COROLLARY 5.3. *The spaces* $B_{bb}(F, F)$ *and* $B_{bb}(F_1, X)$ *are not (DF)-spaces.*

Corollaries 5.3 and 5.2 give an answer to *Question non résolue* 7 in [4].

Remark. After this paper was submitted, Gilles Pisier noticed that an analogue of Proposition 2.1 is valid for $C(0, 1)$ instead of our (E, p). The proof for this case is more elementary; it uses only a form of Grothendieck's theorem.

References

[1] H. Collins and W. Ruess, *Duals of spaces of compact operators*, Studia Math. 74 (1982), 213–245.

[2] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer, 1979.

[3] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[4] —, *Sur les espaces (F) et (DF)*, Summa Brasil. Math. 3 (1954), 57–123.

[5] R. Hollstein, *Extension and lifting of continuous linear mappings in locally convex spaces*, Math. Nachr. 108 (1982), 275–297.

[6] G. Köthe, *Topological Vector Spaces I–II*, Springer, Berlin 1983 and 1979.

[7] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam 1980.

[8] G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. 151 (1983), 181–208.

[9] J. Taskinen, *Counterexamples to "Problème des topologies" of Grothendieck*, Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertationes 63 (1986).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
Hallituskatu 15, 00100 Helsinki, Finland

Received December 10, 1986 (2255)
Revised version September 30, 1987

STUDIA MATHEMATICA, T. XCI. (1988)

The Wold–Cramér concordance problem for Banach-space-valued stationary processes

by

GRAŻYNA HAJDUK-CHMIELEWSKA (Szczecin)

Abstract. The problem of the concordance of the Wold decomposition and the spectral measure decomposition of Banach-space-valued stationary processes is studied. We give a sufficient condition for the concordance in terms of the representation of the process as a process in the space of square Bochner integrable functions on the circle.

0. Introduction. The problem of the Wold–Cramér concordance for q-variate stationary processes was extensively studied (cf. [4]–[6]). In the case of stationary processes with values in a Banach space the only result was given by F. Schmidt. He proved that every such process X admits a unique orthogonal decomposition

$$X(k) = Y(k)+U(k)+V(k)$$

where Y is regular, both U and V are singular and the spectral measures of Y, U are absolutely continuous, while the spectral measure of V is singular with respect to the Lebesgue measure (cf. [7], Theorem 5). In particular, the question of whether there exists a nonsingular process with nonzero U part in the Schmidt decomposition remained open.

In this paper we present a sufficient condition for the concordance of the Wold decomposition and the spectral measure decomposition for Banach-space-valued stationary processes. The proof is based on the isomorphism theorem (cf. [8], Theorem 3.3) which yields a representation of the process under consideration as a process in the space $L^2(K, \mu, H)$ of all μ-square Bochner integrable functions from the circle K to a Hilbert space H. Our condition is formulated in terms of this representation. In Section 2 we establish some properties of this representation we need in the proof of the main theorem. Finally, we give in Section 4 several examples related to our theorem. One of them (Example 4.1) answers positively the question formulated above.

1. Preliminaries. In this paper we use the following notation:

$\boldsymbol{Z}$ – the set of integers,

$\boldsymbol{C}$ – the set of complex numbers,

K – the unit circle of the complex plane C,
m – the normalized Lebesgue measure on K,
$\mathscr{B}(K)$ – the family of Borel subsets of the unit circle K.

For any two normed spaces X and Y let $L(X, Y)$ denote the space of all continuous linear operators from X into Y. We use the letters G, H to denote complex Hilbert spaces, and B will denote a complex Banach space. The space of all continuous linear functionals on B is denoted by B^*. By $\bar{L}^+(B, B^*)$ we will denote the space of all antilinear nonnegative continuous operators from B into B^*.

Let μ be a finite nonnegative measure on $\mathscr{B}(K)$. By $L^2(K, \mu, H)$ we mean the set of all strongly measurable functions f on K with values in H such that $\|f(z)\|_H^2$ is μ-integrable. $L^2(K, \mu, H)$ with the inner product

$$\langle f, g\rangle = \int_K \langle f(z), g(z)\rangle_H \, \mu(dz)$$

is a Hilbert space, where $\langle \cdot, \cdot \rangle_H$ and $\|\cdot\|_H$ denote the inner product and the norm in H. The norm in $L^2(K, \mu, H)$ will be denoted in the sequel by $\|\cdot\|$. By T we shall denote the operator in $L^2(K, \mu, H)$ given by

$$Tf(z) = zf(z). \tag{1}$$

Clearly T is a unitary operator.

A *stationary process* with values in B is a mapping $X\colon \mathbf{Z} \to L(B, G)$ such that the correlation function $R(k, l) = X^*(k)X(l)$ depends only on $l-k$. It is known (cf. [1]) that R has the representation

$$X^*(k)X(l) = R(l-k) = \int_K z^{l-k} F(dz)$$

where F is an additive and weakly countably additive measure on $\mathscr{B}(K)$ with values in $\bar{L}^+(B, B^*)$. The measure F is called the *spectral measure* of the process X.

