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The projective tensor product of Fréchet–Montel spaces

by

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Abstract. We construct a Fréchet–Montel space F for which $F\hat{\otimes}_\pi F$ is not a Montel space. It follows that the spaces $L_b(F, F'_b)$, $B_{bb}(F, F)$ and $F'_b\hat{\otimes}_\varepsilon F'_b$ are not (DF)-spaces. We also show that if X is an L^p-space with $1 < p < \infty$, then there is a Fréchet–Montel space F such that $L_b(F, X')$, $B_{bb}(F, X)$ and $F'_b\hat{\otimes}_\varepsilon X'$ are not (DF)-spaces.

It is well known that the projective tensor product of Fréchet–Schwartz spaces is again a Schwartz space. This fact was already proved by Grothendieck in his thesis. It has been conjectured that the similar statement would also be true for Fréchet–Montel spaces ([6], 45.3). However, in this paper we shall construct an example of a Fréchet–Montel space F for which $F\hat{\otimes}_\pi F$ is not a Montel space.

The preceding question is equivalent to "Problême des topologies" of Grothendieck (see [3], *Question non résolue* 2) and we get a new counterexample to it, too. The first counterexamples for Fréchet spaces were given in [9].

The Fréchet–Montel space F has the approximation property. Hence, we have the duality relation

$$(F'_b\hat{\otimes}_\varepsilon F'_b)'_b = F\hat{\otimes}_\pi F.$$

Using this we get an example of a (DFM)-space F'_b for which $F'_b\hat{\otimes}_\varepsilon F'_b$ is not even a (DF)-space. Thus we get an answer to *Question non résolue* 10 in [3].

The spaces $L_b(F, F'_b)$ and $B_{bb}(F, F)$ are topologically isomorphic to $F'_b\hat{\otimes}_\varepsilon F'_b$ if F is as above. Thus, they are not (DF)-spaces, and we get other counterexamples to the questions of Grothendieck.

Let X be an infinite-dimensional l^p- or L^p-space with $1 < p < \infty$. We shall also show that for a suitable Fréchet–Montel space F the space $F\hat{\otimes}_\pi X$ does not have property (BB), i.e. not all bounded sets B of $F\hat{\otimes}_\pi X$ are contained in sets $\overline{\Gamma(B_1\otimes B_2)}$, where $B_1 \subset F$ and $B_2 \subset X$ are bounded. It follows again that $L_b(F, X')$, $B_{bb}(F, X)$ and $F'_b\hat{\otimes}_\varepsilon X'$ are not (DF)-spaces.

Section 1 contains notation and preliminary results. In Section 2 we

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study certain extension properties of tensors in Banach spaces. As a consequence of Pisier's results in [8] we get a proposition which is later used in the counterexamples. Section 3 contains the main construction and the counterexample concerning Fréchet–Montel spaces. In Section 4 we study the projective tensor product of a Fréchet–Montel space and a Banach space. The other results are given in Section 5.

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1. Preliminaries. For locally convex spaces and tensor products we shall use the notation and the definitions of [6]. Let us, however, recall some of the most important facts. A barrelled locally convex space is *Montel* if all bounded sets are precompact. A locally convex space E is (DF) if it has a fundamental sequence of bounded sets and if every strongly bounded subset of E' which is a union of countably many equicontinuous sets is also equicontinuous. By a *(DFM)-space* we mean a (DF)-space which is also a Montel space. A locally convex space E is said to have the *approximation property* if the identity operator on E can be approximated uniformly on precompact sets by finite rank operators.

We denote by $L(E, F)$ the space of continuous linear mappings $E \to F$ and by $B(E, F)$ the space of continuous bilinear forms on $E \times F$. The topological dual of a locally convex space E is denoted by E'; E'_b is the dual equipped with the strong topology.

The projective and injective tensor product and the ε-products of the spaces E and F are denoted by $E \otimes_\pi F$, $E \otimes_\varepsilon F$, $\varepsilon(E, F)$ and $E\varepsilon F$, respectively (for definitions, see [6]). The completion OF $E \otimes_\pi F$ (resp. $E \otimes_\varepsilon F$) is $E \hat{\otimes}_\pi F$ (resp. $E \hat{\otimes}_\varepsilon F$). For all locally convex spaces E and F, $E\varepsilon F$ is topologically isomorphic to $\varepsilon(E, F)$. Moreover, if one of the spaces E and F has the approximation property, we have

$$E \hat{\otimes}_\varepsilon F = E\varepsilon F. \tag{1.1}$$

In the case E and F are Fréchet–Montel spaces we have the topological isomorphism

$$(E'_b \varepsilon F'_b)'_b = E \hat{\otimes}_\pi F \tag{1.2}$$

(see [6], 45.3(1) and 44.3(8)).

