(x_i),$$

where $\{r_i\}$ is a Rademacher sequence. It is obvious that every Banach space is of Rademacher type 1. A theorem of Maurey and Pisier [11] and Krivine [5] states that a Banach space E is of Rademacher type r for some r, $2 \geq r > p \geq 1$, if and only if l^p is not finitely representable in E. We recall that l^p is *finitely representable* in (E, q) if for every $\varepsilon > 0$ and every $n \in \boldsymbol{N}$ one can find $x_1, \ldots, x_n \in E$ such that for all $\beta_1, \ldots, \beta_n \in \boldsymbol{R}$

$$\Big(\sum_{i=1}^{n} |\beta_i|^p\Big)^{1/p} \leq q\Big(\sum_{i=1}^{n} \beta_i x_i\Big) \leq (1+\varepsilon)\Big(\sum_{i=1}^{n} |\beta_i|^p\Big)^{1/p}.$$

The following result about the behaviour of densities of stable seminorms is taken from [15].

LEMMA 1. *Let (E, q) be a separable Banach space of Rademacher type r, and let μ be a symmetric p-stable measure on E, $0 < p < r \leq 2$. Then the density $F'(t)$ of $F(t) = \mu\{q < t\}$ is bounded on every halfline (t_0, ∞), $t_0 > 0$, and*

$$F'(t) = o(t^{-\alpha}) \quad \text{as } t \to 0, \tag{2}$$

where $\alpha > 1 + pr/(r-p)$. Moreover, if E is a locally convex separable metric vector space then the same is true (with $r = 1$) whenever $0 < p < 1$ and q is a lower semicontinuous seminorm which is finite μ-a.s.

Remark 2. Corollary 3.5 in [15] states that there exists $\alpha > 0$ such that (2) holds but if we put $t_0 = (1/2)t$ in formula (3.7) of [15] and analyse the behaviour of the function $R(t)$ defined before the formulation of Lemma 3.3 in [15] we see that α can be taken as in Lemma 1.

We will also need one more result [8]:

LEMMA 3. *Let E be a locally convex separable linear metric space and let μ be a symmetric p-stable measure on E, $0 < p < 1$. If q is a lower semicontinuous seminorm which is finite μ-a.s. then the density $F'(t)$ of $F(t) = \mu\{q < t\}$ exists for all $t > 0$ and*

$$F'(t) = (p/t) \int_E \big(\mu(U_t) - \mu(U_t + x)\big) \nu(dx), \tag{3}$$

where $U_t = \{q < t\}$ and ν is the Lévy measure of μ.

By $\{\theta_i\}$ we denote the standard p-stable sequence, i.e. the sequence of independent identically distributed random variables with the characteristic function $\exp(-|t|^p)$.

Let (E, q) and (F, q_1) be two Banach spaces. For $\alpha > 1$, a linear isomorphism T from E to F is called an *α-isomorphism* if for $x \in E$ we have $q(x)/\alpha \leq q_1(Tx) \leq \alpha q(x)$. We say that two norms q, $\tilde{q}$ on E are α-equivalent if the identity is an α-isomorphism from (E, q) to $(E, \tilde{q})$.

3. In this section we prove the boundedness of the density for $0 < p < 1$. Let E be a locally convex separable linear metric space with a lower semicontinuous seminorm q. Let μ be a symmetric p-stable measure, $0 < p < 1$, such that $q < \infty$ μ-a.s. Now, we are ready to prove

PROPOSITION 4. *Suppose that the linear span of $\operatorname{supp}\mu \cap \{q > 0\}$ is infinite-dimensional. Then the density $F'(t)$ of $F(t) = \mu\{q < t\}$ is bounded on $\boldsymbol{R}^+$ and for every $n \in \boldsymbol{N}$, $F'(t) = o(t^n)$ as $t \to 0$.*