We shall use the notation

$$M_k(X) = \overline{\operatorname{sp}}\{X(l)b\colon\ l \leqslant k,\ b \in B\},$$

$$M_\infty(X) = \overline{\operatorname{sp}}\{X(l)b\colon\ l \in \mathbf{Z},\ b \in B\}, \qquad M_{-\infty}(X) = \bigcap_{k \in \mathbf{Z}} M_k(X),$$

where $\overline{\operatorname{sp}}\,A$ denotes the closed linear space spanned by $A \subseteq G$. A process X is *singular* if $M_\infty(X) = M_{-\infty}(X)$ and *regular* if $M_{-\infty}(X) = \{0\}$.

Recall that to any stationary process X one associates the unitary shift operator $U\colon M_\infty(X) \to M_\infty(X)$ defined by

$$UX(k) = X(k+1), \qquad k \in \mathbf{Z}.$$

Moreover, if E is the spectral measure of the operator U then E and F are related by the formula

$$F(\Delta) = [X(0)]^* E(\Delta) X(0), \qquad \Delta \in \mathscr{B}(K).$$

Next, we shall need the following special case of the Wold decomposition theorem (cf. [1], Theorem 8.6).

1.1. THEOREM. *Let $X\colon \mathbf{Z} \to L(B, G)$ be a stationary process with the shift operator U. Then there exist two processes X^r and X^s with the same shift operator U so that:*

(i) $X(k) = X^r(k) + X^s(k)$.
(ii) *$M_\infty(X^r)$ and $M_\infty(X^s)$ are orthogonal.*
(iii) *For each $k \in \mathbf{Z}$, $M_k(X^r), M_k(X^s) \subseteq M_k(X)$.*
(iv) *X^r is regular and X^s is singular.*

The above decomposition is unique.

Further, we suppose that the spectral measure F of the stationary process X satisfies the following condition:

(2) There exists a finite nonnegative measure μ on $\mathscr{B}(K)$ such that F is absolutely continuous with respect to μ (notation: $F \ll \mu$).

In particular, (2) always holds when $M_\infty(X)$ is a separable subspace in G (cf. [2], Remark 1).

The following theorem is proved in [3] (even with weaker assumptions).

1.2. THEOREM. *If F satisfies (2) then there exist a Hilbert space H and an operator $Q \in L(B, L^2(K, \mu, H))$ such that for all $b_1, b_2 \in B$*

$$\frac{d\big(F(z)b_1\big)(b_2)}{d\mu} = \langle Qb_1(z), Qb_2(z)\rangle_H,$$

where $d\big(F(z)b_1\big)(b_2)/d\mu$ denotes the density of the scalar measure $\big(F(\cdot)b_1\big)(b_2)$ with respect to μ.

The following isomorphism theorem is a consequence of Theorem 1.2 (cf. [8], Theorem 3.3).

Let Q and H be as in Theorem 1.2 and let E denote the spectral measure of the operator T (T as in (1)), i.e.

$$\big(E(\Delta)f\big)(z) = \mathbf{1}_\Delta(z) f(z), \qquad f \in L^2(K, \mu, H), \tag{3}$$

where $\mathbf{1}_\Delta$ is the indicator of the set $\Delta \in \mathscr{B}(K)$.

1.3. THEOREM. *Let F satisfying (2) be the spectral measure of a stationary process $X\colon \mathbf{Z} \to L(B, G)$. The Hilbert spaces $M_\infty(X)$ and $M(F) = \overline{\operatorname{sp}}\{E(\Delta)Qb\colon \Delta \in \mathscr{B}(K),\ b \in B\} \subseteq L^2(K, \mu, H)$ are isomorphic. This isomorphism takes $X(k)b$ to $z^kQb(z) = T^kQb$.*

2. Representations of stationary processes. Suppose that $X\colon \mathbf{Z} \to L(B, G)$ is stationary and its spectral measure F satisfies (2). Let $Y\colon \mathbf{Z} \to L(B, L^2(K, \mu, H))$ be a stationary process such that $Y(k) = T^kQ$ for T

and Q as in 1.3. We shall refer to Y as *representation of X in $L^2(K, \mu, H)$*. Obviously F is also the spectral measure of the process Y.

2.1. Proposition. *If X is nonsingular, then $m \ll \mu$. Moreover,*

$$m(\Delta) = \int_K \|f(z)\|_H^2 \mu(dz), \quad \Delta \in \mathscr{B}(K), \tag{4}$$

for some function f from the innovation space

$$W = M_0(Y) \ominus M_{-1}(Y)$$

of the process Y.

Proof. By the assumption there exists $f \in W$ such that $\|f\| = 1$. Since T is the shift operator of the process Y, the spaces W and $T^k W$ are orthogonal (cf. [1]). Hence $\langle T^k f, f\rangle = 0$ for $k \neq 0$ and

$$\int_K \langle z^k f(z), f(z)\rangle_H \mu(dz) = \int_K z^k \|f(z)\|_H^2 \mu(dz) = \begin{cases} 0 & k \neq 0, \\ 1, & k = 0. \end{cases} \tag{5}$$

If we define $\nu(\Delta) = \int_\Delta \|f(z)\|_H^2 \mu(dz)$ for $\Delta \in \mathscr{B}(K)$ then ν is a finite nonnegative measure on $\mathscr{B}(K)$ and

$$\int_K z^k \nu(dz) = \begin{cases} 0, & k \neq 0, \\ 1, & k = 0. \end{cases}$$

By the Herglotz theorem, $\nu = m$, which gives (4) and $m \ll \mu$.