If p and q are seminorms in E and F, we set

$$(p \otimes q)(z) = \inf \sum_{i=1}^{n} p(x_i)\, q(y_i),$$

where the infimum is taken over all representations $z = \sum_i x_i \otimes y_i$ with $x_i \in E$ and $y_i \in F$. Similarly

$$(p \otimes_\varepsilon q)(z) = \sup_{(u,v) \in G_1 \times G_2} \Big| \sum_{i=1}^{n} (ux_i)(vy_i) \Big|,$$

where G_1 and G_2 are the polars

$$G_1 = \{x \in E \mid p(x) \leqslant 1\}^\circ \subset E', \qquad G_2 = \{y \in F \mid q(y) \leqslant 1\}^\circ \subset F'$$

and $z = \sum x_i \otimes y_i$. If M and N are subspaces of E and F and $z \in M \otimes N$, we denote

$$\big((p \mid M) \otimes (q \mid N)\big)(z) = \inf \sum_{i=1}^{n} p(x_i)\, q(y_i),$$

where the infimum is taken only over representations $z = \sum x_i \otimes y_i$ with $x_i \in M$ and $y_i \in N$. Thus for all $z \in M \otimes N$ we have $\big((p \mid M) \otimes (q \mid N)\big)(z) \geqslant (p \otimes q)(z)$; see [9], Section 4, for more details. The same notation will be used for the ε-tensor product, too. In this case we have always $(p \mid M) \otimes_\varepsilon (q \mid N) = p \otimes_\varepsilon q$ in $M \otimes N$.

The following theorem is a special case of [8], Theorem 3.2.

THEOREM 1.1. *The Hilbert space l^2 can be isometrically imbedded into a separable Banach space X for which $X \otimes_\pi X$ and $X \otimes_\varepsilon X$ are topologically isomorphic.*

2. On the extension of bilinear forms in Banach spaces. We used already in [9], Section 4, a quantitative analysis of the extension properties of tensors in Banach spaces. To get a counterexample to "Problème des topologies" in the case of Fréchet–Montel spaces we need the best possible results in this field. These can be achieved by using the Banach space X constructed by Pisier.

PROPOSITION 2.1. *There exists a separable Banach space (E, p) and a family of its n-dimensional subspaces $(M_n)_{n \in \mathbf{N}}$ with the following property:*

$$\big((p \mid M_n) \otimes (p \mid M_n)\big)(z_n) > Cn\,(p \otimes p)(z_n) \tag{2.1}$$

for some $z_n \in M_n \otimes M_n$ and a universal positive constant C.

Remark. Given E and (M_n) we can always find a projection P from E onto M_n with $\|P\| \leqslant \sqrt{n}$ (see [7], 28.2). From this it follows that

$$\big((p \mid M_n) \otimes (p \mid M_n)\big)(z) \leqslant n\,(p \otimes p)(z)$$

for all $z \in M_n \otimes M_n$ ([9], 4(1)). Thus, Proposition 2.1 is an optimal result in the obvious sense.

For our purposes it is essential that the coefficient Cn in the right side of (2.1) satisfies

$$\sup_{n \in \mathbf{N}} \left\{ \frac{\varrho_n^2}{Cn} \right\} < \infty,$$

where $\varrho_n = \inf\{\|P\| \mid P$ is a projection from E onto $M_n\}$. However, it is not necessary that (ϱ_n) grows as fast as $\sqrt{n}$ when n tends to infinity; any M_n with unbounded (ϱ_n) would do in this respect.

Proof of 2.1. We take the space X of Theorem 1.1 for E and for M_n the spaces $l_n^2 = \mathrm{sp}((e_k)_{k=1}^n) \subset l^2 \subset E$, where $(e_k)_{k=1}^\infty$ is the natural basis of l^2. By Theorem 1.1 there is a positive constant C such that $p\otimes_\varepsilon p > C(p\otimes p)$.

Consider the tensor $z_n = \sum_{i=1}^n e_i \otimes e_i \in M_n \otimes M_n \subset E\otimes E$. By [6], 42.6(1), $((p\,|\,M_n)\otimes(p\,|\,M_n))(z_n) = n$. On the other hand, $((p\,|\,M_n)\otimes_\varepsilon(p\,|\,M_n))(z_n) = 1$, because z_n can be considered as an isometry $M_n' \to M_n$ and the ε-norm is equal to the operator norm of $L(M_n', M_n)$.

Combining these facts we get

$$((p\,|\,M_n)\otimes(p\,|\,M_n))(z_n) = n((p\,|\,M_n)\otimes_\varepsilon(p\,|\,M_n))(z_n)$$
$$= n(p\otimes_\varepsilon p)(z_n) > Cn(p\otimes p)(z_n)$$

for some constant C. ∎

Proposition 2.1 can also be expressed in terms of bilinear forms. In the following E and M_n are as above.

Corollary 2.2. *There is a bilinear form $b\in B(M_n, M_n)$ such that*

$$\|\tilde{b}\| > Cn\|b\|$$

for every extension $\tilde{b}\in B(E, E)$ of b.

Proof. Let I be the identity mapping $(M_n\otimes M_n,\, p\otimes p) \to M_n\otimes_\pi M_n$. By Proposition 2.1, $\|I'\| = \|I\| > Cn$ (I' is the adjoint of I). There exists thus a $b\in(M_n\otimes_\pi M_n)'$ such that

$$\|b\|_{(M_n\otimes M_n,\, p\otimes p)'} > Cn\|b\|_{(M_n\otimes_\pi M_n)'}.$$

For an arbitrary extension $\tilde{b}\in(E\otimes_\pi E)'$ of b we then have

$$\|\tilde{b}\|_{(E\otimes_\pi E)'} > Cn\|b\|_{(M_n\otimes_\pi M_n)'}.$$

The assertion follows now from the natural isometry of $(E\otimes_\pi E)'$ (resp. $(M_n\otimes_\pi M_n)'$) and $B(E, E)$ (resp. $B(M_n, M_n)$). ∎

3. The projective tensor product of Fréchet–Montel spaces.

3.1. Definitions. Since no Montel space can contain an infinite-dimensional normable subspace, we cannot use the space E of the preceding section as such in the construction of our Fréchet–Montel space. However, it is not difficult to find finite-dimensional subspaces of E which are suitable for our purposes.

