STUDIA MATHEMATICA, T. XCI. (1988)

Spectral trajectories, duality and inductive-projective limits of Hilbert spaces

by

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Abstract. An explicit representation of the topological dual of the inductive limit space $\mathcal{S}_{\mathcal{R}}$, generated by a colllection $\mathcal{R}$ of s.a. operators, has been found in the form of a space of spectral trajectories, i.e., vector-valued measures with the orthogonal scattering property. This paper is a continuation of [5] completing the previous theory. Illustrations of this type of spaces can be derived from distribution theory and Gel'fand triples theory. At the end of Section 5 we give a short summary on these matters.

Introduction. Let H be a separable Hilbert space. Let Σ be a semiring of Borel sets in a locally compact topological Hausdorff space Λ, and let $\sigma(\Sigma)$ be the σ-algebra generated by Σ. In general, $\sigma(\Sigma)$ is essentially smaller than the field of Borel sets in Λ. As a model may serve the σ-algebra of Borel sets generated by cylinders in a Tikhonov product space $\prod \Lambda_\alpha$, $\alpha \in I$. Let E be defined on $\sigma(\Sigma)$ as a projection-valued σ-additive set function. Examples of such a "spectral measure" can be obtained from the joint spectral measure of a strongly commuting family of (unbounded) s.a. operators in a Hilbert space (cf. [1]). In particular, we shall consider generating families of operators and their strong commutants in the sense of [5], and in the sense of commutative group theory.

The basic notion of this paper is the notion of "spectral trajectory" (controlled by a projection-valued measure E), i.e., an H-valued set function ξ defined on the semiring Σ such that for any $\Delta, \Delta' \in \Sigma$

$$E(\Delta)\xi(\Delta') = \xi(\Delta \cap \Delta'). \tag{0.1}$$

For a given measure E the collection of all E-trajectories forms a linear space over C^1. Moreover, this space is a maximal biorthogonal family of countably additive orthogonally scattered measures over Λ and its unique propagating spectral measure coincides with E (cf. [9]).

In our previous paper we have constructed an inductive limit $\mathcal{S}_{\mathcal{R}}$ of Hilbert spaces, originating from the generating family $\mathcal{R}$ of bounded s.a. positive operators in H. In the present paper we characterize the topological

AMS Subject Classification: 46F05, 46F10, 47D40.

dual of the space $\mathcal{S}_{\mathcal{R}}$ in terms of spectral trajectories controlled by the joint spectral measure E of the family $\mathcal{R}$. However, some of the results are interesting on their own. So we begin with a slightly more general setting.

The idea to represent the dual space of a given topological vector space $\mathcal{S}_{\mathcal{R}}$ by means of a space of measures (vector measures in this case) is in fact suggested by the Riesz representation theorem for continuous functionals over Banach spaces of continuous functions.

We believe that the present approach provides an interesting global point of view on the notion of duality in the theory of generalized function spaces.

The main reference is our paper [5], some technicalities come from [4], [9], and [6]. An extensive study of the theory and its bacground can be found in the book [3]. We use freely the general results and facts from the theory of locally convex topological vector spaces, for which the monograph [11] is the most appropriate reference.

The paper is organized as follows.

In Section 1 the basic notion of A-bounded trajectory is introduced and its basic properties are discussed. In Section 2 the essential decomposition of an A-bounded E-trajectory is proved, also assumptions are formulated which link the present approach to our previous results on $\mathcal{S}_{\mathcal{R}}$ spaces [5]. Section 4 contains considerations on topological duality between the space of $\mathcal{R}$-bounded trajectories and the topological dual to the space $\mathcal{S}_{\mathcal{R}}$. There the main result of the paper is formulated establishing a topological identification between the space $T_{\mathcal{R}}$ and the topological dual of the space $\mathcal{S}_{\mathcal{R}}$. It is closely related to distribution theory and Gel'fand triples discussed in [4–8]. Examples are presented in Section 5.

1. A-Bounded trajectories. Let us recall the definition of spectral trajectory (cf. [4]).

Let H be a separable Hilbert space, and let Σ be a semiring of Borel subsets of a locally compact topological (Hausdorff) space Λ, i.e., a family of sets which is closed under the operation of finite intersection and which is such that the monotone difference of any two of its members is a finite disjoint union of elements of Σ. Let $\sigma(\Sigma)$ denote the σ-algebra generated by Σ and let $E\colon \sigma(\Sigma)\to \operatorname{Proj} H$ be a projection-valued σ-additive measure over $\sigma(\Sigma)$. Without loss of generality, and in the light of applications, we can assume that the set $\{E(\Delta)x\colon \Delta\in\Sigma,\ x\in H\}$ is dense in H.

Definition 1.1. A set function $\xi\colon \Sigma\to H$ is called a *spectral trajectory controlled* (or *propagated*) *by the spectral measure* E if for any $\Delta,\ \Delta'\in\Sigma$

$$E(\Delta)\xi(\Delta') = \xi(\Delta\cap\Delta'). \tag{1.2}$$

For brevity's sake we shall say that ξ is an "E-trajectory". The collection of all E-trajectories will be denoted by M_E.

The simplest example of an E-trajectory is

$$\xi_x(\Delta) = E(\Delta)x \quad \text{with } x\in H. \tag{1.3}$$

Observe that the spectral trajectories thus defined are precisely the countably additive orthogonally scattered (c.a.o.s.) measures in the sense of P. Masani [10], i.e., for any $\Delta,\ \Delta'\in\Sigma$ such that $\Delta\cap\Delta' = \emptyset$, $\xi(\Delta)\perp\xi(\Delta')$, and $\xi(\bigcup_i\Delta_i) = \sum_i\xi(\Delta_i)$ whenever $\Delta_i\cap\Delta_j=\emptyset$ for $i\neq j$ and $\bigcup\Delta_i\in\Sigma$. Moreover, M_E is a maximal biorthogonal family of c.a.o.s. measures, i.e., for any $\xi,\ \xi'\in M_E$ and $\Delta,\ \Delta'\in\Sigma$ with $\Delta\cap\Delta'=\emptyset$, we have $\xi(\Delta)\perp\xi'(\Delta')$. However, our terminology (i.e., spectral trajectory) is justified by the explicit use of the spectral measure E.

A spectral trajectory is called *bounded* if $\sup\{\|\xi(\Delta)\|\colon \Delta\in\Sigma\}<\infty$.

Definition 1.4. Let A be a normal (not necessarily bounded) operator in H, commuting strongly with $E(\Delta)$, $\Delta\in\Sigma$. An E-trajectory ξ is called *A-bounded* if:

1) For any $\Delta\in\Sigma$, $\xi(\Delta)\in\mathcal{D}(A)$.
2) $\sup\{\|A\xi(\Delta)\|\colon \Delta\in\Sigma\}<\infty$.

In other words, the set function $\Sigma\ni\Delta\to A\xi(\Delta)\in H$ is well defined and it is a bounded E-trajectory.