Let $h \in H$, $\|h\|_H = 1$. We put

$$W_h = \{f \in L^2(K, \mu, H): f(z) \in \operatorname{sp}\{h\} \text{ for each } z \in K\}.$$

Clearly, if $\varphi(z) \in L^2(K, \mu, C)$ then $f(z) = \varphi(z) h \in W_h$ and

$$\|f(z)\|^2 = \int_K \|\varphi(z) h\|_H^2 \mu(dz) = \int_K |\varphi(z)|^2 \mu(dz) = \|\varphi(z)\|^2.$$

On the other hand, it is obvious that every function $f \in W_h$ has the form $f(z) = \varphi(z) h$, where $\varphi \in L^2(K, \mu, C)$. Hence W_h is isomorphic to $L^2(K, \mu, C)$.

Choose now an orthonormal basis $\{e_i\}_{i \in A}$ of the space H. We show that every function $f \in L^2(K, \mu, H)$ has a representation

$$f(z) = \sum_{i \in A} \varphi_i(z) e_i \tag{6}$$

where $\varphi_i \in L^2(K, \mu, C)$ and the orthonormal series on the right is convergent in $L^2(K, \mu, H)$. In fact, for fixed $z \in K$, let (6) be the Fourier series of $f(z) \in H$ in the basis $\{e_i\}_{i \in A}$. Since every function $f \in L^2(K, \mu, H)$ is separable-valued, we can assume that there exists a sequence $\{i_n\}_{n=1}^\infty \subseteq A$ such that

$$f(z) = \sum_{n=1}^{\infty} \varphi_{i_n}(z) e_{i_n} \tag{7}$$

for each $z \in K$. For any n_0 we have

$$\varphi_{i_{n_0}}(z) = \langle \sum_{n=1}^{\infty} \varphi_{i_n}(z) e_{i_n}, e_{i_{n_0}} \rangle_H = \langle f(z), e_{i_{n_0}} \rangle_H.$$

Hence $\varphi_{i_{n_0}}$ is integrable. Since $|\varphi_{i_{n_0}}(z)|^2 \leqslant \|f(z)\|_H^2$, $\varphi_{i_{n_0}} \in L^2(K, \mu, C)$. Moreover, the series (7) is convergent in $L^2(K, \mu, H)$. We have

$$\lim_{N\to\infty} \int_K \Big\|f(z) - \sum_{n=1}^{N} \varphi_{i_n}(z) e_{i_n}\Big\|_H^2 \mu(dz) = \lim_{N\to\infty} \int_K \Big\|\sum_{n=N+1}^{\infty} \varphi_{i_n}(z) e_{i_n}\Big\|_H^2 \mu(dz)$$
$$= \int_K \Big(\lim_{N\to\infty} \Big\|\sum_{n=N+1}^{\infty} \varphi_{i_n}(z) e_{i_n}\Big\|_H^2\Big) \mu(dz) = 0$$

because $\|\sum_{n=N+1}^{\infty} \varphi_{i_n}(z) e_{i_n}\|_H^2 \leqslant \|f(z)\|_H^2$ for $z \in K$ and we can use the Lebesgue theorem. We have shown (6). Hence

$$L^2(K, \mu, H) = \bigoplus_{i \in A} W_{e_i} = \bigoplus_{i \in A} L^2(K, \mu, C).$$

Denote now by P_i the orthogonal projection from H onto W_{e_i}. Clearly, if $f(z)$ is as in (6), then $P_i f = \varphi_i e_i$. Let

$$Y_i(k) = P_i Y(k).$$

Since W_{e_i} reduces T, the operators T^k and P_i commute. Then

$$Y_i(k) = P_i Y(k) = P_i T^k Y(0) = T^k Y_i(0)$$

which proves that Y_i: $\boldsymbol{Z} \to L(B, L^2(K, \mu, C))$ is a stationary process with the shift operator T. Note that if $Y_{i_0} = 0$, then $W_{e_{i_0}}$ is orthogonal to $M_\infty(Y)$ and hence for every $f \in M_\infty(Y)$ the vector e_{i_0} is orthogonal to $f(z)$ for μ-almost each $z \in K$. Thus we may pass to the subspace of H obtained by removing e_{i_0} from the basis of H. By this remark we may assume that $Y_i \neq 0$ for each $i \in A$.

The following lemma gives a connection between the process Y and the family $\{Y_i\} = \{P_i Y\}$.

2.2. Lemma. (i) $\overline{P_i M_k(Y)} = M_k(Y_i)$, $i \in A$, $k \in \boldsymbol{Z}$ *or* $k = \infty$.

(ii) $M_{-\infty}(Y) \subseteq \bigoplus_{i \in A} M_{-\infty}(Y_i)$.

(iii) $P_i M_{-\infty}(Y) \subseteq M_{-\infty}(Y_i)$.