Proof. In view of Lemma 1 we can assume that $0 < t < 1/2$. By (2) there exist $\alpha > (1-p)^{-1}$ and $M_\alpha > 0$ such that

$$F'(t) < M_\alpha t^{-\alpha}, \quad 0 < t < 1/2. \tag{4}$$

Using formula (3) we have for every positive integer m

$$F'(t) = (p/t) \int_{q \leqslant t^m} (\mu(U_t) - \mu(U_t + x))\, \nu(dx) + (p/t) \int_{q > t^m} (\mu(U_t) - \mu(U_t + x))\, \nu(dx) = I_1 + I_2.$$

Assume that $m \geqslant 2$ is fixed; we specify it later. When $q(x) \leqslant t^m < (1/2)t$ we obtain by (4)

$$\mu(U_t) - \mu(U_t + x) \leqslant \mu(U_t) - \mu(U_{t-q(x)}) = \int_{t-q(x)}^{t} F'(s)\, ds \leqslant 2^\alpha M_\alpha t^{-\alpha} q(x).$$

Therefore by (1) the following estimate holds for I_1:

(5) $$I_1 \leqslant 2^\alpha M_\alpha p t^{-(\alpha+1)} \int_{q \leqslant t^m} q(x)\, \nu(dx) = 2^\alpha M_\alpha p \sigma t^{-(\alpha+1)} \int_0^{t^m} r^{-p}\, dr = \sigma 2^\alpha M_\alpha p (1-p)^{-1} t^{m(1-p)-(\alpha+1)},$$

where $\sigma = p\nu\{q \geqslant 1\}$.

Applying once more (1) we get

(6) $$I_2 \leqslant (p/t)\mu(U_t)\, \nu\{q > t^m\} = p\sigma F(t) t^{-mp-1}.$$

By the result of de Acosta [1], $F(t) = o(t^k)$, $t \to 0$, for every $k \in N$, hence taking m sufficiently large the conclusion follows from (5) and (6).

Remark 5. Denote by E_1 the linear space spanned by $\operatorname{supp} \mu \cap \{q > 0\}$. If E_1 is n-dimensional then examining $F(t)$ we may assume that μ is concentrated on R^n and q is a norm on R^n. Denote by λ_n the Lebesgue measure on R^n. Then $\lambda_n\{q < t\} = \mathrm{const} \cdot t^n$. Since the density of μ with respect to λ_n is bounded, $F'(t) = O(t^{n-1})$ as $t \to 0$.

4. Suppose that (E, q) is a separable Banach space such that l^p, $1 \leqslant p < 2$, is finitely representable in E. In this section we find for any $\eta > 1$ a norm $\tilde{q}$ which is η-equivalent to the norm q and a symmetric p-stable measure μ on E such that $F(t) = \mu\{\tilde{q} < t\}$ has a density which is unbounded. In our approach we adopt methods developed in [13], where a similar result for Gaussian measures on Banach spaces was obtained by Rhee. Let us recall that by Dvoretzky's theorem l^2 is finitely representable in every infinite-dimensional Banach space.

Now we state two facts which are crucial for our construction. The first one is the following lemma which is a direct consequence of the Weak Law of Large Numbers (see also [16]).

LEMMA 6. *Let $\{\theta_i\}$ be a standard p-stable sequence, $1 \leqslant p < 2$. Then for every $a > b > 0$, every $\varepsilon > 0$, and every $n_0 > 0$, there exists a positive integer $n > n_0$ and a positive number m satisfying the condition*

$$P\{a < \|(1/m) \sum_{i=1}^{n} e_i \theta_i\|_p < b\} > 1 - \varepsilon,$$

where $\{e_i\}$ is the standard basis in l^p and $\|\cdot\|_p$ is the standard norm on l^p.

The second fact is the following Banach space result:

PROPOSITION 7. *Let (E, q) be a Banach space such that l^p is finitely representable in E, $1 \leqslant p < 2$. Let F be a finite-dimensional subspace of E, and let $\tau > 1$, $n \in N$, $n > \dim F$. Then there is an n-dimensional subspace G of E which is τ-isomorphic to l_n^p and for $x \in G$, $y \in F$ we have $q(x) \leqslant \tau q(x+y)$.*

Remark 8. The above proposition for $p = 2$ was proved by Rhee [13] with the help of Dvoretzky's theorem. In our proof of Proposition 7 we use the ideas of Pisier [12], where he applied random methods to select subspaces of a Banach space which are very close to l_n^p.