If $\mathcal{R}$ is a collection of normal operators strongly commuting with all projections $E(\Delta)$, $\Delta\in\Sigma$, then we put:

$$T_{\mathcal{R}} = \{\xi\in M_E\colon \xi \text{ is } A\text{-bounded for every } A\in\mathcal{R}\}. \tag{1.5}$$

Lemma 1.6. *Let ξ be a bounded E-trajectory. Then there exists $x\in H$ such that for every $\Delta\in\Sigma$*

$$\xi(\Delta) = E(\Delta)x.$$

Proof. Consider the net $\{\xi(\Delta)\}_{\Delta\in\Sigma}$, where Σ is directed by set inclusion. It is uniformly bounded in H, thus it admits weak cluster points. Let $x\in H$ be one of them. Then $\|x\|\leqslant\sup\{\|\xi(\Delta)\|\colon \Delta\in\Sigma\}$. Let $\Delta\in\Sigma$. Then for every $\varepsilon>0$ and $z\in H$ there exists $\Delta'\in\Sigma$ such that $\Delta\subset\Delta'$ and

$$|(E(\Delta)z,\ x-\xi(\Delta'))|<\varepsilon.$$

Thus $|(z,\ E(\Delta)x-\xi(\Delta))|<\varepsilon$. Since $z\in H$ and $\varepsilon>0$ are arbitrary, $E(\Delta)x = \xi(\Delta)$ for every $\Delta\in\Sigma$. ∎

Although x is not unique, we can force it to be by taking its projection onto the closure of the subspace $\mathcal{D}_\xi = \{\xi(\Delta)\colon \Delta\in\Sigma\}$, and then lifting it back to H. The vector so constructed will be denoted by x_ξ.

Definition 1.7. x_ξ is the unique vector in H such that for every $\Delta\in\Sigma$

$$E(\Delta)x_\xi = \xi(\Delta),\qquad x_\xi\in\mathcal{D}_\xi^{\perp\perp}.$$

Now the next result follows easily:

Proposition 1.8. *Let $\mathscr{R}$ be a family of normal operators which commute strongly with the spectral measure E over the semiring Σ. Then for each $A \in \mathscr{R}$ and $\xi \in T_{\mathscr{R}}$* (cf. 1.5) *there exists a (unique) vector $\xi_A \in H$ such that for any $\Delta \in \Sigma$*

$$E(\Delta)\xi_A = A\xi(\Delta). \tag{1.9}$$

Remark 1.10.

$$\|\xi_A\| = \sup\{\|A\xi(\Delta)\|:\ \Delta \in \Sigma\}.$$

Let us introduce the following seminorm on the linear space $T_{\mathscr{R}}$:

$$T_{\mathscr{R}} \ni \xi \to \|\xi\|_A = \|\xi_A\|. \tag{1.11}$$

The locally convex (possibly non-Hausdorff) topology over $T_{\mathscr{R}}$ generated by the family of seminorms $\| \ \|_A$ will be denoted by τ_{proj}.

Now let us define for every $A \in \mathscr{R}$ the space

$$H_A = \overline{\{\xi_A:\ \xi \in T_{\mathscr{R}}\}}. \tag{1.12}$$

For every $A \in \mathscr{R}$ the space H_A is a Hilbert space in which the norm is induced by the norm from H. On the other hand, for any $A \in \mathscr{R}$ we can introduce a scalar product in the space $T_{\mathscr{R}}$ by the formula:

$$(\xi, \xi')_A = (\xi_A, \xi'_A)_H \quad \text{for } \xi, \xi' \in T_{\mathscr{R}}. \tag{1.13}$$

Clearly $\|\xi\|_A^2 = (\xi, \xi)_A$.

Let us denote by $A^{-1}\cdot H$ the Hilbert space resulting from the completion of $T_{\mathscr{R}}/\mathrm{Ker}\| \ \|_A$ with respect to the norm induced by the canonical map $T_{\mathscr{R}} \to T_{\mathscr{R}}/\mathrm{Ker}\| \ \|_A$ (to make the notation compatible with the one used in Section 3).

Remark 1.14. The Hilbert spaces H_A and $A^{-1}\cdot H$ are isometrically isomorphic.

Remark 1.15. Let A be a normal operator strongly commuting with E over Σ. Let

$$T_{\{A\}} = \{\xi \in M_E:\ \xi \text{ is } A\text{-bounded}\}.$$

Then the set $\mathscr{D}_{T_{\{A\}}} = \{\xi(\Delta):\ \Delta \in \Sigma,\ \xi \in T_{\{A\}}\}$ is dense in H.

Proof. Let E_A be the spectral measure of A. Then for every bounded $\Omega \subset \boldsymbol{C}^1$ and for every $x \in H$, $E_A(\Omega)x \in \mathscr{D}(A)$. Hence the c.a.o.s. measure

$$\Sigma \ni \Delta \to \xi_y(\Delta) = E(\Delta)y, \quad \text{with } y = E_A(\Omega)x,$$

is A-bounded. However, all $E(\Delta)$ commute with $E_A(\Omega)$. Thus $E_A(\Omega)E(\Delta)x = \xi_y(\Delta)$. Suppose now that for some $z \in H$, $(z, E_A(\Omega)E(\Delta)x) = 0$ for all bounded Borel sets Ω and all $\Delta \in \Sigma$. Since $x \in H$ is arbitrary, for each Ω we have $E_A(\Omega)z = 0$. Hence $z = 0$ and the result follows. ■

Corollary 1.16. *If a normal operator A commutes strongly with the spectral measure E over Σ, then for every bounded Borel set $\Omega \subset \boldsymbol{C}^1$ the mapping $\Sigma \ni \Delta \to E(\Delta)E_A(\Omega)x$ is an A-bounded E-trajectory for each $x \in H$.*

2. Factorization theorem. It has been proved in [9] that every (bounded) H-valued c.a.o.s. measure ξ over a semiring Σ can be represented in the form $\xi(\Delta) = E_\xi(\Delta)x$, where $x \in H$ and E_ξ is a projection-valued σ-additive measure over $\sigma(\Sigma)$. However, in the case of E-spectral trajectories this statement reduces to a trivial one, since the measure E_ξ is simply given by the controlling spectral measure E. We shall show that for an E-trajectory ξ which may not be bounded, but merely A-bounded, there is still an analogous representation, however for the price of "smoothing" by means of an "A-bounded" operator L.

For this we need to introduce a class of $\mathscr{R}$-bounded operators.