(iv) *If the process Y is singular then Y_i is singular for each $i \in A$.*

(v) *If Y_i is regular for each $i \in A$ then Y is regular.*

Proof. $P_i M_k(Y) = P_i \overline{\operatorname{sp}}\{Y(l)b: \ l \leqslant k, \ b \in B\} \subseteq \overline{\operatorname{sp}}\{P_i Y(l)b: \ l \leqslant k, b \in B\} = M_k(Y_i)$, thus $\overline{P_i M_k(Y)} \subseteq M_k(Y_i)$ for each $k \in \boldsymbol{Z}$ or $k = \infty$. Conversely, $Y_i(l)b = P_i Y(l) b \in \overline{P_i M_k(Y)}$ for each $b \in B$ and $l \leqslant k$. Hence $M_k(Y_i) \subseteq \overline{P_i M_k(Y)}$, which gives (i). This implies that

$$M_k(Y) \subseteq \bigoplus_{i \in A} P_i M_k(Y) \subseteq \bigoplus_{i \in A} M_k(Y_i).$$

If we take the intersection over all $k \in \boldsymbol{Z}$ on both sides we get

$$M_{-\infty}(Y) \subseteq \bigcap_{k\in Z} \Big(\bigoplus_{i\in A} M_k(Y_i)\Big) = \bigoplus_{i\in A} \Big(\bigcap_{k\in Z} M_k(Y_i)\Big) = \bigoplus_{i\in A} M_{-\infty}(Y_i).$$

Thus we have shown (ii). Conditions (iii) and (v) are immediate consequence of (ii).

We shall prove (iv). If $M_{-\infty}(Y) = M_\infty(Y)$, then

$$Y_i(k)b = P_i Y(k) b \in P_i M_\infty(Y) = P_i M_{-\infty}(Y) \subseteq M_{-\infty}(Y)$$

for each $b \in B$ and $k \in \boldsymbol{Z}$. Hence $M_\infty(Y_i) \subseteq M_{-\infty}(Y_i)$, which implies the singularity of Y_i. The lemma is proved.

3. A concordance theorem. Now we show that if the spectral measure of a stationary process satisfies (2) then it decomposes into an absolutely continuous part and a singular part with respect to m.

3.1. LEMMA. *Let F be a weakly countably additive measure on $\mathscr{B}(K)$ with values in $\bar{L}^+(B, B^*)$ and suppose $F \ll \mu$ for a finite nonnegative measure μ on $\mathscr{B}(K)$. Then there exists a unique decomposition $F = F_a + F_s$, where F_a and F_s are weakly countably additive measures on $\mathscr{B}(K)$ with values in $\bar{L}^+(B, B^*)$ such that $F_a \ll m$ and F_s is singular with respect to m.*

Proof. Let Δ_0 be the support of the measure μ_a, the absolutely continuous part of μ with respect to m. Define

$$F_a(\Delta) = F(\Delta \cap \Delta_0), \qquad F_s(\Delta) = F(\Delta \cap \Delta_0^C),$$

where Δ^C denotes the complement of the set $\Delta \in \mathscr{B}(K)$. It is obvious that F_s is singular with respect to m and $F = F_a + F_s$. We show that $F_a \ll m$. Let $g_b(z)$ denote the density of the nonnegative measure $(F(\Delta)b)(b)$ with respect to μ. For $\Delta \in \mathscr{B}(K)$ we have

$$(F_a(\Delta)b) = (F(\Delta \cap \Delta_0)b)(b) = \int_{\Delta\cap\Delta_0} g_b(z)\,\mu(dz) = \int_\Delta g_b(z)\,\mu_a(dz).$$

Hence for each $b \in B$, $(F_a(\cdot)b)(b)$ is absolutely continuous with respect to m. Moreover, the condition $F(\Delta) = 0$ is equivalent to $(F(\Delta)b_1)(b_2) = 0$ for all $b_1, b_2 \in B$ and equivalent to $(F(\Delta)b)(b) = 0$ for each $b \in B$. This implies $F_a \ll m$. The uniqueness of the Lebesgue decomposition of the scalar measure $(F(\cdot)b)(b)$ implies the uniqueness of the decomposition $F = F_a + F_s$.

Now we prove the concordance of the Wold decomposition and the spectral measure decomposition for the process $Y_i = P_i Y$.

3.2. LEMMA. *Suppose that T is the shift operator of a stationary process $X\colon \boldsymbol{Z} \to L(B, L^2(K, \mu, \boldsymbol{C}))$ and $X = X^r + X^s$ is the Wold decomposition of X. If X is nonsingular, then F_a is the spectral measure of X^r and F_s is the spectral measure of X^s.*

Proof. Let E denote the spectral measure of the operator T (E has the form (3)). By F_{X^r} (resp. F_{X^s}) we denote the spectral measure of X^r (resp. X^s). For each $b \in B$ we have

$$(F(\Delta)b)(b) = \|E(\Delta)X(0)b\|^2 = \int_\Delta |X(0)b|^2\,d\mu, \tag{8}$$

hence $F \ll \mu$ and

$$(F_{X^r}(\Delta)b)(b) = \int_\Delta |X^r(0)b|^2\,d\mu, \tag{9}$$

$$(F_{X^s}(\Delta)b)(b) = \int_\Delta |X^s(0)b|^2\,d\mu. \tag{10}$$

We define

$$S_1 = \{f \in L^2(K, \mu, \boldsymbol{C})\colon \operatorname{supp} f \subseteq \Delta_0\}, \quad S_2 = \{f \in L^2(K, \mu, \boldsymbol{C})\colon \operatorname{supp} f \subseteq \Delta_0^C\},$$

where Δ_0 is as in the proof of Lemma 3.1. It is obvious that $S_1 \oplus S_2 = L^2(K, \mu, \boldsymbol{C})$ and S_1, S_2 reduce the operator T. By Theorem 10.2 in [1] we know that $F_{X^r} \ll m$. This implies that $X^r(0)b \in S_1$. Indeed, otherwise there exists a set $\Delta_1 \subseteq \Delta_0^C$, $\mu(\Delta_1) > 0$, such that $X^r(0)b \neq 0$ on Δ_1. Then $m(\Delta_1) = 0$ and by (9), $(F_{X^r}(\Delta_1)b)(b) > 0$, which contradicts the absolute continuity of F_{X^r}. Since S_1 reduces T, $X^r(k)b = T^k X^r(0)b \in S_1$ for all $b \in B$. Hence $M_\infty(X^r) \subseteq S_1$.