Before proving Proposition 7 we recall some facts needed in the proof. Let $\{\alpha_i\}$ be a sequence of i.i.d. exponential random variables, i.e. $P\{\alpha_i > \lambda\} = e^{-\lambda}$ for any $\lambda > 0$. Write $\Gamma_j = \sum_{i=1}^{j} \alpha_i$. The next lemma is a special case of the series representation of stable vectors in Banach spaces (for details, see [10] or [7]).

LEMMA 9. *Assume that $0 < p < 2$ and that $\{\theta_i\}$ is a standard p-stable sequence. Let E be a Banach space, and let $x_1, \ldots, x_n \in E$. Then there is a number $C_p > 0$ depending only on p such that*

$$n^{-1/p} \sum_{i=1}^{n} \theta_i x_i$$

has the same distribution as

$$C_p \sum_{j=1}^{\infty} \Gamma_j^{-1/p} V_j$$

where $\{V_j\}$ is a sequence of i.i.d. random vectors independent of the sequence $\{\alpha_j\}$, and the distribution of V_1 is $(1/(2n)) \sum_{i=1}^{n} (\delta_{x_i} + \delta_{-x_i})$.

Now, we state some basic inequalities from Pisier's work [12]. Note that they were established in a more explicit form than the one presented here, but the latter is sufficient for our purposes.

LEMMA 10. *Let (E, q) be a Banach space and let $1 \leqslant p < 2$. Assume that $\{V_i\}$, $\{\Gamma_i\}$ are two sequences with the properties as above with $q(V_i) \leqslant 1$. Write*

$$A_p = \left(Eq^r\left(\sum_{i=1}^{\infty} \Gamma_i^{-1/p} V_i\right)\right)^{1/r},$$

where $r = 1/2$ *for* $p = 1$, *and* $r = 1$ *for* $p > 1$. *There exist two functions* $\psi_p(\delta, s)$, $\varphi_p(\delta, s)$ *depending only on p, and a sequence* $\{X_i\}$ *of E-valued random vectors depending only on* $\{V_i\}$ *such that:*

(i) *For* $0 < \delta < 1/2$, $\lim_{s\to\infty}\psi_p(\delta, s) = \infty$ *and* $\lim_{s\to\infty}\varphi_p(\delta, s) = 0$.

(ii) *If* $k \leqslant \psi_p(\delta, A_p)$ *then for* $\beta = (\beta_1, \ldots, \beta_k) \in R^k$, $\|\beta\|_p = 1$,

$$P\{|q(\sum_{i=1}^{k} \beta_i X_i) - A_p| > \delta A_p\} \leqslant \varphi_p(\delta, A_p).$$

Proof of Proposition 7. Let $0 < \delta < 1/2$ be fixed. Let H be a complement of F. We introduce a new norm $\tilde q$ on H: $\tilde q(x) = \inf\{q(x+y)\colon y \in F\}$. Of course, q and $\tilde q$ are equivalent on H, so $\eta q \leqslant \tilde q \leqslant q$ for some $0 < \eta < 1$. Let L be a finite δ-net on the unit sphere of l_n^p. Then by Lemma 10 we can choose $s_0 > 0$ so large that for $s \geqslant s_0$

$$n \leqslant \psi_p(\delta, s), \quad \operatorname{card} L\cdot\varphi_p(\delta, s) < 1/2. \tag{7}$$

Now, consider a p-stable random vector in l^p of the form

$$\frac{1}{2m^{1/p}} \sum_{i=1}^{m} \theta_i e_i = Y_m.$$

Since $E\big((1/k)\sum_{i=1}^{k}|\theta_i|^p\big)^{1/p} \to \infty$ if $p > 1$ and $E\big((1/k)\sum_{i=1}^{k}|\theta_i|\big)^{1/2} \to \infty$ if $p = 1$, it follows that for some $m \in N$

$$(E\|Y_m\|_p^r)^{1/r} \geqslant 2\eta^{-1} s_0 C_p.$$

Next, let us note that l^p is finitely representable in (H, q). Therefore we can pick $x_1, \ldots, x_m \in H$ such that

$$q(x_i) \leqslant 1, \quad 1 \leqslant i \leqslant m, \quad \big(Eq^r(m^{-1/p}\sum_{i=1}^{m} x_i\theta_i)\big)^{1/r} \geqslant \eta^{-1} C_p s_0.$$

