[9] M. Cwikel and P. Nilsson, *Interpolation of Marcinkiewicz spaces*, Math. Scand. 56 (1985), 29–42.
[10] M. Cwikel and J. Peetre, *Abstract K and J spaces*, J. Math. Pures Appl. 60 (1981), 1–50.
[11] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. 42 (1978), 289–305.
[12] J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. 60 (1977), 33–59.
[13] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow 1978 (in Russian; English transl.: Amer. Math. Soc., Providence 1982).
[14] G. Ya. Lozanovskii, *Transformations of ideal Banach spaces by means of concave functions*, in: Qualitative and Approximate Methods for the Investigation of Operator Equations, Yaroslavl'. Gos. Univ., 1978, 122–148 (in Russian).
[15] M. Mastyło, *The K-functional for quasi-normed couples* $(A_0, (A_0, A_1)_E^K)$ *and* $((A_0, A_1)_E^K, A_1)$, Funct. Approx. 15 (1986), 59–72.
[16] —, *On some questions concerning the real interpolation method for symmetric spaces*, in: Qualitative and Approximate Methods for the Investigation of Operator Equations, Yaroslavl'. Gos. Univ., 1986, 47–57 (in Russian).
[17] —, *On the K-monotonicity of symmetric spaces*, in: Theory of Functions of Several Variables, Yaroslavl'. Gos. Univ., 1986, 49–55 (in Russian).
[18] P. Nilsson, *Reiteration theorems for real interpolation and approximation spaces*, Ann. Mat. Pura Appl. 132 (1982), 291–330.
[19] —, *Interpolation of Calderón pairs and Ovčinnikov pairs*, ibid. 134 (1983), 201–232.
[20] —, *Interpolation of Banach lattices*, Studia Math. 82 (1985), 135–154.
[21] V. I. Ovchinnikov, *Interpolation theorems resulting from an inequality of Grothendieck*, Funktsional. Anal. i Prilozhen. 10 (4) (1976), 45–54 (in Russian).
[22] J. Peetre, *Generalizing Ovčinnikov's theorem*, technical report, Lund 1981.

INSTYTUT MATEMATYKI UNIWERSYTETU im. ADAMA MICKIEWICZA
INSTITUTE OF MATHEMATICS, ADAM MICKIEWICZ UNIVERSITY
Matejki 48/49, 60-769 Poznań, Poland

Received December 1, 1986 (2251)
Revised version April 15, 1987

STUDIA MATHEMATICA, T. XC. (1988)

On the ergodic power function for invertible positive operators

by

RYOTARO SATO (Okayama)

Abstract. Let T be an invertible positive linear operator on L_p, $1 < p < \infty$, of a σ-finite measure space, and suppose T^{-1} is also positive. For $1 < r < \infty$, the *ergodic r-th power function* $P_r f$ of $f \in L_p$ (with respect to T) is defined by

$$P_r f = \Big[\sum_{k=0}^{\infty} |T_{k+1,0} f - T_{k,0} f|^r + |T_{0,k+1} f - T_{0,k} f|^r\Big]^{1/r}$$

where $T_{n,k} f = (n+k+1)^{-1} \sum_{i=-n}^{k} T^i f$ with $n, k \geqslant 0$. In this paper it is proved that if $T_{k,k}$ are uniformly bounded operators on L_p then $\|P_r f\|_p \leqslant C \|f\|_p$ for all $f \in L_p$. This generalizes a recent result of F. J. Martín-Reyes. An application is also given.

1. Introduction and the theorem. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T an invertible linear operator on $L_p = L_p(X, \mathcal{F}, \mu)$, with $1 < p < \infty$. If both T and T^{-1} are positive, then, as is well known (see e.g. [3]), T and T^{-1} are Lamperti operators, and there exists an invertible positive linear operator S acting on measurable functions such that S is multiplicative and $S1 = 1$, and a sequence $\{g_i\}_{i=-\infty}^{\infty}$ of positive measurable functions on X such that for each integer i, T^i has the form

(1) $$T^i f(x) = g_i(x) S^i f(x).$$

It is immediately seen that

(2) $$g_{i+j}(x) = g_i(x) S^i g_j(x) \quad \text{a.e. on } X.$$

Further, by the Radon–Nikodym theorem there exists a sequence $\{J_i\}_{i=-\infty}^{\infty}$ of positive measurable functions on X such that

(3) $$\int J_i(x) S^i f(x)\, d\mu = \int f(x)\, d\mu \quad \text{for each } i \text{ and } f \in L_1.$$