Next, we shall show that $M_\infty(X^r) = S_1$. Since $M_\infty(X^r)$ also reduces T, there exists a set $\sigma_0 \in \mathscr{B}(K)$ such that

$$M_\infty(X^r) = \{f \in L^2(K, \mu, \boldsymbol{C})\colon \operatorname{supp} f \subseteq \sigma_0\}$$

because only subspaces of this form reduce the operator T in $L^2(K, \mu, \boldsymbol{C})$. The inclusion $M_\infty(X^r) \subseteq S_1$ says that $\sigma_0 \subseteq \Delta_0$. To prove the equality it is sufficient to show that $\operatorname{supp} f = \Delta_0$ for some function $f \in M_\infty(X^r)$. The existence of such a function follows from Proposition 2.1. In fact, if $f \in M_0(X) \ominus M_{-1}(X)$ then $f \in M_\infty(X^r)$. By the equality

$$m(\Delta) = \int_\Delta |f(z)|^2\,\mu(dz)$$

f does not vanish on a set of positive Lebesgue measure, thus also f does not vanish on a set of positive μ_a-measure. Therefore $M_\infty(X^r) = S_1$ and $M_\infty(X^s) \subseteq S_2$. Hence for all $b \in B$, $\operatorname{supp} X^s(0)b \subset \Delta_0^C$ and (10) implies that F_{X^s} is singular with respect to m. Furthermore, for all $b \in B$ we have

$$(F(\Delta)b)(b) = \|E(\Delta)[X^r(0)b + X^s(0)b]\|^2 = \|E(\Delta)X^r(0)b\|^2 + \|E(\Delta)X^s(0)b\|^2$$
$$= (F_{X^r}(\Delta)b)(b) + (F_{X^s}(\Delta)b)(b),$$

because $E(\Delta)X^r(0)b \in S_1$, $E(\Delta)X^s(0)b \in S_2$ and the subspaces S_1 and S_2 are orthogonal. Consequently, $F = F_{X^r}+F_{X^s}$ and from Lemma 3.1 we have $F_{X^r} = F_a$ and $F_{X^s} = F_s$.

Now we prove our main result.

3.3. THEOREM. *Let $X\colon \mathbf{Z} \to L(B, G)$ be a stationary process, X^r the regular part and X^s the singular part of X. Suppose that the spectral measure F of X satisfies (2) and there exists a representation $Y\colon \mathbf{Z} \to L(B, L^2(K, \mu, H))$ of X such that for some orthonormal basis $\{e_i\}_{i\in A}$ in H*

(11) *The process $P_i Y$ is nonsingular for each $i \in A$.*

Then F_a is the spectral measure of X^r and F_s is the spectral measure of X^s.

Proof. As above we denote by S_1 the set of all functions from $L^2(K, \mu, H)$ with supports in Δ_0 and by S_2 the set of all functions with supports in Δ_0^C. The equalities analogous to (8)–(10) are also true, i.e.

$$(F(\Delta)b)(b) = \int_\Delta \|Y(0)b\|_H^2\,d\mu, \quad (F_{Y^r}(\Delta)b)(b) = \int_\Delta \|Y^r(0)b\|_H^2\,d\mu,$$

$$(F_{Y^s}(\Delta)b)(b) = \int_\Delta \|Y^s(0)b\|_H^2\,d\mu.$$

Since $F_{Y^r} \ll m$ (cf. [1]), the argument used in the proof of 3.2 gives $M_{-\infty}(Y^r) \subseteq S_1$. By Lemma 2.2(iii) it follows that $P_i Y^s(0)b \in M_{-\infty}(Y_i) = M_\infty(Y_i^s)$. Since $Y_i = P_i Y$ is not singular, by Lemma 3.2, $P_i Y^s(0)b \in S_2$ in $L^2(K, \mu, \mathbf{C})$ for all $i \in A$, $b \in B$. Hence $Y^s(0)b \in S_2$ in $L^2(K, \mu, H)$ (because by (6), $Y^s(0)b = \sum_{i\in A} P_i Y^s(0)b$) and F_{Y^s} is singular with respect to m. Since S_2 reduces T, we have $M_\infty(Y^s) \subseteq S_2$. Again as in 3.2 we get $F = F_{Y^r}+F_{Y^s}$, which implies $F_{Y^r} = F_a$, $F_{Y^s} = F_s$. Obviously $F_{Y^r} = F_{X^r}$ and $F_{Y^s} = F_{X^s}$, which completes the proof.

3.4. Remark. The assumption (11) depends on the choice of the orthonormal basis in H (cf. 4.2). We may formulate the condition independently of the choice of the basis in the following way:

(12) For each $h \in H$ there exists a function $f_h \in M_\infty(Y)$ such that $P_h f_h \neq 0$ and $P_h f_h$ is orthogonal to $P_h M_0(Y)$ (P_h denotes the orthogonal projection onto W_h).