Definition 2.1 (cf. [5], Def. 2.1). Let $\mathscr{R}$ be a family of normal operators in a Hilbert space H. A densely defined operator L is *$\mathscr{R}$-bounded* if the operator LA is densely defined and bounded in H for every $A \in \mathscr{R}$. The collection of $\mathscr{R}$-bounded operators will be denoted by $\mathscr{R}\mathscr{B}(H)$. If $\mathscr{R} = \{A\}$, then we say that L is *A-bounded.*

Definition 2.2 (cf. [5], Def. 2.4). Let $\mathscr{R}$ be a family of normal operators in H with a joint dense domain. Let $\mathscr{K} \subset \mathscr{R}\mathscr{B}(H)$. The set

$$\mathscr{K}^c = \{K \in \mathscr{R}\mathscr{B}(H):\ \text{for each } K' \in \mathscr{K} \text{ and } A \in \mathscr{R},$$
$$KK',\ K'K \in \mathscr{R}\mathscr{B}(H) \text{ and } KK'A = K'KA\}$$

is called the *$\mathscr{R}$-commutant* of $\mathscr{K}$, and the set $\mathscr{K}^{cc} = (\mathscr{K}^c)^c$ is called the *$\mathscr{R}$-bicommutant* of $\mathscr{K}$.

To prove the main result of this section we have to strengthen the previous assumption that the set $\{E(\Delta)x:\ \Delta \in \Sigma,\ x \in H\}$ is dense in H. Also the semiring Σ should be endowed with some additional structure, which imitates the properties of bounded Borel sets in $\boldsymbol{R}^n$. This provides a possibility to control the "asymptotic" behavior of unbounded operators. Here we follow our previous paper [5].

(2.3) Assumption I. In Σ there exists a countable family $\Sigma_0 = \{\Delta_n\} \subset \Sigma$ such that:

(a) The family Σ is locally Σ_0-finite, i.e., for every $\Delta \in \Sigma$ there exists $n_0 < \infty$ such that $\Delta \subset \bigcup_{n=1}^{n_0} \Delta_n$.

(b) The set $\{E(\Delta_n)x:\ \Delta_n \in \Sigma_0,\ x \in H\}$ is dense in H.

This apparently natural assumption is *not* automatically satisfied, not even in the case that Σ originates from the spectral sets of an (uncountable) generating family $\mathscr{R}$ in the sense of [5].

Remark 2.4. If Assumption I is satisfied then $\sup E(\Delta_n) = 1_H$, thus we can assume that $E(\Delta_n)E(\Delta_m) = 0$ for $n \neq m$, and $\sum_n E(\Delta_n) = 1_H$.

Proof. If $\sup E(\Delta_n) \neq 1_H$, then there exists $z \in H$ such that $E(\Delta_n)z = 0$ for every n. Thus $z \perp \{E(\Delta_n)x \colon x \in H,\ \Delta_n \in \Sigma_0\}$, and hence $z = 0$. The result follows. ■

Suppose now that the family $\mathscr{R}$ consists of (not necessarily bounded) operators related to the semiring Σ with properties analogous to the ones for a generating family of operators in the sense of [5]. These properties form the operator-theoretic counterparts of the properties of a semigroup of functions with a well-defined asymptotic behavior (see [6]).

(2.5) Assumption II. 1) For every $A \in \mathscr{R}$ there exists $B \in \mathscr{R}$ strongly commuting with A such that for all $f \in \mathscr{D}(A) \cap \mathscr{D}(B)$

$$|(f, Af)| \leqslant \|Bf\|^2 \quad (\text{or } \|A^{1/2}f\| \leqslant \|Bf\|).$$

2) For every $n = 1, 2, \ldots$, and every $x \in H$, $E(\Delta_n)x \in \mathscr{D}(A)$ for each $A \in \mathscr{R}$.

3) For every $n = 1, 2, \ldots$, and every $A \in \mathscr{R}$, there exists $B \in \mathscr{R}$ and a constant $c > 0$ such that

$$n^2 \|AE(\Delta_n)\| \leqslant c \inf_{\|y\|=1} \|BE(\Delta_n)y\|.$$

4) For every $n = 1, 2, \ldots$, there exist $A \in \mathscr{R}$ and a constant $c' > 0$ such that for every $f \in \mathscr{D}(A^{1/2})$ with $A^{1/2}f \neq 0$

$$\|E(\Delta_n)f\| \leqslant c' \|A^{1/2}f\|.$$

5) For every $n = 1, 2, \ldots$, $E(\Delta_n) \in \mathscr{R}^{cc}$.

The above Assumption II is a generalization of the notion of generating family of operators in the sense of [5] to noncommuting unbounded s.a. operators. In [5] all members of the family $\mathscr{R}$ are assumed to be bounded, positive and commuting. In general, it follows from (2.3) and (2.5) that for every $\Delta \in \Sigma$ the operator $E(\Delta)$ is $\mathscr{R}$-bounded. In fact, the strong Assumption II.5 is satisfied in all interesting cases. Without this assumption, however, we should extend the family $\mathscr{R}$ by $\{E(\Delta_n)\}$. We could as well assume that all $E(\Delta_n)$ belong to the center of the von Neumann algebra $\mathscr{R}''$ generated by the spectral projections of the elements of $\mathscr{R}$. However, this condition would be more difficult to check.

Lemma 2.6. *Let $f \in \mathscr{D}(A)$ for every $A \in \mathscr{R}$. Then $Af = 0$ for every $A \in \mathscr{R}$ if and only if $f = 0$.*

Proof. Since by (2.5) for every $n = 1, 2, \ldots$ there exists $A \in \mathscr{R}$ such that $\|E(\Delta_n)f\|^2 \leqslant |(f, Af)|$, we have $E(\Delta_n)f = 0$ for every n. Thus, by the density assumption (2.3), $f = 0$. ■

Lemma 2.7. *Let the semiring Σ have the property that for each $\Delta \in \Sigma$ and each $x \in H$, $E(\Delta)x \in \mathscr{D}(A)$ for all $A \in \mathscr{R}$, and let $A, B \in \mathscr{R}$ be two s.a. operators which strongly commute (i.e., their spectral projections commute). Then for every $\xi \in T_{\mathscr{R}}$*

$$A\xi_B = B\xi_A.$$

Proof. First we note that both sides of the above equality make sense, i.e., $\xi_A \in \mathscr{D}(B)$ and $\xi_B \in \mathscr{D}(A)$. Indeed, for every $\Delta \in \Sigma$, $E(\Delta)\xi_A = A\xi(\Delta)$ and $\xi(\Delta) \in \mathscr{D}(A) \cap \mathscr{D}(B)$. Thus $BE(\Delta)\xi_A = BA\xi(\Delta) = AB\xi(\Delta)$.

Consider the linear functional on the set $\mathscr{D}_{T_{\mathscr{R}}} = \{\xi'(\Delta) \colon \xi' \in T_{\mathscr{R}},\ \Delta \in \Sigma\}$:

$$\xi'(\Delta) \to (A\xi'(\Delta), \xi_B) = (E(\Delta)A\xi'(\Delta), \xi_B) = (A\xi'(\Delta), B\xi(\Delta)) = (\xi'(\Delta), AB\xi(\Delta)).$$

This functional is bounded on the whole space $H = \overline{\mathscr{D}}_{T_{\mathscr{R}}}$ (cf. Remark 1.14). Hence $\xi_B \in \mathscr{D}(A)$. Similarly $\xi_A \in \mathscr{D}(B)$.