Let (E, p), (M_n) and (z_n) be as in Proposition 2.1. For each z_n we take a finite representation $z_n = \sum_{i=1}^{m_n} a_{in}\otimes b_{in}$ such that

$$\sum_i p(a_{in})\,p(b_{in}) \leqslant 2(p\otimes p)(z_n).$$

Let us define for all $n\in N$ the subspace $E_n = \mathrm{sp}\{a_{in}, b_{in}\,|\, i = 1, \ldots, m_n\} + M_n$ of E. Proposition 2.1 implies now

(3.1) $\quad ((p\,|\,M_n)\otimes(p\,|\,M_n))(z_n) \geqslant Cn(p\otimes p)(z_n) \geqslant \frac{1}{2}Cn((p\,|\,E_n)\otimes(p\,|\,E_n))(z_n).$

We choose a projection Q_n from (E_n, p) onto (M_n, p) with $\|Q_n\| \leqslant \sqrt{n}$ (this is possible by [7], 28.2) and set $R_n = \mathrm{id}_{E_n} - Q_n$ and $N_n = R_n(E_n)$.

We are now ready to construct the Fréchet–Montel space F we need in the counterexample. For all $n\in N$ let us denote.

$$A_n := M_n \oplus \bigoplus_{t=1}^{n} N_{tn},$$

where $N_{tn} = N_n$ for $1 \leqslant t \leqslant n$. We define a family of seminorms in A_n as follows: if $z = x + \sum y_t \in A_n$ with $x\in M_n$, $y_t\in N_{tn}$, then for $k \geqslant 1$

$$(p_k\,|\,A_n)(z) = n^{(k-1)/(2k)}\,p\Big(x + \sum_{t=1}^{n} y_t\Big) + \sum_{t=1}^{n} t^k p(y_t) + \sum_{t=1}^{n\wedge k} n^k p(y_t).$$

Here $n\wedge k = \min\{n, k\}$. Note that $x + \sum_t y_t$ and y_t are considered as elements of E_n in a natural way when taking the seminorm p. Finally, F will be the space

$$F = \{z = (z_n)_{n\in N}\,|\, z_n\in A_n,\ p_k(z) := \sum_{n\in N}(p_k\,|\,A_n)(z_n) < \infty\}.$$

It is clear that F is a Fréchet space when topologized by the seminorms $p_1 \leqslant p_2 \leqslant p_3 \leqslant \ldots$

We denote the closed unit ball of p_k by V_k.

Let us give some explanations for the preceding definitions. We shall choose tensors $z_n\in M_n\otimes M_n$ using formula (3.1) applied to the terms $n^{(k-1)/(2k)}\,p(x+\sum y_t)$ in the definition of $p_k\,|\,A_n$. It turns out that for each $k < n$ the tensor z_n has a representation $z_n = \sum a_i\otimes b_i$ in $(M_n\oplus N_{tn})\otimes(M_n\oplus N_{tn})$, $t > k$, for which

$$(p_k\otimes p_k)(z_n) \leqslant \sum p_k(a_i)\,p_k(b_i) < C_k.$$

Here $C_k > 0$ does not depend on n. On the other hand, we shall show that $(p_2\otimes p_2)(z_n) > C > 0$ for all $n\in N$ and a constant C. It follows that $\mathrm{sp}\{z_n\,|\, n\in N\}$ is isomorphic to a normed space. It is essential that the "good" representations $\sum a_i \otimes b_i$ of z_n are "very different" for different p_k. This causes the different behaviour of F and $F \hat{\otimes}_\pi F$ with respect to the Montel property.

The purpose of the terms $n^k p(y_t)$ in the definition of $p_k | A_n$ is to ensure that $\bigoplus_{n \in N} N_{tn}$, t fixed, is not normable; the sums $\sum t^k p(y_t)$ do the same in every $\bigoplus_{n \in N} N_{nf(n)}$, where f is an arbitrary unbounded function $N \to N$.

PROPOSITION 3.2. *The space F is Montel.*

Proof. It is enough to show that every bounded set $B \subset F$ is precompact. We may assume that B is of the form $\bigcap_{l=1}^\infty r_l V_l$ with $r_l \geq 1$. We fix some $\varepsilon > 0$ and V_k, $k \geq 2$, and choose $n_0 \in N$ such that

$$n^{(k-1)/(2k)} < \frac{\varepsilon}{4r_{k+1}} n^{k/(2(k+1))} \tag{3.2}$$

for $n \geq n_0$ and

$$t^k < \frac{\varepsilon}{4r_{k+1}} t^{k+1} \tag{3.3}$$

for $t \geq n_0$. Then we take an $s \in N$ with $s \geq \max\{n_0, k+1\}$ and $n_1 \in N$, $n_1 > n_0$, such that

$$n^k < \frac{\varepsilon}{8r_s} n^s \tag{3.4}$$

for $n \geq n_1$.

Let $z \in B$ be arbitrary. We denote the A_n-coordinate of z by $z_n = x_n + \sum_{t=1}^n y_{tn}$, where $x_n \in M_n$ and $y_{tn} \in N_{tn}$.