The condition (12) is equivalent to the fact that (11) holds for any choice of the basis. An example of a nonregular process satisfying (12) is given in 4.3. Notice that $P_h f(z) = \langle f(z), h\rangle_H h$ for every $f \in L^2(K, \mu, H)$, $h \in H$, $\|h\| = 1$.

3.5. COROLLARY. *If there exists a representation of X satisfying (11), then for all $b \in B$, $X^r(\cdot)b$ is the regular part and $X^s(\cdot)b$ is the singular part of the one-dimensional process $X(\cdot)b$.*

Proof. For each $b \in B$, $X(\cdot)b$ is a univariate stationary process and $(F(\cdot)b)(b)$ is its spectral measure. Moreover, $X(\cdot)b = X^r(\cdot)b+X^s(\cdot)b$. If $X^r(\cdot)b = 0$, then $X(\cdot)b = X^s(\cdot)b$ is singular since by Theorem 3.3 its spectral measure is singular with respect to m. Suppose now that $X^r(\cdot)b \neq 0$. By Theorem 3.3 its spectral measure is $(F_a(\cdot)b)(b)$. Lemma 8.8 in [1] implies that $X^r(\cdot)b$ is regular, hence $(F_a(\cdot)b)(b)$ as the spectral measure of a regular process satisfies

$$\int_K \log\frac{(dF_a(z)b)(b)}{dm} > -\infty. \tag{13}$$

Since $(F_a(\cdot)b)(b)$ is the absolutely continuous part of the measure $(F(\cdot)b)(b)$, (13) implies that $X(\cdot)b$ is nonsingular. As for a one-dimensional nonsingular process its decompositions are concordant, $(F_a(\cdot)b)(b)$ is the spectral measure of the regular part and $(F_s(\cdot)b)(b)$ is the spectral measure of the singular part of $X(\cdot)b$, which completes the proof.

3.6. COROLLARY. *If T is the shift operator of a process $X\colon \mathbf{Z} \to L(B, L^2(K, \mu, \mathbf{C}))$ and $X(\cdot)b$ is regular for all $b \in B$, then X is either singular or regular.*

Proof. In this case the nonsingularity of X means that the condition (11) holds. Corollary 3.5 implies that $X(k) = X^r(k)$, hence X is regular.

3.7. COROLLARY. *If the spectral measure of a stationary process $Y\colon \mathbf{Z} \to L(B, L^2(K, \mu, H))$ is absolutely continuous with respect to m, then either Y is regular or for any choice of an orthonormal basis $\{e_i\}_{i\in A}$ in H there exists $i \in A$ such that $P_i Y$ is singular.*

Proof. This follows immediately from Theorem 3.3.

4. Examples. In this section we will give several examples related to Theorem 3.3. The first shows that the nonsingularity of a process X is not sufficient for the concordance of the Wold decomposition with its spectral measure decomposition.

4.1. EXAMPLE. Let $B = \overline{\mathrm{sp}}\{z^k\colon k \geq 0\} \subseteq L^2(K, m, \mathbf{C})$. B is a Hilbert space with the orthonormal basis $\{z^k\colon k \geq 0\}$. Let $X_1(0)(z^k) = z^{-k}$, $k \geq 0$. Clearly $X_1(0)$ uniquely extends to a linear isometry $X_1(0)\colon B \to L^2(K, m, \mathbf{C})$. Let $X_2(0)f = f$, $f \in B$. If we put $X_1(k) = T^k X_1(0)$, $X_2(k) = T^k X_2(0)$, $k \in \mathbf{Z}$, then both X_1, X_2 are stationary processes. It is easy to show that $M_\infty(X_1) = M_\infty(X_2) = L^2(K, m, \mathbf{C})$ and

$$M_k(X_1) = \overline{\mathrm{sp}}\{T^l z^{-n}\colon l \leq k,\ n \geq 0\} = \overline{\mathrm{sp}}\{z^{l-n}\colon l \leq k,\ n \geq 0\} = \overline{\mathrm{sp}}\{z^l\colon l \leq k\},$$

$$M_k(X_2) = \overline{\mathrm{sp}}\{z^{l+n}\colon l \leq k,\ n \geq 0\} = \overline{\mathrm{sp}}\{z^m\colon m \in \mathbf{Z}\} = L^2(K, m, \mathbf{C}) = M_\infty(X_2).$$

Hence X_1 is regular and X_2 is singular. Now we put

$$X(k)f = (X_1(k)f, X_2(k)f), \quad f \in B.$$

Then $X\colon \boldsymbol{Z} \to L(B, L^2(K, m, \boldsymbol{C}^2))$, $X(k) = T^k X(0)$. We have

$$M_k(X) = \overline{\operatorname{sp}}\{(z^{l-n}, z^{l+n})\colon l \leqslant k,\ n \geqslant 0\}.$$

Hence all functions of the form $(0, z^l - z^{l-2})$, $l \in \boldsymbol{Z}$, $(z^l - z^{l-2}, 0)$, $l \leqslant k$, belong to $M_k(X)$. We claim that

(14) $$\overline{\operatorname{sp}}\{z^k - z^{k-2}\colon k \in \boldsymbol{Z}\} = L^2(K, m, \boldsymbol{C}).$$

Suppose that $f \in L^2(K, m, \boldsymbol{C})$ is orthogonal to $z^k - z^{k-2}$ for all $k \in \boldsymbol{Z}$. Then