Now, let $\{V_i\}$ and $\{\Gamma_i\}$ be as in Lemma 9. Then

$$q(V_i) \leqslant 1, \quad A_p = \big(Eq^r(\sum_{i=1}^{\infty} \Gamma_i^{-1/p} V_i)\big)^{1/r} \geqslant \eta^{-1} s_0 \geqslant s_0, \tag{8}$$

and consequently

$$\tilde q(V_i) \leqslant 1, \quad \tilde A_p = \big(E\tilde q^r(\sum_{i=1}^{\infty} \Gamma_i^{-1/p} V_i)\big)^{1/r} \geqslant s_0. \tag{9}$$

Therefore we can apply Lemma 10 for both norms q and $\tilde q$, and for $\beta = (\beta_1, \ldots, \beta_n) \in L$ we get

$$P\{|q(\sum_{i=1}^{n} \beta_i X_i) - A_p| > \delta A_p\} \leqslant \varphi_p(\delta, A_p),$$

$$P\{|\tilde q(\sum_{i=1}^{n} \beta_i X_i) - \tilde A_p| > \delta \tilde A_p\} \leqslant \varphi_p(\delta, \tilde A_p).$$

The last two inequalities together with (7)–(9) imply

$$P\{\sup_{\beta\in L}|q(\sum_{i=1}^{n} \beta_i X_i) - A_p| > \delta A_p\} \leqslant \operatorname{card} L\cdot\varphi_p(\delta, A_p) < \tfrac12,$$

$$P\{\sup_{\beta\in L}|q(\sum_{i=1}^{n} \beta_i X_i) - \tilde A_p| > \delta \tilde A_p\} < \tfrac12.$$

Writing $c = A_p/\tilde A_p$ we infer that there is ω such that for any $\beta \in L$

$$1-\delta \leqslant q\big(\sum_{i=1}^{n} \beta_i (X_i(\omega)/A_p)\big) \leqslant 1+\delta,$$

$$1-\delta \leqslant c\tilde q\big(\sum_{i=1}^{n} \beta_i (X_i(\omega)/A_p)\big) \leqslant 1+\delta.$$

By the well-known argument (cf. e.g. [4], Lemma 2.5) we can extend the above inequalities to the whole unit sphere of l_n^p. Namely, there exist $\varepsilon(\delta) > 0$ with $\varepsilon(\delta) \to 0$ as $\delta \to 0$ such that for any $\beta \in l_n^p$

$$(1+\varepsilon)^{-1}\|\beta\|_p \leqslant q(\sum_{i=1}^{n} \beta_i x_i) \leqslant (1+\varepsilon)\|\beta\|_p, \tag{10}$$

$$(1+\varepsilon)^{-1}\|\beta\|_p \leqslant c\tilde q(\sum_{i=1}^{n} \beta_i x_i) \leqslant (1+\varepsilon)\|\beta\|_p, \tag{11}$$

where $x_i = X_i(\omega)/A_p$.

Let $G = \operatorname{span}\{x_i\colon i = 1, \ldots, n\}$. The inequality (10) states that (G, q) is $(1+\varepsilon)$-isomorphic to l_n^p. Now, we estimate the constant c. Since $\dim G > \dim F$ it follows from [9], Lemma 2.8 C, that there exists $x_0 \in G$ with $q(x_0) = \tilde q(x_0) = 1$. Hence by (10) and (11) we get $c \leqslant (1+\varepsilon)^2$ and finally

$$q(x) \leqslant (1+\varepsilon)^4 \tilde q(x), \quad \text{for all } x \in G.$$

By the choice of an appropriately small δ we obtain the conclusion.

Now, we are ready to formulate and prove the main result of this section.