I) We first consider the coordinates z_n, $n \geq n_1$. Using (3.2) and (3.3) we get

$$\begin{aligned} & n^{(k-1)/(2k)} p\Big(x_n + \sum_{t=1}^n y_{tn}\Big) + \sum_{t=n_0}^n t^k p(y_{tn}) \\ & \quad < \frac{\varepsilon}{4r_{k+1}} \Big(n^{k/(2(k+1))} p\Big(x_n + \sum_{t=1}^n y_{tn}\Big) + \sum_{t=n_0}^n t^{k+1} p(y_{tn})\Big) \\ & \quad \leq \frac{\varepsilon}{4r_{k+1}} (p_{k+1} | A_n)(z_n). \end{aligned} \tag{3.5}$$

Similarly by (3.4) (note that $n_0 < n$, $n_0 < s$ and $k < s$)

$$\begin{aligned} & \sum_{t=1}^{n_0-1} t^k p(y_{tn}) + \sum_{t=1}^{n \wedge k} n^k p(y_{tn}) \\ & \quad \leq \sum_{t=1}^{n \wedge s} 2n^k p(y_{tn}) < \frac{\varepsilon}{4r_s} \sum_{t=1}^{n \wedge s} n^s p(y_{tn}) \leq \frac{\varepsilon}{4r_s} (p_s | A_n)(z_n). \end{aligned} \tag{3.6}$$

By (3.5) and (3.6) we get

$$\begin{aligned} p_k((0, \ldots, 0, z_{n_1}, z_{n_1+1}, \ldots)) &= \sum_{n=n_1}^\infty (p_k | A_n)(z_n) \\ &= \sum_{n=n_1}^\infty \Big(n^{(k-1)/(2k)} p\Big(x_n + \sum_{t=1}^n y_{tn}\Big) + \sum_{t=n_0}^n t^k p(y_{tn}) + \sum_{t=1}^{n_0-1} t^k p(y_{tn}) + \sum_{t=1}^{n \wedge k} n^k p(y_{tn})\Big) \\ &< \sum_{n=n_1}^\infty \Big(\frac{\varepsilon}{4r_{k+1}} (p_{k+1} | A_n)(z_n) + \frac{\varepsilon}{4r_s} (p_s | A_n)(z_n)\Big) \\ &\leq \frac{\varepsilon}{4r_{k+1}} p_{k+1}(z) + \frac{\varepsilon}{4r_s} p_s(z) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned} \tag{3.7}$$

II) The subspace $\bigoplus_{n=1}^{n_1-1} A_n$ of F is finite-dimensional. Thus we can choose a set $(v_i)_{i=1}^m \subset F$ such that for all $z \in B = \bigcap_l r_l V_l$

$$(z_1, \ldots, z_{n_1-1}, 0, 0, \ldots) \in B \cap \Big(\bigoplus_{n=1}^{n_1-1} A_n\Big) \subset \bigcup_{i=1}^m \Big(v_i + \frac{\varepsilon}{2} V_k \cap \Big(\bigoplus_{n=1}^{n_1-1} A_n\Big)\Big).$$

Combining this with (3.7) we get

$$B \subset \bigcup_{i=1}^m (v_i + \varepsilon V_k),$$

which completes the proof. ∎

THEOREM 3.3. *There exists a Fréchet–Montel space F such that l^1 is topologically isomorphic to a subspace of $F \hat{\otimes}_\pi F$.*

Proof. Let F be as above. We first define suitable tensors in the spaces $A_n \otimes A_n$. By formula (3.1) there exists $z_n \in M_n \otimes M_n \subset E_n \otimes E_n$ such that

$$((p | M_n) \otimes (p | M_n))(z_n) \geq Cn((p | E_n) \otimes (p | E_n))(z_n).$$

(We have redefined C to remove the unnecessary number 1/2.) We may assume $((p | M_n) \otimes (p | M_n))(z_n) = 1$. Let $\sum_{i=1}^{m_n} \lambda_{in} x_{in} \otimes y_{in}$ be a representation of z_n in $E_n \otimes E_n$ for which $\sum |\lambda_{in}| \leq 2/C$, $p(x_{in}) \leq 1/\sqrt{n}$ and $p(y_{in}) \leq 1/\sqrt{n}$. We define for $1 \leq t \leq n$

$$\tilde{x}_{int} = Q_n x_{in} + R_n x_{in} \in A_n, \quad \tilde{y}_{int} = Q_n y_{in} + R_n y_{in} \in A_n,$$

where $Q_n x_{in} \in M_n$ and $R_n x_{in} \in N_{tn}$ and similarly for $\tilde{y}_{int}$. By the inclusion $M_n \hookrightarrow A_n$ we can consider z_n as an element of $A_n \otimes A_n$. Moreover, z_n has the representations

$$z_n = \sum_{i=1}^{m_n} \lambda_{in} \tilde{x}_{int} \otimes \tilde{y}_{int} \tag{3.8}$$

for all $1 \leq t \leq n$.