For any $\Delta \in \Sigma$, $z \in H$, we have

$$\big(E(\Delta)z, A\xi_B\big) = \big(E(\Delta)z, AB\xi(\Delta)\big) = \big(E(\Delta)z, BA\xi(\Delta)\big) = \big(E(\Delta)z, BE(\Delta)\xi_A\big) = \big(E(\Delta)z, B\xi_A\big).$$

Since the set $\{E(\Delta)z \colon \Delta \in \Sigma,\ z \in H\}$ is dense in H, the result follows. ■

Corollary 2.8. *If Assumptions I and II are satisfied, then $\xi_A \in \mathscr{D}(A)$ for all $A \in \mathscr{R}$ and all $\xi \in T_{\mathscr{R}}$.*

Theorem 2.9. *Let Assumptions I and II hold for a family of normal operators $\mathscr{R}$ which strongly commute with the measure E over Σ. Let ξ be a c.a.o.s. measure over Σ. Then $\xi \in T_{\mathscr{R}}$ if and only if there exist a s.a. operator L in $\mathscr{R}^{cc}$ and an $x \in H$ such that for every $\Delta \in \Sigma$, $\xi(\Delta) = LE(\Delta)x$.*

Proof. By the assumption, $E(\Delta) \in \mathscr{R}^{c}$ for every $\Delta \in \Sigma$. So by (2.5), $\xi(\Delta) = LE(\Delta)x$ is an E-trajectory. Since L is $\mathscr{R}$-bounded, so is ξ, i.e., $\xi \in T_{\mathscr{R}}$.

Conversely, let $\xi \in T_{\mathscr{R}}$. Consider the sequence $y_n = \xi(\Delta_n)$ with $\Delta_n \in \Sigma_0$, $\Delta_n \cap \Delta_m = \emptyset$ for $m \neq n$, and $\sum_n E(\Delta_n) = 1_H$ (see Remark 2.4). Define $r_n = \|y_n\|$. Since

$$\sum_{\substack{n=1\\ r_n \neq 0}}^{\infty} \|(1/(nr_n))y_n\|^2 \leqslant \sum 1/n^2 < \infty$$

the series

(2.10) $$y = \sum_{\substack{n=1\\ r_n \neq 0}}^{\infty} (1/(nr_n))y_n$$

converges in H, and so $y \in H$. Define

(2.11) $$L = \sum_{n=1}^{\infty} n r_n E(\Delta_n).$$

We shall show that L is $\mathscr{R}$-bounded. Let $A \in \mathscr{R}$. Then LA is densely defined (on $\{E(\Delta_n)H:\ n \in N\}$). We have

$$\sup_n (n r_n \|AE(\Delta_n)\|) \leqslant c \sup_n ((1/n) r_n \|B^{-1} E(\Delta_n)\|^{-1})$$

where $B \in \mathscr{R}$ and $c > 0$ are chosen as in Assumption II.3. Further, we have

$$r_n = \|B^{-1} E(\Delta_n) \xi_B\| \leqslant \|B^{-1} E(\Delta_n)\| \|\xi_B\|.$$

This makes sense by virtue of Proposition 1.8. Thus, by (2.11), we have

$$\|LA\| \leqslant c \|\xi_B\| < \infty,$$

i.e., L is $\mathscr{R}$-bounded. It obviously belongs to $\mathscr{R}^{cc}$ (even to $\mathscr{R}^{cc}{}_+$). Now an easy computation shows that $\xi(\Delta) = LE(\Delta) y$. ∎

Remark 2.12. If the family $\mathscr{R}$ does not satisfy Assumption II, we still obtain the formula $\xi(\Delta) = LE(\Delta) x$, however, the operator L given by (2.11) will not be $\mathscr{R}$-bounded, which makes the results much less interesting.

3. Locally convex topologies on $T_{\mathscr{R}}$. As in Section 1 let $\mathscr{R}$ be a family of normal operators in H strongly commuting with the spectral measure E over the semiring Σ. Let $T_{\mathscr{R}}$ be the space of $\mathscr{R}$-bounded trajectories in H as before. We consider $T_{\mathscr{R}}$ as a locally convex topological vector space endowed with the topology τ_{proj} generated by the family of seminorms $\|\ \|_A$, $A \in \mathscr{R}$ (cf. (1.11)). To make the topology τ_{proj} Hausdorff it is enough to assume that the family $\mathscr{R}$ is separating, i.e., for every $f \in H$, if $f \in \mathscr{D}(A)$ and $Af = 0$ for all $A \in \mathscr{R}$, then it follows that $f = 0$. So, if Assumptions I and II are satisfied, this condition holds and τ_{proj} is Hausdorff (cf. Lemma 2.6). Under the assumption that $\mathscr{R}$ is separating, we have the following easy observation.

Proposition 3.1. *The space $T_{\mathscr{R}}$ with the (Hausdorff) l.c. topology τ_{proj} is homeomorphic to the projective limit space of the family of the Hilbert spaces H_A, i.e., to the projective limit of the family $\{H_A\}_{A \in \mathscr{R}}$ with respect to the family of projections π_A: $T_{\mathscr{R}} \to H_A$, where $\pi_A(\xi) = \xi_A$.*

The main result of this section is the representation of the space $(T_{\mathscr{R}}, \tau_{\text{proj}})$ as an *inductive limit* of a family of Hilbert spaces. However, to obtain meaningful results we have to assume that the semiring Σ and the family $\mathscr{R}$ satisfy not only Assumptions I and II, but also the following one. (Let as before $\mathscr{R}''$ denote the bicommutant of the family $\mathscr{R}$ in $B(H)$, i.e., the von Neumann algebra with unit generated by the spectral projections of the elements of $\mathscr{R}$.)

(3.2) Assumption III. Let

$$\mathscr{R}^{\dagger\dagger} = \{L \in \mathscr{R}'':\ \text{for every } L' \in \mathscr{R}^{cc},\ L' L \in \mathscr{R}''\}.$$

Then for every $Q \in \mathscr{R}^{\dagger\dagger}$ there exist $A \in \mathscr{R}$ and a number $c > 0$ such that $Q^* Q \leqslant cA^2$ (weakly, on a suitable domain).

Elements of $\mathscr{R}^{\dagger\dagger}$ can be seen as "smoothing operators" which behave "nicely" at infinity with respect to the spectral measure E and the semiring Σ. Thus, owing to Assumption II.4, the elements of $\mathscr{R}^{\dagger\dagger}$ map $T_{\mathscr{R}}$ into $\mathscr{S}_{\mathscr{R}}$.

Using the factorization theorem 2.9 we can represent the space $T_{\mathscr{R}}$ as the union of the linear spaces $L \cdot H$, $L \in \mathscr{R}^{cc}{}_+$ ($\mathscr{R}^{cc}{}_+ = \{L \in \mathscr{R}^{cc}:\ L \geqslant 0\}$), defined as follows:

(3.2) $$L \cdot H = \{\xi \in T_{\mathscr{R}}:\ \xi(\Delta) = LE(\Delta) y \text{ for all } \Delta \in \Sigma \text{ and some } y \in H\}.$$

Let, for $L \in \mathscr{R}\mathscr{B}(H)$, $p(L)$ be the orthogonal projection in H onto the closure of the linear span of the set $\{LAH:\ A \in \mathscr{R}\}$.