THEOREM 11. *Let* (E, q) *be a separable Banach space, and let* $1 \leqslant p < 2$. *The following conditions are equivalent:*

(i) l^p *is finitely representable in E.*

(ii) *For every* $\eta > 1$, *and every function* $f\colon R^+ \to R^+$, *there exist a symmetric p-stable measure* μ *on E, a new norm* $\tilde q$ *on E which is* η*-equivalent to q, and two sequences of positive numbers* a_j, $\delta_j \to 0$ *as* $j \to \infty$, *such that*

$$\mu\{a_j \leqslant \tilde q < a_j + \delta_j\} > f(a_j)\delta_j, \quad j \geqslant 1. \tag{12}$$

It is worth while to notice that if we take f such that $\lim_{t\to 0^+} f(t) = \infty$, then (12) yields that the density of $F(t) = \mu\{\tilde q < t\}$ is unbounded.

Proof. (i) $\Rightarrow$ (ii). Let $\{\beta_k\}$ be a sequence with $\beta_k > 1$ and $\prod_{k\geq 1}\beta_k < \eta < 2$. By induction on k we shall construct subsets B_k of E, positive integers n_k, p-stable symmetric measures ϱ_k on E, and two positive sequences a_k, $\delta_k \leq 2^{-k-1}$ satisfying the following conditions:

(A) B_k is convex and balanced; B_1 is the unit ball of (E, q), and for $k \geq 2$, $B_{k-1} \subset B_k \subset \beta_k B_{k-1}$.

(B) ϱ_k is supported in a finite-dimensional subspace of E and $\varrho_k\{q \geq 2^{-k+1}\} < 2^{-k}$, for $k \geq 1$.

(C) If $\mu_k = \varrho_1 * \ldots * \varrho_k$ and q_k is the Minkowski functional of B_k, then we have

(13) $$\mu_k\{a_j \leq q_k < a_j+\delta_j\} > f(a_j)\delta_j, \quad 1 \leq j \leq k.$$

We begin with the first step of the construction. Let $a_1 = 1/4$, and let $0 < \delta_1 < 1/4$ be such that $f(a_1)\delta_1 < 1$. If $\alpha > 1$ is close enough to 1, then by Lemma 6 there exist n_1 and a symmetric p-stable measure ν_1 on $l^p_{n_1}$ such that

(14) $$\nu_1\{a_1\alpha \leq \|\cdot\|_p < \alpha^{-1}(a_1+\delta_1)\} > \max\{f(a_1)\delta_1, 2^{-1}\}.$$

Since l^p is finitely representable in E, there exist a subspace G of E and an α-isomorphism T from $l^p_{n_1}$ to G. Therefore, if we take ϱ_1 to be the image of the measure ν_1 by T, we have from (14)

$$\varrho_1\{a_1 \leq q < a_1+\delta_1\} > f(a_1)\delta_1, \quad \varrho_1\{q > 2^{-1}\} \leq 2^{-1}.$$

Since $q = q_1$, this completes the first step of the construction.

Now, assume that we have carried out our construction up to step k. There exist positive numbers α, b satisfying

(15) $$\mu_k\{(a_j+b)\alpha \leq q_k < a_j+\delta_j-b\} > f(a_j)\delta_j(1+b), \quad 1 \leq j \leq k.$$

We can also assume that $1 < \alpha < \beta_k$ and $b \leq 2^{-k-1}$. Let us choose a_{k+1}, $\delta_{k+1} > 0$ such that $a_{k+1}+\delta_{k+1} < b/2$ and

(16) $$\mu_k\{q < (\alpha-1)2^{-1}a_{k+1}\} > 2f(a_{k+1})\delta_{k+1}.$$

Let $\tau = ((\alpha+1)/2)^{1/3}$. If $F = \operatorname{supp}\mu_k$, then F is finite-dimensional. By Lemma 6 there exist $n_{k+1} > \dim F$ and a symmetric p-stable measure ν_{k+1} on $l^p_{n_{k+1}}$ such that

(17) $$\nu_{k+1}\{a_{k+1} \leq \tau^{-2}\|\cdot\|_p < a_{k+1}+\delta_{k+1}\} > 1/(1+b).$$