I) We form upper bounds for the seminorms $(p_k \otimes p_k)(z_n)$. Since, by definition, $\|R_n\| \leqslant \sqrt{n}+1 < 2\sqrt{n}$, we have

$$p(R_n x_{in}) \leqslant 2\sqrt{n}\, p(x_{in}) \leqslant 2\sqrt{n} \cdot \frac{1}{\sqrt{n}} = 2$$

and similarly for $R_n y_{in}$. For $k < t \leqslant n$ we thus have

(3.9) $$\begin{aligned}(p_k \otimes p_k)(z_n) &= ((p_k \mid A_n) \otimes (p_k \mid A_n))(z_n) \\ &\leqslant \sum_i |\lambda_{in}| (p_k \mid A_n)(\tilde{x}_{in})(p_k \mid A_n)(\tilde{y}_{in}) \\ &= \sum_i |\lambda_{in}| \big(n^{(k-1)/(2k)} p(Q_n x_{in} + R_n x_{in}) + t^k p(R_n x_{in})\big) \\ &\qquad \times \big(n^{(k-1)/(2k)} p(Q_n y_{in} + R_n y_{in}) + t^k p(R_n y_{in})\big) \\ &\leqslant \sum_i |\lambda_{in}| \big(n^{1/2} p(x_{in}) + t^k p(R_n x_{in})\big) \\ &\qquad \times \big(n^{1/2} p(y_{in}) + t^k p(R_n y_{in})\big) \\ &\leqslant \sum_i |\lambda_{in}| (1+2t^k)^2 \leqslant 2(1+2t^k)^2/C.\end{aligned}$$

We choose $t = k+1$ in (3.9). Then

(3.10) $$(p_k \otimes p_k)(z_n) \leqslant \max_{m=1,\dots,k} \left\{(p_k \otimes p_k)(z_m), \frac{2(1+2(k+1)^k)^2}{C}\right\}$$

for all z_n. The right-hand side of (3.10) is clearly independent of n.

II) On the other hand, we need a lower bound for $(p_2 \otimes p_2)(z_n)$. For all $w = a + \sum_{t=1}^{n} b_t \in A_n$ we have

(3.11) $$n^{1/4} p\Big(a + \sum_{t=1}^{n} b_t\Big) + \sum_{t=1}^{n} t^k p(b_t) \geqslant \tfrac{1}{2} p(a),$$

so that

(3.12) $$(p_2 \mid A_n)(w) \geqslant \tfrac{1}{2} p(a) = \tfrac{1}{2} p(Q_n w).$$

Formula (3.11) follows immediately if $p\big(a+\sum b_t\big) \geqslant \frac{1}{2} p(a)$. In the other case we get by the triangle inequality

$$\sum_t p(b_t) \geqslant p\Big(\sum_t b_t\Big) \geqslant \tfrac{1}{2} p(a),$$

which also implies (3.11).

It now follows from (3.12) and the normalization of z_n that

(3.13) $$\begin{aligned}(p_2 \otimes p_2)(z_n) &= ((p_2 \mid A_n) \otimes (p_2 \mid A_n))(z_n) \\ &= \inf_{\sum a_i \otimes b_i = z_n} \sum_i (p_2 \mid A_n)(a_i)(p_2 \mid A_n)(b_i) \\ &\geqslant \inf_{\sum a_i \otimes b_i = z_n} \sum_i \tfrac{1}{4} p(Q_n a_i)\, p(Q_n b_i) = \tfrac{1}{4}((p \mid M_n) \otimes (p \mid M_n))(z_n) = \tfrac{1}{4},\end{aligned}$$

because $z_n \in M_n \otimes M_n$.

Let J be a finite subset of N and $a_n \in \mathrm{K}$ for all $n \in J$. One verifies immediately that

(3.14) $$(p_k \otimes p_k)\Big(\sum_{n\in J} a_n z_n\Big) = \sum_{n \in J} |a_n| ((p_k \mid A_n) \otimes (p_k \mid A_n))(z_n)$$

for all k.

We define a continuous linear mapping Ψ from $\overline{\mathrm{sp}(\{z_n\})} \subset F \hat{\otimes}_\pi F$ onto l^1 by $\Psi(z_n) = e_n$, where (e_n) is the natural basis of l^1. By (3.10), (3.13) and (3.14), Ψ is a topological isomorphism. ∎

Corollary 3.4. *There exists a Fréchet–Montel space F for which $F \hat{\otimes}_\pi F$ is not Montel.*

Corollary 3.5. *Let F be as above. The space $F \hat{\otimes}_\pi F$ does not have property* (BB).

4. The space $F \hat{\otimes}_\pi l^2$. Using the space F constructed in Section 3 we get another counterexample to "Problème des topologies": the space $F \hat{\otimes}_\pi l^2$ does not have property (BB).

We begin with the following remark: If p, E_n, M_m and z_n are as in formula (3.1), then by (3.1)

(4.1) $$\begin{aligned}((p \mid M_n) \otimes (p \mid M_n))(z_n) &\geqslant Cn((p \mid E_n) \otimes (p \mid E_n))(z_n) \\ &\geqslant C\sqrt{n}((p \mid E_n) \otimes (p \mid M_n))(z_n).\end{aligned}$$

Here we have again redefined the constant C. The last inequality follows from the existence of a projection from E_n onto M_n with $\|P\| \leqslant \sqrt{n}$.

It is certainly clear that we would not need the complex construction of Pisier to get an example of spaces E_n and M_n which satisfy (4.1). We discuss later the question to which Banach spaces our construction can be generalized.

We define the space F as in the previous section and we use the same notation.

Theorem 4.1. *The space $F \hat{\otimes}_\pi l^2$ does not have property* (BB).