Proposition 3.3. *Let ξ, $\xi' \in L \cdot H$ for some $L \in \mathscr{R}^{cc}{}_+$. Then the formula $(\xi, \xi')_L = (y, y')_H$, where $\xi(\Delta) = LE(\Delta) y$ and $\xi'(\Delta) = LE(\Delta) y'$, defines a Hermitian scalar product in the space $L \cdot H$. Thus $L \cdot H$ becomes a Hilbert space.*

Proof. It is easy to observe that if $\xi(\Delta) = LE(\Delta) y_1 = LE(\Delta) y_2$ for all $\Delta \in \Sigma$, then $p(L) y_1 = p(L) y_2$, i.e., the scalar product $(\xi, \xi')_L$ is well defined.

To show that $L \cdot H$ is complete we note first that by virtue of (2.5) for every $L \in \mathscr{R}^{cc}{}_+$ and $\Delta \in \Sigma$, $LE(\Delta)$ is a bounded operator in H. For each $f \in \mathscr{D}(LE(\Delta))$ we have

$$\|LE(\Delta) f\| = \|E(\Delta) LE(\Delta) f\| \leqslant c' \|A^{1/2} LE(\Delta) f\|$$
$$\leqslant c' \|BLE(\Delta) f\| \leqslant c' \|LB\| \|f\|.$$

Thus for every $x \in H$ the mapping $\Sigma \ni \Delta \to LE(\Delta) x \in H$ is a well-defined $\mathscr{R}$-bounded E-trajectory. In particular, if a sequence $\{\xi_k\}$ in $L \cdot H$ is convergent in the norm $\|\ \|_L$, then $\xi(\Delta) = LE(\Delta) \lim_k (p(L) y_k)$, where $\xi_k(\Delta) = LE(\Delta) y_k$, is its $\|\ \|_L$-limit, and the result follows. ∎

Now using the (algebraic) identification

$$T_{\mathscr{R}} = \bigcup_{L \in \mathscr{R}^{cc}{}_+} L \cdot H$$

we can introduce in $T_{\mathscr{R}}$ an inductive limit topology τ_{ind}, induced by the family of canonical embeddings of the Hilbert spaces $L \cdot H$ into $T_{\mathscr{R}}$, directed by the usual ordering of s.a. positive operators from $\mathscr{R}^{cc}$. Recall that a convex circled set $O \subset T_{\mathscr{R}}$ is open in the inductive limit topology τ_{ind} if and only if the set $O \cap L \cdot H$ is open in the Hilbert space $L \cdot H$. Since for every $A \in \mathscr{R}$ and $L \in \mathscr{R}^{cc}{}_+$ the operator LA is bounded, the seminorms $T_{\mathscr{R}} \ni \xi \to \|\xi\|_A$ are continuous in the topology τ_{ind}. Thus τ_{ind} is stronger than τ_{proj}. To prove the converse we need the following crucial result.

Proposition 3.4 (cf. [7], Proposition 2.2). *Let Assumptions* I–III *hold, and let* $O \subset T_R$ *be a convex null neighborhood such that for every* $L \in \mathscr{RB}_+(H)$ *the set* $O \cap L \cdot H$ *is open in* $L \cdot H$. *Then there exist* $A \in \mathscr{R}$ *and a number* $\delta > 0$ *such that the set*

$$V_{\delta,A} = \{\xi \in T_{\mathscr{R}}:\ \|\xi\|_A < \delta\}$$

is contained in O.

Proof. We give here only an outline of the rather technical proof which in main lines coincides with that in [4–7] (e.g., cf. Lemma 5.5 in [7]).

Our aim is to construct a suitable element $A \in \mathscr{R}$ and to find $\delta > 0$ such that $V_{\delta,A} \subset O$. Notice that $O \cap L \cdot H$ is open for every $L \in \mathscr{R}^{cc}{}_+$. It follows that for every $\Delta_n \in \Sigma_0$, $O \cap E(\Delta_n) H$ is also open, where we identify $E(\Delta_n) H$ with its image $E(\Delta_n) \cdot H$ in $T_{\mathscr{R}}$ under the natural embedding $H \ni x \to \xi_x \in T_{\mathscr{R}}$ (cf. 1.3). Let r_n denote the radius of the largest open ball in $E(\Delta_n) H$ contained in $O \cap E(\Delta_n) H$, i.e.,

$$r_n = \sup\{\varrho > 0:\ \text{if } \xi \in E(\Delta_n) H \text{ and } \sup\|E(\Delta_n)\xi(\Delta)\| < \varrho, \text{ then } \xi \in O\}.$$

Now let us define the operator

$$Q = \sum_{n=1}^{\infty} 2(n^2/r_n) E(\Delta_n).$$

After easy calculation and extensive use of Assumptions I and II, it can be proved that $Q \in \mathscr{R}^{\dagger\dagger}$ (cf. Lemma 5.5 in [5]). Now, owing to the "ad hoc" Assumption III, we can find $A \in \mathscr{R}$ such that $Q \leqslant cA$ (in the weak sense on a suitable domain). Define $\delta = 1/c$ and take

$$V_{\delta,A} = \{\xi \in T_{\mathscr{R}}:\ \|\xi\|_A = \|\xi_A\| < \delta\}.$$

Observe that since

$$n^2 \sup_{\Delta \in \Sigma} \|E(\Delta_n)\xi(\Delta)\| = \tfrac{1}{2} r_n \sup_{\Delta \in \Sigma} \|QE(\Delta_n)\xi(\Delta)\| \leqslant \tfrac{1}{2} c r_n \|\xi\|_A < r_n,$$

we have, for every $n \in N$, $n^2 E(\Delta_n)\xi \in O$. Thus, once again using the standard method originating from [8] (cf. [4]), we can prove that every $\xi \in V_{\delta,A}$ splits for each $n_0 \in N$ into the convex combination of elements of O:

$$\xi_1 = \sum_{n=1}^{n_0} \big(1/(2n^2)\big) 2n^2 E(\Delta_n)\xi,$$

and the rest ξ_2 which tends to 0 in $L \cdot H$ as $n_0 \to \infty$. Since $O \cap L \cdot H$ is a convex neighborhood of zero, the result follows, i.e., $\xi \in O$. ∎

Taking into account Proposition 3.4 and the previously stated fact that $\tau_{\text{ind}} > \tau_{\text{proj}}$ we can formulate the following result.