(15) $$\int_K f\bar{z}^k\,dm = \int_K f\bar{z}^{k-2}\,dm.$$

Let $f = \sum_{k=-\infty}^{\infty} a_k z^k$ be the Fourier series of f. Then $a_k = \int_K f\bar{z}^k\,dm$ and (15) implies that

$$\ldots = a_{-2} = a_0 = a_2 = a_4 = \ldots, \qquad \ldots = a_{-1} = a_1 = a_3 = \ldots$$

Since $\|f\|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2$, $a_k = 0$ for all $k \in \boldsymbol{Z}$. Hence $f = 0$, which gives (14). This implies $M_\infty(X) = L^2(K, m, \boldsymbol{C}^2)$ and $M_{-\infty}(X) = \{0\} \times L^2(K, m, \boldsymbol{C})$. Therefore X is nonsingular, $X^{\mathrm{r}}(k) = (X_1(k), 0)$ and $X^{\mathrm{s}}(k) = (0, X_2(k))$. It is easy to show that both X^{r} and X^{s} have absolutely continuous spectral measures, hence this yields the example we were looking for. Notice that for this process the condition (11) does not hold.

The next example shows that the condition (11) depends on the choice of an orthonormal basis in H.

4.2. EXAMPLE. We define the measure μ on $\mathscr{B}(K)$ by

$$\mu(\Delta) = \begin{cases} m(\Delta) & \text{if } 1 \notin \Delta, \\ 1 & \text{if } \Delta = \{1\}. \end{cases}$$

Let

$$f_0(z) = \begin{cases} 1 & \text{if } z \neq 1, \\ 0 & \text{if } z = 1. \end{cases}$$

The system of functions $\{1 - f_0, z^k f_0\colon k \in \boldsymbol{Z}\}$ is an orthonormal basis in $L^2(K, \mu, \boldsymbol{C})$. Let $B = \overline{\operatorname{sp}}\{1 - f_0, z^k f_0\colon k \geqslant 0\}$. We define a pair of stationary processes with values in B. Let $X_1(0)(1 - f_0) = 0$, $X_1(0)(z^k f_0) = z^{-k} f_0$, $X_2(0)(1 - f_0) = 1 - f_0$, $X_2(0)(z^k f_0) = z^{-k} f_0$ for all $k \geqslant 0$. $X_1(0)$ and $X_2(0)$ uniquely extend to linear continuous operators from B into $L^2(K, \mu, \boldsymbol{C})$. As

above, $X_1(k) = T^k X_1(0)$, $X_2(k) = T^k X_2(0)$ for each $k \in \boldsymbol{Z}$. If we put $X(k)f = (X_1(k)f, X_2(k)f)$, $f \in B$, we get

$$M_k(X) = \overline{\operatorname{sp}}\{(0, 1 - f_0), (z^n f_0, z^n f_0)\colon n \leqslant k\}$$

(notice that $T^l(1 - f_0) = 1 - f_0$) and hence $M_{-\infty}(X) = \overline{\operatorname{sp}}\{(0, 1 - f_0)\} = \{0\} \times S_2$, where S_2 is as in the proof of Lemma 3.2. Therefore, for each $k \in \boldsymbol{Z}$,

$$X^{\mathrm{s}}(k)(1 - f_0) = (0, 1 - f_0), \qquad X^{\mathrm{s}}(k)(z^n f_0) = (0, 0), \qquad n \geqslant 0,$$

$$X^{\mathrm{r}}(k)(1 - f_0) = (0, 0), \qquad X^{\mathrm{r}}(k)(z^n f_0) = (z^{-n+k} f_0, z^{-n+k} f_0), \qquad n \geqslant 0.$$

Let $f \in B$. Then $f = a(1 - f_0) + \sum_{k=0}^{\infty} a_k f_0 z^k$ and

$$(F(\Delta)f)(f) = \|E(\Delta)X(0)f\|^2 = \Big\|E(\Delta)\Big[X(0)\big(a(1 - f_0)\big) + X(0)\Big(\sum_{k=0}^{\infty} a_k f_0 z^k\Big)\Big]\Big\|^2$$

$$= \int_\Delta \Big(\Big|\sum_{k=0}^{\infty} a_k f_0 z^{-k}\Big|^2 + \Big|\sum_{k=0}^{\infty} a_k f_0 z^{-k}\Big|^2\Big)\,dm + |a|^2 \int_\Delta d\mu_s$$

$$= 2\int_1 |f_1|^2\,dm + |f(1)|^2 \mu_s(\Delta),$$

where $f_1(z) = \sum_{k=0}^{\infty} a_k f_0 z^{-k}$. In the same way we get

$$(F_{X^{\mathrm{r}}}(\Delta)f)(f) = 2\int_1 |f_1|^2\,dm, \qquad (F_{X^{\mathrm{s}}}(\Delta)f)(f) = |f(1)|^2 \mu_s(\Delta),$$

where μ_s denotes the singular part of μ with respect to m. Hence $F_{X^{\mathrm{r}}} = F_{\mathrm{a}}$, $F_{X^{\mathrm{s}}} = F_{\mathrm{s}}$.