Proof. The beginning of the proof is analogous to that of Theorem 3.3. The norm of l^2 is denoted by q and the closed unit ball by U. We fix for each n an n-dimensional subspace of l^2 and using formula (4.1) we choose tensors $z_n \in M_n \otimes l_n^2 \subset E_n \otimes l^2$ such that

$$((p \mid M_n)\otimes q)(z_n) \geqslant C\sqrt{n}((p \mid E_n)\otimes q)(z_n).$$

We normalize again $((p \mid M_n)\otimes q)(z_n) = 1$ and choose a representation

$$z_n = \sum_{i=1}^{m_n} \lambda_{in} x_{in} \otimes y_{in}$$

with $\sum |\lambda_{in}| \leqslant 2/C$, $x_{in} \in E_n$, $p(x_{in}) \leqslant 1/\sqrt{n}$, $y_{in} \in l^2$, $q(y) \leqslant 1$. We get representations

$$z_n = \sum_{i=1}^{m_n} \lambda_{in} \tilde{x}_{itn} \otimes y_{in}$$

in $A_n \otimes l^2 \subset F \otimes l^2$ by defining

$$\tilde{x}_{itn} = Q_n x_{in} + R_n x_{in} \in A_n,$$

where $Q_n x_{in} \in M_n$ and $R_n x_{in} \in N_{tn}$.

Analogously to the proof of Theorem 3.3 we see that $(p_k \otimes q)(z_n) < C_k$ for all $n \in N$ and some positive constants C_k. Thus, the set $B = (z_n)_{n \in N}$ is bounded. By antithesis, let

$$B \subset \overline{\Gamma\big(\big(\bigcap_{k \in N} r_k V_k\big) \otimes U\big)}, \tag{4.2}$$

where $r_k > 0$. Since $B \subset F \otimes l^2$, the closure in (4.2) may be taken in $F \otimes_\pi l^2$. It follows from (4.2) that for a fixed V_{k_0} every z_n has a representation

$$z_n = \sum_{i=1}^{m_1} \varrho_{in} a_{in} \otimes b_{in} + \sum_{i=1}^{m_2} \sigma_{in} c_{in} \otimes d_{in}, \tag{4.3}$$

where $\sum_i |\varrho_{in}| \leqslant 1$, $a_{in} \in \bigcap r_k V_k$, $b_{in} \in U$, $\sum |\sigma_{in}| \leqslant 1$, $d_{in} \in U$ and $c_{in} \in (3n)^{-1} V_{k_0}$. We may assume that a_{in}, $c_{in} \in A_n$ for all i and n.

We now consider a_{in} and c_{in} also as elements of (E_n, p) in the natural way (see the remark after the definition of $(p_k \mid A_n)$ in Section 3.1). Because $z_n \in M_n \otimes l^2$,

$$z_n = \sum_i \varrho_{in}(Q_n a_{in}) \otimes b_{in} + \sum_i \sigma_{in}(Q_n c_{in}) \otimes d_{in}.$$

It now follows from the normalization of z_n and the definition of the projective tensor norm that for each n there is an i such that $p(Q_n a_{in}) \geqslant 1/2$ or $p(Q_n c_{in}) \geqslant 1/2$. But the latter is not possible. Indeed,

$$p(Q_n c_{in}) \leqslant \sqrt{n}\, p(c_{in}) \leqslant \sqrt{n}(p_{k_0} \mid A_n)(c_{in}) = \sqrt{n}\, p_{k_0}(c_{in}),$$

since $\|Q_n\| \leqslant \sqrt{n}$ in (E_n, p). On the other hand, $p_{k_0}(c_{in}) \leqslant 1/(3n)$ for all n. Thus, for each n

$$p(Q_n a_n) \geqslant \tfrac{1}{2}, \tag{4.4}$$

where $a_n = a_{in}$ for some i.

By assumption the set $\{p_2(a_n)\}_{n \in N}$ is bounded above. Formula (4.4) implies

$$p_2(a_n) \geqslant n^{1/4} p\big(Q_n a_n + \sum_{t=1}^{n} \tilde{a}_{tn}\big) \geqslant n^{1/4} \big|\tfrac{1}{2} - p\big(\sum_{t=1}^{n} \tilde{a}_{tn}\big)\big|,$$

where $\tilde{a}_{tn}$ is the N_{tn}-component of a_n. Thus, for n large enough, say $n > n_0$,

$$\sum_{t=1}^{n} p(\tilde{a}_{tn}) \geqslant p\big(\sum_{t=1}^{n} \tilde{a}_{tn}\big) \geqslant \tfrac{1}{4}; \tag{4.5}$$

otherwise $\{p_2(a_n)\}_{n \in N}$ is not bounded. Therefore there are indices t_0 such that

$$\sum_{t=t_0}^{n} p(\tilde{a}_{tn}) \geqslant \tfrac{1}{8}.$$

For $n > n_0$ we set

$$D_n = \max\big\{t_0 \in N,\ 1 \leqslant t_0 \leqslant n \,\big|\, \sum_{t=t_0}^{n} p(\tilde{a}_{tn}) \geqslant \tfrac{1}{8}\big\}.$$

I) Suppose first that $\{D_n\}_{n > n_0}$ is bounded, say $D_n < C$ for some $C > 0$. For $k > C$ and $n > \max\{n_0, k\}$ we have

$$p_k(a_n) \geqslant \sum_{t=1}^{n \wedge k} n^k p(\tilde{a}_{tn}) \geqslant n^k \sum_{t=1}^{C+1} p(\tilde{a}_{tn}) \geqslant n^k/8. \tag{4.6}$$

The last step follows from (4.5) and the definition of D_n. Clearly, (4.6) contradicts the assumption $p_k(a_n) < r_k$ $\forall n$ in formula (4.3).

