On the ergodic power function for invertible positive operators

by

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Abstract. Let $T$ be an invertible positive linear operator on $L_p$, $1 < p < \infty$, of a $\sigma$-finite measure space, and suppose $T^{-1}$ is also positive. For $1 < r < \infty$, the ergodic $r$-th power function $P_r f$ of $f \in L_p$ (with respect to $T$) is defined by

$$P_r f = \frac{1}{n} \sum_{i=1}^{n} [T_i f - T_{i-1} f] + [T_{n+r-1} f - T_{n+r} f]$$

where $T_i f = (n+k+1)^{-1} \sum_{j=1}^{k} T^j f$ with $n, k \geq 0$. In this paper it is proved that if $T_{ik}$ are uniformly bounded operators on $L_p$ then $\|P_r f\|_p \leq C \|f\|_p$ for all $f \in L_p$. This generalizes a recent result of F. J. Martín-Reyes. An application is also given.

1. Introduction and the theorem. Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $T$ an invertible linear operator on $L_p = L_p(X, \mathcal{F}, \mu)$, with $1 < p < \infty$. If both $T$ and $T^{-1}$ are positive, then, as is well known (see e.g. [23]), $T$ and $T^{-1}$ are Lamperti operators, and there exists an invertible positive linear operator $S$ acting on measurable functions such that $S$ is multiplicative and $S 1 = 1$, and a sequence $\{g_i\}_{i=-\infty}^{\infty}$ of positive measurable functions on $X$ such that for each integer $i$, $T^i$ has the form

$$T^i f(x) = g_i(x) S^i f(x).$$

It is immediately seen that

$$g_{i+1}(x) = g_i(x) S g_i(x) \quad \text{a.e. on } X.$$

Further, by the Radon–Nikodym theorem there exists a sequence $\{J_i\}_{i=-\infty}^{\infty}$ of positive measurable functions on $X$ such that

$$\int J_i(x) S^i f(x) d\mu = \int f(x) d\mu \quad \text{for each } i \text{ and } f \in L_1.$$

Clearly,

$$J_{i+1}(x) = J_i(x) S J_i(x) \quad \text{a.e. on } X.$$
On the other hand, we now consider functions $f$ on the integers. The maximal function $f^*$ of $f$ is defined by

$$f^*(i) = \sup_{n,k \geq 0} (n+k+1)^{-1} \sum_{j=-n}^{n} |f(i+j)|.$$ 

For a positive real function $u$ on the integers the following is known (see e.g. [6] for a proof):

$$\sum_{i=-\infty}^{\infty} (f^*(i))^p u(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p u(i) \quad \text{for all } f$$

if and only if $u$ satisfies the condition

$$(A_p) \quad \sum_{j=0}^{k} u(i+j)\left(\sum_{j=0}^{k} u(i+j)^{p-1}\right)^{1/p-1} \leq C(1+k)^p$$

for all $i, k$, with $k \geq 0$.

Here, and in the sequel, $C$ will denote a constant that may be different at each occurrence.

Using these properties, Martin-Reyes and de la Torre [5] proved a dominated ergodic theorem. That is, they proved

**Theorem A.** If both $T$ and $T^{-1}$ are positive linear operators on $L_p$, $1 < p < \infty$, and if $\sup_{x \in X} \|T_x\|_p < \infty$, then the following are equivalent:

(i) $\|Mf\|_p \leq C\|f\|_p$ for all $f \in L_p$, where $Mf$ is the ergodic maximal function defined by

$$Mf = \sup_{n,k \geq 0} \|T_{x,n}\|_p.$$ 

(ii) $\|T_{x,n}\|_p \leq C\|f\|_p$ for all $k \geq 0$ and $f \in L_p$.

(iii) For almost all $x \in X$, the function $u_n(x) = g^*_n(x)J_n(x)$ defined on the integers satisfies $(A_p)$ with a constant independent of $x$, where $\{g_n\}_{n=-\infty}^{\infty}$ and $\{J_n\}_{n=-\infty}^{\infty}$ are the sequences of functions determined by (1) and (3), respectively.