Consider now the following orthonormal basis in $\boldsymbol{C}^2$:

$$e_1 = \frac{1}{\sqrt{2}}(-1, 1), \qquad e_2 = \frac{1}{\sqrt{2}}(1, 1).$$

Then

$$P_{e_1} X(0)(1 - f_0) = \frac{1}{2}\big(-(1 - f_0), 1 - f_0\big), \qquad P_{e_1} X(0)(z^k f_0) = (0, 0),$$

$$P_{e_2} X(0)(1 - f_0) = \frac{1}{2}(1 - f_0, 1 - f_0), \qquad P_{e_2} X(0)(z^k f_0) = (z^{-k} f_0, z^{-k} f_0).$$

This shows that the projection of X onto W_{e_1} is a singular process. Hence (12) is not satisfied in spite of the concordance of the decomposition.

Finally, we show an example of a nonregular process satisfying (12).

4.3. EXAMPLE. Suppose that μ, B and $X_2(k)$ are the same as in 4.2. We put $X_1(0)(z^k f_0) = z^{-2k} f_0$, $X_1(0)(1-f_0) = 1-f_0$. As above, $X_1(k) = T^k X_1(0)$ and $X(k) = (X_1(k)f, X_2(k)f)$, $f \in B$. We have

$$M_\infty(X) = \overline{\operatorname{sp}}\{(1-f_0, 1-f_0), (z^{2n+l} f_0, z^{n+l} f_0): l \in \boldsymbol{Z}, n \leqslant 0\}.$$

It is easy to verify that all functions of the form $(f_0(z^k - z^{k-1}), 0)$ and $(0, f_0(z^k - z^{k-1}))$, $k \in \boldsymbol{Z}$, belong to $M_\infty(X)$. As in 4.1 one can prove that $\overline{\operatorname{sp}}\{f_0(z^k - z^{k-1}): k \in \boldsymbol{Z}\} = L^2(K, m, C) = S_1$. Hence

$$M_\infty(X) = (S_1 \times S_1) \oplus \overline{\operatorname{sp}}\{(1-f_0, 1-f_0)\}.$$

In the same way we get

$$M_0(X) = (\overline{\operatorname{sp}}\{z^k f_0: k \leqslant -1\} \times \overline{\operatorname{sp}}\{z^k f_0: k \leqslant -1\}) \oplus \overline{\operatorname{sp}}\{(f_0, f_0)\}$$

$$\oplus \overline{\operatorname{sp}}\{(1-f_0, 1-f_0)\}$$

and $M_{-\infty}(X) = \operatorname{sp}\{(1-f_0, 1-f_0)\}$. Therefore X is nonregular.

Let $(a, b) \in C^2$, $|a|^2 + |b|^2 = 1$, and $f = (f_1, f_2) \in M_\gamma(X)$. The projection $P_{(a,b)}$ onto the subspace $W_{(a,b)}$ has the following form:

$$P_{(a,b)} f = (a^2 f_1 + ab f_2, ab f_1 + b^2 f_2)$$

(in the general case in $L^2(K, \mu, H)$, $P_h f(z) = \langle f(z), h \rangle_H h$, $h \in H$, $\|h\| = 1$). If we take $f(z) = (z f_0, z^2 f_0) \in M_\infty(X)$, then $P_{(a,b)} f \neq 0$. The formula for $M_0(X)$ easily implies that

$$P_{(a,b)} f = (a^2 z f_0 + abz^2 f_0, abz f_0 + b^2 z^2 f_0)$$

is orthogonal to $P_{(a,b)} M_0(X)$ for every $(a, b) \in \boldsymbol{C}^2$, $|a|^2 + |b|^2 = 1$.

Similarly to 4.2 one can verify that for this process the concordance of its decompositions holds.

References

[1] S. A. Chobanyan and A. Weron, *Banach-space-valued stationary process and their linear prediction*, Dissertationes Math. 125 (1975).

[2] A. Makagon, *An isomorphism theorem for Banach space valued stationary stochastic sequences*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978), 169–173.

[3] A. Makagon and F. Schmidt, *A decomposition theorem for densities of positive operator-valued measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (1980), 41–43.

[4] A. Makagon and A. Weron, *Wold–Cramér concordance theorems for interpolation of q-variate stationary processes over locally compact abelian groups*, J. Multivariate Anal. 6 (1976), 123–137.

[5] J. B. Robertson, *Orthogonal decomposition of multivariate weakly stationary processes*, Canad. J. Math. 20 (1986), 386–383.

[6] H. Salehi and J. K. Scheidt, *Interpolation of q-variate weakly stationary processes over a locally compact abelian group*, J. Multivariate Anal. 2 (1972), 307–331.

[7] F. Schmidt, *Banach space valued stationary processes with absolutely continuous spectral function*, in: Lecture Notes in Math. 656, Springer, 1978, 237–244.

[8] A. Weron, *Second order stochastic processes and the dilation theory in Banach spaces*, Ann. Inst. H. Poincaré 16 (1980), 29–38.

INSTYTUT MATEMATYKI POLITECHNIKI SZCZECIŃSKIEJ
INSTITUTE OF MATHEMATICS, SZCZECIN TECHNICAL UNIVERSITY
Al. Piastów 17, 70-310 Szczecin, Poland

Received December 22, 1986 (2260)

Added in proof (March 1988). It turns out that Lemma 3.2 is a special case of Corollary 3.8 of the recent paper by A. Makagon and H. Salehi, *Infinite dimensional stationary sequence with multiplicity one*, Ann. Acad. Sci. Fenn. Ser. AI Math. 1 (1987), 135–150.