To construct the set B_{k+1} we repeat the reasoning of Rhee [13]. We apply Proposition 7 for (E, q_k), so there exist an n_{k+1}-dimensional subspace G of E and a τ-isomorphism T from $l^p_{n_{k+1}}$ to G such that for $x \in G$, $y \in F$ we have

(18) $$q_k(x) \leq \tau q_k(x+y).$$

We define B_{k+1} as the closed convex hull of the set

$$B_k \cup \{x+y\colon x \in G,\ y \in F,\ \|T^{-1}x\|_p = \tau^2,\ q(y) \leq (\alpha-1)/2\}.$$

It is rather easy to notice that

(19) $$B_k \subset B_{k+1} \subset \alpha B_k.$$

Now, we use the following property which follows from (18) and can be proved in exactly the same way as Fact in Construction in [13]:

(20) $$q_{k+1}(x+y) = \tau^{-2}\|T^{-1}x\|_p \quad \text{provided } x \in G,\ y \in F \text{ and } q(y) \leq ((\alpha-1)/2)\tau^{-2}\|T^{-1}x\|_p.$$

Next, if we choose the measure ϱ_{k+1} as the image of ν_{k+1} by the isomorphism T, we can restate (17) as

(21) $$\varrho_{k+1}\{a_{k+1} \leq \tau^{-2}\|T^{-1}x\|_p < a_{k+1}+\delta_{k+1}\} > 1/(1+b).$$

Since $\tau^3 \leq 2$ and $a_{k+1}+\delta_{k+1} < b/2 < 2^{-k-2}$ the inequality (21) implies

(22) $$\varrho_{k+1}\{\|T^{-1}x\|_p \leq b\tau^{-1}\} > 1/(1+b),$$

(23) $$\varrho_{k+1}\{q_k \geq 2^{-k-1}\} \leq 2^{-k-1}.$$

Let us now assume that $1 \leq j \leq k$ and define

$$A_j = \{y \in F\colon \alpha(a_j+b) \leq q_k(y) < a_j+\delta_j-b\},$$
$$B = \{x \in G\colon \tau\|T^{-1}x\|_p \leq b\},$$
$$C_j = \{z \in E\colon a_j \leq q_{k+1}(z) < a_j+\delta_j\},$$
$$D = \{y \in F\colon q(y) \leq ((\alpha-1)/2)a_{k+1}\},$$
$$\tilde{D} = \{x \in G\colon a_{k+1} \leq \tau^{-2}\|T^{-1}x\|_p < a_{k+1}+\delta_{k+1}\}.$$

For $y \in A_j$ and $x \in B$, since $q_{k+1}(x) \leq q_k(x) \leq b$, we have by (19)

$$q_{k+1}(x+y) \leq q_k(x)+q_k(y) < a_j+\delta_j,$$
$$q_{k+1}(x+y) \geq q_{k+1}(y)-q_k(x) \geq \alpha^{-1}q_k(y)-q_k(x) \geq a_j.$$

These two inequalities together with (15) and (22) give

$$\mu_{k+1}(C_j) = \mu_k * \varrho_{k+1}(C_j) \geq \mu_k(A_j)\varrho_{k+1}(B) > f(a_j)\delta_j.$$

We now suppose that $y \in D$ and $x \in \tilde{D}$. Then we have $q(y) \leq ((\alpha-1)/2)\tau^{-2}\|T^{-1}x\|_p$ and (20) implies that $q_{k+1}(x+y) = \tau^{-2}\|T^{-1}x\|_p$. Therefore by virtue of (16) and (21) we have

$$\mu_{k+1}\{a_{k+1} \leq q_{k+1} < a_{k+1}+\delta_{k+1}\} \geq \mu_k(D)\varrho_{k+1}(\tilde{D}) \geq 2(1+b)^{-1}f(a_{k+1})\delta_{k+1} > f(a_{k+1})\delta_{k+1}.$$

This completes our construction.