On the other hand, Martin-Reyes [4] recently studied good weights for the ergodic power function associated with an invertible measure preserving transformation, and generalized the results of Jones [2]. In particular, the following lemma is due to Martin-Reyes [4].

**Lemma A.** Let $u$ be a positive real function on the integers and suppose $u$ satisfies $(A_p)$ with a constant $C$. Then for each $1 < r < \infty$ there exists a constant $C$, depending only on the constant $C$ such that

$$\sum_{i=-\infty}^{\infty} |P_i^* f(i)|^p u(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p u(i)$$

for all functions $f$ on the integers, where

$$P_i^* f(i) = \left[ \sum_{k=-\infty}^{\infty} ((k+1)^{-1})^r |f(i+k)|^r \right]^{1/r}.$$ 

In this paper, using Theorem A and Lemma A, we shall generalize a recent result of Martin-Reyes [4] to the operator-theoretic level. In the next section we shall apply the result obtained in this section to give a unified proof of the almost everywhere and $L_p$-norm convergence of the ergodic averages $\frac{1}{n} \sum_{t=0}^{n-1} T_t f(x)$ and the ergodic series $\sum_{k=1}^{\infty} k^{-1} T_k f(x)$, where $T$ is an invertible Lamperti operator on $L_p$ satisfying a certain norm condition. The theorem we are going to prove in this section is as follows.

**Theorem 1.** If both $T$ and $T^{-1}$ are positive linear operators on $L_p$, $1 < p < \infty$, and if $\sup_{x \in X} \|T_x\|_p < \infty$, then for any $r$ with $1 < r < \infty$ there exists a constant $C_r$ such that $\|P_r f\|_p \leq C_r\|f\|_p$ for all $f \in L_p$, where $P_r f$ is the ergodic $r$-th power function of $f$ defined in the abstract.

**Proof.** From the relation $T_{x+n} f - T_{x+k} f = (k+2)^{-1} [T^{k+1} f - T_{0,k} f]$, it follows that

$$P_r f \leq \left[ \sum_{k=-\infty}^{\infty} ((k+1)^{-1})^r |T^{k+1} f|^r \right]^{1/r} + C(Mf).$$

Hence Theorem A implies that for the proof of the theorem it suffices to prove $\|Q_r f\|_p \leq C\|f\|_p$ for all $f \in L_p$, where

$$Q_r f = \left[ \sum_{k=-\infty}^{\infty} ((k+1)^{-1})^r |T^k f|^r \right]^{1/r}.$$ 

To do this, we now fix an $N \geq 1$ and let

$$Q_N f = \left[ \sum_{k=-N}^{N} ((k+1)^{-1})^r |T^k f|^r \right]^{1/r}.$$ 

From (1), (2) and (3) we see that

$$\int |Q_N f|^p d\mu = \frac{1}{2L+1} \int_{L} \sum_{i=L-1}^{L} S^i(Q_N f)^r J_i d\mu$$

$$= \frac{1}{2L+1} \int_{L} \sum_{i=L-1}^{L} \left( \sum_{k=-N}^{N} \frac{|S^k f|^r}{|k+1|^r} \right)^{1/r} J_i d\mu$$

$$= \frac{1}{2L+1} \int_{L} \sum_{i=L-1}^{L} \left( \sum_{k=-N}^{N} \frac{|g_{i+k} S^k f|^r}{|k+1|^r} \right)^{1/r} g_{i+k} J_i d\mu.$$
On the other hand, since $\|T_k f\|_p \leq C \|f\|_p$ for all $k \geq 0$ and $f \in L_p$, by hypothesis, Theorem A states that for almost all $x$ in $X$ the function $w_0(x) = g_i^r(x) J_i(x)$ on the integers satisfies $w_0(x)$ with a constant $C$ independent of $x$. Hence by Lemma A, for almost all $x$ in $X$,

$$\sum_{i=-L}^{L} \sum_{k=-N}^{N} \left( \frac{g_i^r(x) S_i^k f(x)}{|k|+1} \right)^p \leq C_r \sum_{i=-L}^{L} \sum_{k=-N}^{N} S_i^k f(x) J_i(x)$$

and thus

$$\|Q_n f\|_p \leq \frac{1}{2L+1} C_r \frac{1}{2L+1} \sum_{i=-L}^{L} \sum_{k=-N}^{N} S_i^k f(x) J_i(x)$$

Letting $L \uparrow \infty$ and then $N \uparrow \infty$, we have $\|Q_n f\|_p \leq C_r \|f\|_p$, and the proof is complete.