Theorem 3.5 (cf. Theorem 5.19 in [5]). *The l.c. space* $(T_{\mathscr{R}}, \tau_{\text{proj}})$ *is homeomorphic to the inductive limit of the directed family of Hilbert spaces* $\{L \cdot H\}_{L \in \mathscr{R}^{cc}{}_+}$ *provided Assumptions* I–III *are satisfied.*

Now we present another useful result which follows from Theorem 3.5:

Theorem 3.6. *Let* $\mathscr{R}$ *be a generating family of operators satisfying Assumptions* I–III. *A set* $\mathscr{B} \subset T_{\mathscr{R}}$ *is bounded in* τ_{proj} *if and only if there exists* $L \in \mathscr{R}^{cc}$ *such that* $\mathscr{B} \subset L \cdot H$ *and* $\mathscr{B}$ *is bounded in the space* $L \cdot H$.

Proof. The idea of the proof is roughly the same as of the proof of Theorem 2.3 in [6]. Therefore we present only its main steps.

Suppose first that a set $\mathscr{B}$ in $T_{\mathscr{R}}$ is contained in $L \cdot H$ for some $L \in \mathscr{R}^{cc}{}_+$. Then $\mathscr{B}$ is τ_{ind}-bounded in $T_{\mathscr{R}}$ whenever it is bounded in $L \cdot H$, because the embedding $L \cdot H \to T_{\mathscr{R}}$ is τ_{ind}-continuous. Thus, to prove the theorem it is enough to show that every bounded set $\mathscr{B}$ in $T_{\mathscr{R}}$ is contained in $L \cdot H$ for some $L \in \mathscr{R}^{cc}{}_+$. For every $n \in N$ we define the numbers

$$s_n = \sup_{\xi \in B} \|A_n^{-1} E(\Delta_n)\xi_{A_n}\|,$$

where the A_n are chosen as in Assumption II.4. Now we define the unbounded operator L by

$$L = \sum_{n=1}^{\infty} n s_n E(\Delta_n).$$

It follows from Assumption I.3 that $L \in \mathscr{RB}(H)$, and hence $L \in \mathscr{R}^{cc}{}_+$.

For every $\xi \in \mathscr{B}$ the series

$$\sum_{\substack{n=1 \\ s_n \neq 0}}^{\infty} (1/(ns_n)) A_n^{-1} E(\Delta_n)\xi_{A_n}$$

is convergent in H. Denote its limit by x_ξ. Since $\|x_\xi\| \leqslant \sum 1/n^2$, the set $\mathscr{B}_0 = \{x_\xi \in H:\ \xi \in \mathscr{B}\}$ is uniformly bounded in H. It is easy to see that $\mathscr{B} = L \cdot \mathscr{B}_0 \subset L \cdot H$. ∎

4. The inductive limit space $\mathscr{S}_{\mathscr{R}}$ and duality. At the beginning of this section we shortly recall our construction of the inductive limit space $\mathscr{S}_{\mathscr{R}}$. Here $\mathscr{R}$ is a given generating family of commuting bounded positive operators satisfying Assumption II with respect to a given semiring Σ. In our previous paper [5] the semiring Σ consisted of measurable subsets of the joint spectrum Λ of the family $\mathscr{R}$, with the following property: for every $\Delta \in \Sigma$ there exists $A \in \mathscr{R}$ such that $E(\Delta) \leqslant c'' A$, for some positive number c''.

In the present paper the measure E is the joint spectral measure of the family $\mathscr{R}$. Let us recall the precise definition of a generating family of operators:

DEFINITION 4.1 (cf. [5], Sect. 1). Let $\mathscr{R} \subset B(H)$ consist of bounded selfadjoint operators in the Hilbert space H. $\mathscr{R}$ is called a *generating family of operators* if:

1) The above-defined semiring Σ satisfies Assumption I.
2) $\mathscr{R}$ satisfies Assumptions II and III with respect to the semiring Σ.
3) All members of $\mathscr{R}$ mutually commute, are positive and bounded by $\mathbf{1}_H$.
4) $\mathscr{R}$ is directed by the usual ordering relation in the cone of positive bounded operators.

If $\mathscr{R} \subset B(H)$ is a generating family of operators then we can construct an inductive limit of Hilbert spaces

$$\mathscr{S}_{\mathscr{R}} = \bigcup_{A\in\mathscr{R}} AH,$$

where $AH = \{Ax\colon x \in H\}$ is a Hilbert space with norm given by $AH \ni s \to \|s\|_A = \|r(A)x\|$. Here $r(A)$ is the right support of the operator A (cf. [5]). The present notation is not misleading since we can embed $\mathscr{S}_{\mathscr{R}}$ into $T_{\mathscr{R}}$ putting $\operatorname{emb}(s)(\Delta) = E(\Delta)r(A)x$ for any $s \in \mathscr{S}_{\mathscr{R}}$ of the form $s = Ax$, $A \in \mathscr{R}$, $x \in H$.

In view of the results of [5] it is enough to consider $\mathscr{S}_{\mathscr{R}}$ as a locally convex topological vector space with the l.c. topology given explicitly by the family of seminorms $\|s\|_L = \|Ls\|$, $L \in \mathscr{R}^{cc}{}_+$ ($\mathscr{R}^{cc}$ is a GB*-algebra, cf. [5]). The space $\mathscr{S}_{\mathscr{R}}$ thus constructed is bornological, barrelled, sequentially complete and reflexive.

In this section we discuss the duality between the spaces $\mathscr{S}_{\mathscr{R}}$ and $T_{\mathscr{R}}$. In first instance, only under Assumptions I and II, we show an algebraic identification of the space $T_{\mathscr{R}}$ with the strong dual space $\mathscr{S}'_{\mathscr{R}}$ of the space $\mathscr{S}_{\mathscr{R}}$, and vice versa.

However, to prove the topological identification of these spaces we have to strengthen the assumptions by imposing also Assumption III on the family $\mathscr{R}$ and the ring Σ.

Now let us define the following pairing between the spaces $\mathscr{S}_{\mathscr{R}}$ and $T_{\mathscr{R}}$.

DEFINITION 4.2. Let $\xi \in T_{\mathscr{R}}$ and $s \in \mathscr{S}_{\mathscr{R}}$ with $s = Ax$ for some $A \in \mathscr{R}$ and $x \in H$. Define

$$\langle \xi, s\rangle = (\xi_A, x)_H,$$

where ξ_A is given by Proposition 1.8.

It is easy to see that the numbers $\langle \xi, s\rangle$ are well defined and do not depend on the decomposition $s = Ax$.

PROPOSITION 4.3. *The function*

$$T_{\mathscr{R}} \times \mathscr{S}_{\mathscr{R}} \ni (\xi, s) \to \langle \xi, s\rangle \in \mathbf{C}^1$$

is a nondegenerate sesquilinear form.

We omit the rather technical proof based on the fact that the vectors of the form $E(\Delta)x$, $\Delta \in \Sigma$, $x \in H$, constitute a dense set in H.

Now we shall formulate a result which is an analog of the Riesz theorem concerning the representation of continuous linear functionals on spaces of continuous functions by (Radon) measures.