II) Suppose then that there exist elements D_{n_j}, $j \in N$, with $D_{n_j} > j$. In this case

$$p_2(a_{n_j}) \geqslant \sum_{t=1}^{n_j} t^2 p(\tilde{a}_{tn_j}) \geqslant \sum_{t=j}^{n_j} t^2 p(\tilde{a}_{tn_j})$$
$$\geqslant j^2 \sum_{t=j}^{n_j} p(\tilde{a}_{tn_j}) \geqslant j^2 \sum_{t=D_{n_j}}^{n_j} p(\tilde{a}_{tn_j}) \geqslant j^2/8.$$

For j large enough this again contradicts the assumption $p_2(a_n) < r_2$ for all n. ∎

Other pathologies following from this example are discussed in Section 5.

A similar counterexample can be constructed for other l^p- or L^p-spaces, $p > 1$, as well. (We denote the space by G.) We take the duals of 1-complemented n-dimensional l_n^p-subspaces of G for M_n and we embed the spaces M_n isometrically in $C(0, 1)$. It is well known in Banach space theory that the projection constants $\varrho_n := \inf\{\|P_n\| \mid P_n$ is a projection from $C(0, 1)$ onto $M_n\}$ tend to infinity as n grows up. Using Lemma 4.1 of [9] we can choose tensors z_n such that

$$((h \mid M_n) \otimes q)(z_n) \geqslant \frac{\varrho_n}{2} (h \otimes q)(z_n),$$

where h and q denote the norms of $C(0, 1)$ and G, respectively. Finally, using a trick analogous to that at the beginning of Section 3 we find finite-dimensional subspaces E_n of $C(0, 1)$ contaning M_n such that

$$(4.7) \qquad ((h_n \mid M_n) \otimes q)(z_n) \geqslant \frac{\varrho_n}{4} ((h \mid E_n) \otimes q)(z_n)$$

and such that there are projections Q_n from E_n onto M_n with $\|Q_n\| \leqslant 2\varrho_n$. We can now use the preceding construction with (4.1) replaced by (4.7) and G in the role of l^2. In the definition of $p_k \mid A_n$ we take $(\varrho_n)^{(k-1)/k}$ instead of $n^{(k-1)/(2k)}$. The fact that the sequence ϱ_n grows up to infinity is needed in the proof that F is a Montel space. Finally, in (4.3) we take $c_{in} \in \varrho_n^{-2} V_{k_0}/3$ instead of $(3n)^{-1} V_{k_0}$.

5. Other counterexamples. As easy consequences of the preceding constructions we get counterexamples to some old questions of Grothendieck.

It follows directly from the definitions that all of the spaces F in Sections 3 and 4 have a so-called *finite-dimensional decomposition* (for definition, see [2], Ch. VI.1). Hence, the spaces F have the approximation property. Since Fréchet–Montel spaces are reflexive, the same is also true for the spaces F'_b ([6], 43.4(9)).

We now prove

THEOREM 5.1. *There exists a (DFM)-space G for which $G \hat{\otimes}_\varepsilon G$ is not a (DF)-space. Moreover, if X is an l^p- or L^p-space with $1 < p < \infty$, then there is a (DFM)-space G such that $G \hat{\otimes}_\varepsilon X$ is not a (DF)-space.*

Proof. I) We first consider $G \hat{\otimes}_\varepsilon G$. Let F be as in Theorem 3.3 and set $G = F'_b$. By (1.1) and (1.2), $(G \hat{\otimes}_\varepsilon G)'_b = (G \varepsilon G)'_b = (\varepsilon(G, G))'_b = F \hat{\otimes}_\pi F$. It is enough to construct a countable bounded set $B \subset F \hat{\otimes}_\pi F$ which is not equicontinuous. In view of the preceding results it suffices to show that the equicontinuous sets of $F \hat{\otimes}_\pi F$ are subsets of the sets $\overline{\Gamma(B_1 \otimes B_2)}$ with $B_1 \subset F$, $B_2 \subset F$ bounded: then we can take a bounded set B which is not contained in any set $\overline{\Gamma(B_1 \otimes B_2)}$ and choose a dense countable subset N of B. As a consequence, N is bounded but not equicontinuous.

Any neighbourhood W of 0 in $\varepsilon(G, G)$ contains a set $(U^\circ \otimes V^\circ)^\circ$, where U, $V \subset G$ are absolutely convex neighbourhoods of 0 and $U^\circ \otimes V^\circ$ is considered as a subset of $G' \otimes G' = F \otimes F$; $\langle F \otimes F, \varepsilon(G, G) \rangle$ is a dual pair as in [6], 44.3(6). Note that U° and V° are bounded in F. We now form the polar of $(U^\circ \otimes V^\circ)^\circ$ in $F \hat{\otimes}_\pi F = (\varepsilon(G, G))'_b$ and denote it by $(U^\circ \otimes V^\circ)^{\circ\circ}$. The bipolar may be considered to be taken in the dual pair $\langle \varepsilon(G, G), F \hat{\otimes}_\pi F \rangle$. Thus, by the theorem of bipolars

$$(5.2) \qquad W^\circ \subset (U^\circ \otimes V^\circ)^{\circ\circ} = \overline{\Gamma(U^\circ \otimes V^\circ)},$$

where the closure is taken in the weak topology of $F \hat{\otimes}_\pi F$ with respect to the dual pair $\langle \varepsilon(G, G), F \hat{\otimes}_\pi F \rangle$. But $(F \hat{\otimes}_\pi F)'$ is algebraically isomorphic to $\varepsilon(G, G)$ (see [6], 45.3(1)). Thus, the closure in (5.2) may be taken in the original topology of $F \hat{\otimes}_\pi F$. We have shown that every equicontinuous set of $F \hat{\otimes}_\pi F$ is contained in a set $\overline{\Gamma(B_1 \otimes B_2)}$, $B_i \subset F$ bounded, which completes the proof.