Remark. It seems interesting to note that the converse of Theorem 1 is not true even if $T$ is induced by a point transformation; an example can be found in [4].

2. An application. In this section let $T$ be an invertible Lamperti operator on $L_p$ with $1 < p < \infty$. Thus, as is easily seen (cf. [3]), $T^{-1}$ is also a Lamperti operator on $L_p$, and there exists an invertible positive linear operator $S$ acting on measurable functions such that $S$ is multiplicative and $S1 = 1$, and a sequence $\{h_i\}_{i=0}^{\infty}$ of measurable functions on $X$ such that for each $i$, $T^i f(x) = h_i(x) S^i f(x)$. Clearly, $h_i(x) = h_i(x) S^i h_i(x)$ a.e. on $X$ for any $i$ and $j$. Let

$$\tau^i = |h_i| S^i f.$$ 

It then follows that $\tau$ is an invertible positive linear operator on $L_p$ and, for each integer $i$, $\tau^i f(x) = |h_i(x)| S^i f(x)$. We call $\tau$ the linear modulus of $T$. In this section we apply the result obtained in the preceding section to prove the following theorem due to Martin-Reyes and de la Torre [5] and the author [7].

**Theorem 2.** If $T$ is an invertible Lamperti operator on $L_p$, $1 < p < \infty$, and if the linear modulus $\tau$ of $T$ satisfies $\sup_{x \in X} \|\tau x\|_p < \infty$, then for any $f \in L_p$ the limits

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} T^i f \quad \text{and} \quad \lim_{n \to \infty} \sum_{i=0}^{n-1} k^{-1} (T^i f - T^{-i} f)$$

exist almost everywhere and in the norm topology of $L_p$.

Proof. We first notice that for any $f \in L_p$, $\sum_{k=1}^{\infty} (1/|k|)^r T^k f < \infty$ a.e. on $X$.

$$\lim_{n \to \infty} \sum_{k=1}^{n} (1/|k|)^r T^k f = 0 \quad \text{a.e. on $X$}.$$  

In fact, letting $r = p$ in the proof of Theorem 1, we see that

$$\sum_{k=1}^{n} \frac{1}{|k|^{1/p}} T^k f \leq C \sum_{k=1}^{n} \frac{1}{|k|^{1/p}} T^k f$$

from which (5) follows. Further, since $\sum_{k=1}^{\infty} k^{-1/r} |T^k f|^{1/p} < \infty$ a.e. on $X$, Hölder's inequality implies that

$$\sum_{k=1}^{\infty} k^{-2/r} |T^k f|^{1/p} \leq \left( \sum_{k=1}^{\infty} k^{-1/p} |T^k f|^{1/p} \right)^{1/r} \left( \sum_{k=1}^{\infty} k^{-1} \right)^{1/\alpha} < \infty$$

a.e. on $X$, where $1/p + 1/\alpha = 1$. Thus (6) follows.

Since $L_p$ is a reflexive Banach space and

$$\sup_{x \in X} \|H^* f\|_r \leq C \|f\|_p,$$

a mean ergodic theorem (see e.g., Theorem VIII.5.1 in [1]) together with (5) proves the norm convergence of the ergodic averages $n^{-1} \sum_{i=0}^{n-1} T^i f$. It follows that the set $\{\tau^i f : \tau \in L_p\}$ is a dense subset of $L_p$.

Hence, by Theorem A and (5), we may apply Banach's convergence theorem (see e.g., Theorem IV.11.2 in [1]) to infer that the ergodic averages converge a.e. on $X$ for any $f \in L_p$.