THEOREM 4.4. *Let $\mathscr{R}$ be a generating family of operators (as defined in Def. 4.1) satisfying Assumptions* I *and* II. *Then the spaces $\mathscr{S}'_{\mathscr{R}}$ and $T_{\mathscr{R}}$ are algebraically isomorphic. The same holds for the spaces $T'_{\mathscr{R}}$ and $\mathscr{S}_{\mathscr{R}}$.*

Proof. We shall prove the existence of the following antilinear injections:

$$\alpha_1\colon \mathscr{S}_{\mathscr{R}} \to T'_{\mathscr{R}}, \tag{4.5}$$

$$\alpha_2\colon T'_{\mathscr{R}} \to \mathscr{S}_{\mathscr{R}}, \tag{4.6}$$

$$\beta_1\colon T_{\mathscr{R}} \to \mathscr{S}'_{\mathscr{R}}, \tag{4.7}$$

$$\beta_2\colon \mathscr{S}'_{\mathscr{R}} \to T_{\mathscr{R}}, \tag{4.8}$$

and the identities

$$\alpha_1\alpha_2 = \mathrm{id}_{T'_{\mathscr{R}}}, \quad \alpha_2\alpha_1 = \mathrm{id}_{\mathscr{S}_{\mathscr{R}}}, \tag{4.9}$$

$$\beta_1\beta_2 = \mathrm{id}_{\mathscr{S}'_{\mathscr{R}}}, \quad \beta_2\beta_1 = \mathrm{id}_{T_{\mathscr{R}}}. \tag{4.10}$$

Let $s \in \mathscr{S}_{\mathscr{R}}$, $\xi \in T_{\mathscr{R}}$. Define

$$\alpha_1(s)(\xi) = \langle \xi, s\rangle^* = (x, \xi_A)_H,$$

where $s = Ax$, $A \in \mathscr{R}$, $x \in H$. It is easy to see that the antilinear mapping

$$\mathscr{S}_{\mathscr{R}} \ni s \to \alpha_1(s)(\cdot) \in T^*_{\mathscr{R}}$$

is well defined, where $T^*_{\mathscr{R}}$ is the algebraic dual of the space $T_{\mathscr{R}}$. It is obvious that for every $s \in \mathscr{S}_{\mathscr{R}}$ the mapping $T_{\mathscr{R}} \ni \xi \to \alpha_1(s)(\xi)$ is τ_{proj}-continuous, hence $\alpha_1(s) \in T'_{\mathscr{R}}$.

To construct α_2 let us consider $\varphi \in T'_{\mathscr{R}}$. There exists $A \in \mathscr{R}$ such that for all $\xi \in T_{\mathscr{R}}$, $|\varphi(\xi)| \leqslant c\|\xi\|_A$, for some constant $c > 0$. The formula $\tilde{\varphi}(\xi_A) = \varphi(\xi)$ defines a bounded linear functional on the pre-Hilbert space $H_A = \{\xi_A \in H\colon \xi \in T_{\mathscr{R}}\}$. H_A is dense in $r(A)H$ and thus there exists $x \in r(A)H$ such that $\tilde{\varphi}(\xi_A) = (x, \xi_A)_H = \varphi(\xi)$. Put $\alpha_2(\varphi) = Ax$. It is well defined and $\alpha_2(\varphi) \in \mathscr{S}_{\mathscr{R}}$.

Now the identities (4.9) can be directly verified.

Let us construct β_1. Take $\xi \in T_{\mathscr{R}}$ and $s \in \mathscr{S}_{\mathscr{R}}$. Define

$$\beta_1(\xi)(s) = \langle \xi, s\rangle.$$

Then $\beta_1(\xi)(\cdot) \in \mathscr{S}'_{\mathscr{R}}$.

Now let $l \in \mathscr{S}'_{\mathscr{R}}$. Embedding $E(\Delta)H$ into AH for a suitable $A \in \mathscr{R}$, we can represent $l|_{E(\Delta)H}$ by means of a unique vector $\Phi(\Delta) \in E(\Delta)H$ such that

$$l(E(\Delta)x) = (\Phi(\Delta), E(\Delta)x)_H.$$

Evidently, the H-valued set function Φ is an E-trajectory. To see that $\Phi \in T_{\mathscr{R}}$ let us pick out an A in $\mathscr{R}$. Then

$$\sup_{\Delta \in \Sigma} \|A\Phi(\Delta)\| \leqslant \sup_{\Delta \in \Sigma} \sup_{\|x\|=1} |l(E(\Delta)Ax)| \leqslant \|l|_{AH}\|.$$

Now put $\beta_2(l) = \Phi$. Since

$$(\beta_2(l)(\Delta), Ax)_H = l(E(\Delta)Ax),$$

we have $\beta_2\colon \mathscr{S}'_{\mathscr{R}} \to T_{\mathscr{R}}$. A straightforward computation shows the identities (4.10). ∎

To prove the last (and main) result of this section we recall

PROPOSITION 4.6 (cf. [5], Lemma 5.8). *Let the family $\mathscr{R}$ satisfy Assumptions* I–III. *Then a set $\mathscr{B} \subset \mathscr{S}_{\mathscr{R}}$ is bounded if and only if there exists $A \in \mathscr{R}$ such that $\mathscr{B}$ is a bounded subset of the Hilbert space AH.*

Now the main result of this section easily follows:

THHEOREM 4.7. *If the family $\mathscr{R}$ satisfies Assumptions* I–III, *then the spaces $\mathscr{S}'_{\mathscr{R}}$ and $T_{\mathscr{R}}$ are topologically isomorphic, and moreover the same is valid for the spaces $\mathscr{S}_{\mathscr{R}}$ and $T'_{\mathscr{R}}$. The isomorphisms are given by the duality defined in Definition* 4.2.

Proof. The proof is based on the explicit characterization of the topology in $\mathscr{S}_{\mathscr{R}}$ in terms of the seminorms $\|\ \|_L$, $L \in \mathscr{R}^{\infty}{}_{+}$, on Theorem 3.5, and on the above Proposition 4.6. ∎

5. Examples. Choosing different families of sets Σ and related spectral measures we can easily produce a wide variety of classes of examples, such as sequence spaces (cf. [2]), generalized function spaces and Gel'fand triples. The most useful method of the construction of these examples is based on the following idea.

1. Let Φ be a generating family of Borel functions on $\boldsymbol{R}^1$ in the sense of [5, 7], and let A be a s.a. positive operator in a Hilbert space H. We set $\mathscr{R} = \{f(A)\colon f \in \Phi\}$. Then every element ξ of the trajectory space $T_{\mathscr{R}}$ can be identified with the following H-valued measure over $\boldsymbol{R}^1$:

$$\xi(\Delta) = \int^{\oplus} \chi_{\Delta}(\lambda) f(\lambda) y(\lambda)\, d\nu(\lambda),$$

where $d\nu$ is the spectral measure of the operator A, $y \in L_2(\boldsymbol{R}^1, d\nu)$, $f \in \Phi^{\dagger}$, and χ_{Δ} is the characteristic function of the bounded Borel set Δ.