II) Let X be as in the hypothesis. Using the results of Section 4 we choose a Fréchet–Montel space F such that $F \hat{\otimes}_\pi X'$ does not have property (BB). By assumptions, X' is separable. We set again $G = F'_b$. Because G is a Montel space, $G \hat{\otimes}_\varepsilon X = G \varepsilon X = L_b(G'_b, X) = L_b(F, X)$, where $L_b(F, X)$ is the space $L(F, X)$ equipped with the topology of uniform convergence on bounded sets of F. It follows from [1], Corollary 2.8, that $(L_b(F, X))'_b$ is topologically isomorphic to $F \hat{\otimes}_\pi X'$. As in case I we see that there are countable bounded sets in $F \hat{\otimes}_\pi X'$ which are not equicontinuous; in the proof we again need [1], Corollary 2.8 to deduce the algebraic isomorphism of $(F \hat{\otimes}_\pi X')'$ and $\varepsilon(G, X)$. ∎

This result answers *Question non résolue* 10 in [3]. In fact, we have shown that the spaces of Theorem 5.1 are not quasi-barrelled.

We get immediately another Grothendieck counterexample:

COROLLARY 5.2. *For suitable Fréchet–Montel spaces F and F_1 and a Banach space X the spaces $L_b(F, F'_b)$ and $L_b(F_1, X)$ are not (DF)-spaces.*

Proof. The claim follows from Theorem 5.1 and the fact that $L_b(F, F'_b) = F'_b \hat{\otimes}_\varepsilon F'_b$ and $L_b(F_1, X) = (F_1)'_b \hat{\otimes}_\varepsilon X$. ∎

For any Fréchet spaces F and H the space $B(F, H)$ is algebraically isomorphic to $L(F, H'_b)$. We endow $B(F, H)$ with the bibounded topology, defined by the 0-neighbourhoods

$$U = \{w \in B(F, H) \mid |w(B_1, B_2)| \leqslant 1\},$$

where $B_1 \subset F$, $B_2 \subset H$ are arbitrary bounded sets. We denote this space by $B_{bb}(F, H)$. The topologies of $B_{bb}(F, H)$ and $L(F, H'_b)$ coincide and we get by choosing F, F_1 and X as above

Corollary 5.3. *The spaces $B_{bb}(F, F)$ and $B_{bb}(F_1, X)$ are not (DF)-spaces.*

Corollaries 5.3 and 5.2 give an answer to *Question non résolue* 7 in [4].

Remark. After this paper was submitted, Gilles Pisier noticed that an analogue of Proposition 2.1 is valid for $C(0, 1)$ instead of our (E, p). The proof for this case is more elementary; it uses only a form of Grothendieck's theorem.

References

[1] H. Collins and W. Ruess, *Duals of spaces of compact operators*, Studia Math. 74 (1982), 213–245.

[2] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer, 1979.

[3] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[4] —, *Sur les espaces (F) et (DF)*, Summa Brasil. Math. 3 (1954), 57–123.

[5] R. Hollstein, *Extension and lifting of continuous linear mappings in locally convex spaces*, Math. Nachr. 108 (1982), 275–297.

[6] G. Köthe, *Topological Vector Spaces I–II*, Springer, Berlin 1983 and 1979.

[7] A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam 1980.

[8] G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. 151 (1983), 181–208.

[9] J. Taskinen, *Counterexamples to "Problème des topologies" of Grothendieck*, Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertationes 63 (1986).

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The Wold–Cramér concordance problem for Banach-space-valued stationary processes

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Abstract. The problem of the concordance of the Wold decomposition and the spectral measure decomposition of Banach-space-valued stationary processes is studied. We give a sufficient condition for the concordance in terms of the representation of the process as a process in the space of square Bochner integrable functions on the circle.

0. Introduction. The problem of the Wold–Cramér concordance for q-variate stationary processes was extensively studied (cf. [4]–[6]). In the case of stationary processes with values in a Banach space the only result was given by F. Schmidt. He proved that every such process X admits a unique orthogonal decomposition

$$X(k) = Y(k) + U(k) + V(k)$$

where Y is regular, both U and V are singular and the spectral measures of Y, U are absolutely continuous, while the spectral measure of V is singular with respect to the Lebesgue measure (cf. [7], Theorem 5). In particular, the question of whether there exists a nonsingular process with nonzero U part in the Schmidt decomposition remained open.

In this paper we present a sufficient condition for the concordance of the Wold decomposition and the spectral measure decomposition for Banach-space-valued stationary processes. The proof is based on the isomorphism theorem (cf. [8], Theorem 3.3) which yields a representation of the process under consideration as a process in the space $L^2(K, \mu, H)$ of all μ-square Bochner integrable functions from the circle K to a Hilbert space H. Our condition is formulated in terms of this representation. In Section 2 we establish some properties of this representation we need in the proof of the main theorem. Finally, we give in Section 4 several examples related to our theorem. One of them (Example 4.1) answers positively the question formulated above.

1. Preliminaries. In this paper we use the following notation:

$\boldsymbol{Z}$ – the set of integers,

$\boldsymbol{C}$ – the set of complex numbers,