To prove the rest of the theorem, we need the following result due to the author [7] (cf. Lemma in [7]): For all $f \in L_p$,

$$\|H^* f\|_p \leq C \|f\|_p,$$

where $H^* f = \sum_{x \in X} \sum_{k=1}^{\infty} k^{-1} (T^k f - T^{-k} f)$.

By (7) and Lebesgue's convergence theorem, it suffices to prove the a.e. convergence of the ergodic series $\sum_{i=0}^{\infty} k^{-1} (T^i f - T^{-i} f)$. Since $\{\tau^i f : \tau \in L_p\}$ is a dense subset of $L_p$, we again apply Banach's convergence theorem together with (7) and see that it suffices to prove the a.e.
convergence of
\[ \sum_{k=1}^{n} k^{-1} \left[ T^k(f - Tf) - T^{-k}(f - Tf) \right] \]
\[ = f + Tf - \frac{1}{n} \left( T^{n+1} f + T^{-n-1} f - \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} \right) (T^{n+1} f + T^{-n-1} f) \]
as \( n \to \infty \); by (5) and (6) we see that the limit
\[ \lim_{n \to \infty} \sum_{k=1}^{n} k^{-1} \left[ T^k(f - Tf) - T^{-k}(f - Tf) \right] \]
exists a.e. on \( X \). This completes the proof.

References


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On the geometry of spaces of \( C_0 \) \( K \)-valued operators

by

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Abstract. Let \( K \) be a locally compact Hausdorff space and \( X \) a Banach space. We consider operator spaces \( W \subset L(X, C(K)) \) which contain the compact operators and have the property that \( T \in W \) implies \( M_2 \circ T : W \) for every bounded continuous scalar-valued function \( f \) on \( K \) (\( M_2 \) denotes the multiplication operator \( f \mapsto M_2 f \) on \( C_0(K) \)).

Our main results concern the \( M \)-structure properties of such spaces \( W \). We characterize the centralizer of \( W \) if the centralizer of \( X \) is small, and for many classes of Banach spaces \( X \) (including e.g. the \( L^{p} \)-preduals) we are able to describe all \( M \)-ideals of \( W \), at least in the case of compact \( K \).

These characterizations generalize results of Flinn and Smith who discussed the case \( W = L(K, C(K)) \) if the scalars are complex.

With our methods we also can treat questions as "Is \( K \) determined by \( W \)" or "When can \( W \) be a dual space?". We are able to derive answers which generalize recent results of Cambern and Greim.

I. Introduction. Let \( X \) be a real or complex Banach space (the scalar field, \( R \) or \( C \), will be denoted by \( K \) in the sequel). The following basic definitions from \( M \)-structure theory will be of importance:

1.1. Definition. (i) Let \( J \subset X \) be a closed linear subspace. \( J \) is called an \( M \)-summand (resp. \( L \)-summand) if there is a closed subspace \( J^* \subset X \) such that \( X = J \oplus J^* \); \( J \) is algebraically and \( \| x + x' \| = \max \{ \| x \|, \| x' \| \} \) (resp. \( = \| x \| + \| x' \| \) whenever \( x, x' \in J \)). \( J \) is called an \( M \)-ideal if \( J^* \), the annihilator of \( J \) in \( X^* \), is an \( L \)-summand.

(ii) Let \( T : X \to X \) be an operator. \( T \) is called a multiplier if for every extreme functional \( p \) (i.e. for every extreme point \( p \) of the dual unit ball) there is a scalar \( a_p(p) \) such that \( p \circ T = a_p(p) \). \( M(X) \) will denote the collection of all multipliers.

A multiplier \( S \) is called the adjoint of a multiplier \( T \) (and we write \( S = T^* \) in this case) if \( a_S(p) \) is the complex conjugate of \( a_T(p) \) for every \( p \).

\( Z(X) \), the centralizer of \( X \), is the set of all multipliers which admit an adjoint.

These definitions have been introduced by Cunningham ([8, 9]) and Alfsen–Effros ([11]); for a systematic introduction the reader is referred to Behrends ([2]).

Here we only note that \( Z(X) \) and \( M(X) \) are always commutative