Let X_k be a sequence of independent random vectors and suppose each X_k has distribution ϱ_k. Then by (23) the series $\sum_{k\geqslant 1} X_k$ is convergent a.s. to a symmetric p-stable random vector S. Denote by μ the distribution of S. Of course $\mu = \lim_{k\to\infty} \mu_k$. If $\tilde{q} = \lim_{k\to\infty} q_k$ then from (A) we have $\eta^{-1} q \leqslant \tilde{q} \leqslant q$. It is also easy to notice that $\lim_{k\to\infty} q_k(S_k) = q(S)$, where $S_k = \sum_{i=1}^{k} X_i$. This fact together with (13) implies (12). The proof of (i) $\Rightarrow$ (ii) is complete.

(ii) $\Rightarrow$ (i). If l^p is not finitely representable in (E, q), then one can find r, $r > p$, such that (E, q) is of Rademacher type r. Let $\tilde{q}$ be any norm on E equivalent to q. It is clear that $(E, \tilde{q})$ is again of Rademacher type r. If μ is any symmetric p-stable measure on E then by Lemma 1 the density of the μ-distribution of $\tilde{q}$ is $o(t^{-\alpha})$ as $t\to 0$, for any $\alpha > 1+rp/(r-p)$. Therefore, the renorming like in (ii) is impossible for E.

Until now we do not know of any example of a Banach space (E, q) of Rademacher type r, $p < r \leqslant 2$, and a symmetric p-stable measure μ on E with the property that the density of $F(t) = \mu\{q < t\}$ is unbounded. In view of the preceding theorem we conjecture that this is not possible. Since any Banach space is of Rademacher type 1, Proposition 4 says that our conjecture is valid for $p < 1$.

References

[1] A. de Acosta, *Stable measures and seminorms*, Ann. Probab. 3 (1975), 865–875.
[2] T. Byczkowski and K. Samotij, *Absolute continuity of stable seminorms*, ibid. 14 (1986), 299–312.
[3] V. S. Cirelson, *The density of the distribution of the maximum of a Gaussian process*, Theor. Probab. Appl. 20 (1975), 847–855.
[4] T. Figiel, J. Lindenstrauss and V. D. Milman, *The dimensions of almost spherical sections of convex bodies*, Acta Math. 139 (1977), 53–94.
[5] J. L. Krivine, *Sous-espaces de dimension finie des espaces de Banach réticulés*, Ann. of Math. 104 (1976), 1–29.
[6] J. Kuelbs and T. Kurtz, *Berry–Essen estimates in Hilbert space and an application to the law of iterated logarithm*, Ann. Probab. 2 (1974), 387–407.
[7] R. LePage, M. Woodroofe and J. Zinn, *Convergence to a stable distribution via order statistics*, ibid. 9 (1981), 624–632.
[8] M. Lewandowski and T. Żak, *On the density of the distribution of p-stable seminorms*, $0 < p < 1$, Proc. Amer. Math. Soc. 100 (1987), 345–351.
[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, vol. 1, Springer, 1977.
[10] M. B. Marcus and G. Pisier, *Characterisation of almost surely continuous p-stable random Fourier series and strongly stationary processes*, Acta Math. 152 (1984), 245–301.
[11] B. Maurey et G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et géométrie des espaces de Banach*, Studia Math. 58 (1976), 45–90.
[12] G. Pisier, *On the dimension of the l_p^n-subspaces of Banach spaces, for* $1 \leqslant p < 2$, Trans. Amer. Math. Soc. 276 (1983), 201–211.
[13] W. Rhee, *On the distribution of the norm for a Gaussian measure*, Ann. Inst. H. Poincaré Probab. Statist. 20 (1984), 277–286.

[14] W. Rhee and M. Talagrand, *Bad rates of convergence for the central limit theorem in Hilbert space*, Ann. Probab. 12 (1984), 843–850.
[15] M. Ryznar, *Density of stable seminorms*, Bull. Polish Acad. Sci. Math. 33 (1985), 431–440.
[16] R. Sztencel, *On the lower tail of stable seminorms*, ibid. 32 (1984), 715–719.

INSTYTUT MATEMATYKI POLITECHNIKI WROCŁAWSKIEJ
INSTITUTE OF MATHEMATICS, WROCLAW TECHNICAL UNIVERSITY
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

Received April 4, 1987 (2294)