More tangible examples are provided in [4] and [2]. Here are some of them:

2. Let $H = L_2(\boldsymbol{R}^1)$ and $A = \frac{1}{2}(x^2 - d^2/dx^2)$. The family $\mathscr{R}$ is now defined by means of the generating family of functions

$$\Phi = \{f\colon \sup_{\lambda} |f(\lambda)(1+\lambda^2)^n| < \infty,\ n \in N\},$$

i.e., $\mathscr{R} = \{f(A)\colon f \in \Phi\}$. Now $\mathscr{S}_{\mathscr{R}} = \mathscr{S}$, and $T_{\mathscr{R}} = \mathscr{S}'$ (Schwartz spaces). Thus for every $\xi \in T_{\mathscr{R}}$ we have the formula

$$\xi(\Delta) = \int^{\oplus} g(\lambda) K_{\Delta}(\lambda, \gamma) u(\gamma)\, d\gamma\, d\lambda,$$

where $u \in H$, $g \in \Phi^{\dagger\dagger}$ (cf. [6]), and K_{Δ} is the suitable kernel representing the spectral family of A. Eventually, representing H as the direct integral $H = \int^{\oplus} H(\lambda)\, d\nu(\lambda)$, where ν is the spectral measure of the operator A, we obtain the elegant symbolic formula:

$$\langle s, \xi\rangle = \int (s(\lambda), \xi(\lambda))_{H(\lambda)}\, d\nu(\lambda) = \int (s(\lambda), \xi(d\lambda)),$$

where $s \in \mathscr{S}$ and $\xi \in \mathscr{S}'$.

3. If the spectrum of the operator A involved in the construction of Ex. 1 is discrete, then we obtain as the space $\mathscr{S}_{\Phi(A)}$ a certain sequence space (for instance, if the family Φ consists of characteristic functions of bounded Borel sets we can obtain the space φ of all finite complex numerical sequences). Thus the spectral measure gives rise to the counting measure (possibly with some weight). The presented theory provides us with an explicit *topological* representation of the dual space of $\mathscr{S}_{\Phi(A)}$. An interesting observation found in [2] is that the completeness of the involved sequence spaces is equivalent to Assumption III (3.2).

Acknowledgements. The authors are grateful to Prof. J. de Graaf for his keen interest in the presented work.

References

[1] Yu. M. Berezanskii, *A projective spectral theorem*, Uspekhi Mat. Nauk 39 (4) (1984), 1–52 (in Russian).

[2] S. J. L. van Eijndhoven, *Projective-inductive limits of Hilbertian sequence spaces...*, THE-Report 86-06, Eindhoven University of Technology, 1986.

[3] S. J. L. van Eijndhoven and J. de Graaf, *Trajectory Spaces, Generalized Functions and Unbounded Operators*, Lecture Notes in Math. 1162, Springer, Berlin 1985.

[4] S. J. L. van Eijndhoven, J. de Graaf and P. Kruszyński, *Dual systems of inductive-projective limits of Hilbert spaces originating from self-adjoint operators*, Nederl. Akad. Wetensch. Proc. Ser. A 88 (3) (1985), 277–297.

[5] S. J. L. van Eijndhoven and P. Kruszyński, *GB*-Algebras associated with inductive limits of Hilbert spaces*, Studia Math. 85 (2) (1985), 107–123.

[6] —, —, *On Gel'fand triples originating from algebras of unbounded operators*, EUT-Report 84-WSK-02, Eindhoven University of Technology, 1984.

[7] S. J. L. van Eijndhoven and P. Kruszyński, *Spectral trajectories and duality*, EUT-Report, Eindhoven University of Technology, 1986.
[8] J. de Graaf, *A theory of generalized functions based on holomorphic semigroups*, Nederl. Akad. Wetensch. Proc. Ser. A 86 (4) (1983), 407–420; 87 (2) (1984), 173–187.
[9] P. Kruszyński, *Orthogonally scattered measures on relatively orthocomplemented lattices*, preprint, Univ. of Toronto, 1985.
[10] P. Masani, *Orthogonally scattered measures*, Adv. in Math. 2 (1968), 61–177.
[11] H. Schaefer, *Topological Vector Spaces*, Springer, New York 1971.

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Received December 30, 1986 (2265)

Added in proof (March 1988). In a recent paper by A. F. M. ter Elst, *On the connection between a symmetry condition and several nice properties of the space $S_{\Phi(A)}$ and $T_{\Phi(A)}$*, preprint, Eindhoven University of Technology, 1987, it is proved that Assumption III (3.2) is equivalent to a lot of topological properties of the spaces $T_{\Phi(A)}$ and $S_{\Phi(A)}$ constructed in [4].

STUDIA MATHEMATICA, T. XCI. (1988)

Geometrical properties of Banach spaces and the distribution of the norm for a stable measure

by

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Abstract. Let μ be a symmetric p-stable measure, $0 < p < 1$, on a locally convex separable metric linear space E and let q be a lower semicontinuous seminorm on E which is finite μ-a.s. We prove that the density of $F(t) = \mu\{q < t\}$ is bounded. If $1 \leqslant p < 2$ and (E, q) is a Banach space containing l_p^n's uniformly, then for every $\eta > 1$ we find a symmetric p-stable measure on E and a norm $\tilde{q}$ which is η-equivalent to the norm q such that the density of $F(t) = \mu\{\tilde{q} < t\}$ is unbounded.

1. Let μ be a symmetric p-stable measure, $0 < p \leqslant 2$, on a locally convex separable metric linear space E, with a measurable seminorm q. Then the distribution function $F(t) = \mu\{q < t\}$ is absolutely continuous apart from a possible jump (for $p = 2$, i.e., for the Gaussian case see [3], and for $0 < p < 2$, see [2]).

In this note we examine whether the density of $F(t)$ is bounded. This information is very essential to estimate the rate of convergence in CLT. It is well known that if E is a Hilbert space and q is the standard Hilbertian norm then, in the Gaussian case, the density is bounded [6]. However, as was shown by Rhee and Talagrand [14], a small change of the Hilbertian norm may spoil the boundedness. This result was recently generalized to all separable Banach spaces by Rhee [13]. She proved that for any infinite-dimensional Banach space (E, q) and every $\eta > 1$ there exists a new norm $\tilde{q}$ which is η-equivalent to q and a Gaussian measure μ such that the density of the μ-distribution of $\tilde{q}$ is unbounded.

In the first part of this note we consider the case of symmetric p-stable measures, $0 < p < 1$. Applying the explicit formula for the density proved in [8] we show that it is bounded whenever q is lower semicontinuous.

For $1 \leqslant p < 2$ we constructed in [15] some examples of p-stable measures μ on c_0 such that the density of $F(t) = \mu\{\|\cdot\|_\infty < t\}$ is unbounded. In this note we provide such examples in Banach spaces in which l^p is finitely representable. If (E, q) is a Banach space of this type, then for every function $f\colon \mathbf{R}^+ \to \mathbf{R}^+$ with $\lim_{t\to 0^+} f(t) = \infty$, we are able to find an equivalent norm $\tilde{q}$, and a symmetric p-stable measure μ such that the density $F'(t)$ of $F(t)$