Clearly,

(4) $$J_{i+j}(x) = J_i(x) S^i J_j(x) \quad \text{a.e. on } X.$$

On the other hand, we now consider functions f on the integers. The *maximal function* f^* of f is defined by

$$f^*(i) = \sup_{n,k \geq 0} (n+k+1)^{-1} \sum_{j=-n}^{k} |f(i+j)|.$$

For a positive real function u on the integers the following is known (see e.g. [6] for a proof):

$$\sum_{i=-\infty}^{\infty} (f^*(i))^p u(i) \leq C \sum_{i=-\infty}^{\infty} |f(i)|^p u(i) \quad \text{for all } f$$

if and only if u satisfies the condition

$$(A_p) \qquad \Big(\sum_{j=0}^{k} u(i+j)\Big)\Big(\sum_{j=0}^{k} u(i+j)^{-1/p-1}\Big)^{p-1} \leq C(k+1)^p$$

for all i and k, with $k \geq 0$.

Here, and in the sequel, C will denote a constant that may be different at each occurrence.

Using these properties, Martín-Reyes and de la Torre [5] proved a dominated ergodic theorem. That is, they proved

THEOREM A. *If both T and T^{-1} are positive linear operators on L_p, $1 < p < \infty$, then the following are equivalent:*

(i) $\|Mf\|_p \leq C\|f\|_p$ *for all* $f \in L_p$, *where* Mf *is the ergodic maximal function defined by*

$$Mf = \sup_{n,k \geq 0} |T_{n,k} f|.$$

(ii) $\|T_{k,k} f\|_p \leq C\|f\|_p$ *for all* $k \geq 0$ *and* $f \in L_p$.

(iii) *For almost all* $x \in X$, *the function* $u_x(i) = g_i^{-p}(x) J_i(x)$ *defined on the integers satisfies* (A_p) *with a constant independent of* x, *where* $\{g_i\}_{i=-\infty}^{\infty}$ *and* $\{J_i\}_{i=-\infty}^{\infty}$ *are the sequences of functions determined by* (1) *and* (3), *respectively.*

On the other hand, Martín-Reyes [4] recently studied good weights for the ergodic power function associated with an invertible measure preserving transformation, and generalized the results of Jones [2]. In particular, the following lemma is due to Martín-Reyes [4].

LEMMA A. *Let u be a positive real function on the integers and suppose u satisfies (A_p) with a constant C. Then for each $1 < r < \infty$ there exists a constant C_r depending only on the constant C such that*

$$\sum_{i=-\infty}^{\infty} |P_r^* f(i)|^p u(i) \leq C_r \sum_{i=-\infty}^{\infty} |f(i)|^p u(i)$$

for all functions f on the integers, where

$$P_r^* f(i) = \Big[\sum_{k=-\infty}^{\infty} (|k|+1)^{-r} |f(i+k)|^r\Big]^{1/r}.$$

In this paper, using Theorem A and Lemma A, we shall generalize a recent result of Martín-Reyes [4] to the operator-theoretic level. In the next section we shall apply the result obtained in this section to give a unified proof of the almost everywhere and L_p-norm convergence of the ergodic averages $n^{-1}\sum_{i=0}^{n-1} T^i f(x)$ and the ergodic series $\sum_{k=1}^{\infty} k^{-1}(T^k f(x) - T^{-k} f(x))$, when T is an invertible Lamperti operator on L_p satisfying a certain norm condition. The theorem we are going to prove in this section is as follows.

THEOREM 1. *If both T and T^{-1} are positive linear operators on L_p, $1 < p < \infty$, and if $\sup_{k \geq 0} \|T_{k,k}\|_p < \infty$, then for any r with $1 < r < \infty$ there exists a constant C_r such that $\|P_r f\|_p \leq C_r \|f\|_p$ for all $f \in L_p$, where $P_r f$ is the ergodic r-th power function of f defined in the abstract.*

Proof. From the relation $T_{0,k+1} f - T_{0,k} f = (k+2)^{-1}[T^{k+1} f - T_{0,k} f]$, it follows that

$$P_r f \leq \Big[\sum_{k=-\infty}^{\infty} (|k|+1)^{-r} |T^k f|^r\Big]^{1/r} + C(Mf).$$

Hence Theorem A implies that for the proof of the theorem it suffices to prove $\|Qf\|_p \leq C\|f\|_p$ for all $f \in L_p$, where

$$Qf = \Big[\sum_{k=-\infty}^{\infty} (|k|+1)^{-r} |T^k f|^r\Big]^{1/r}.$$

To do this, we now fix an $N \geq 1$ and let

$$Q_N f = \Big[\sum_{k=-N}^{N} (|k|+1)^{-r} |T^k f|^r\Big]^{1/r}.$$

From (1), (2) and (3) we see that

$$\int |Q_N f|^p \, d\mu = \frac{1}{2L+1} \int \sum_{i=-L}^{L} S^i(|Q_N f|^p) J_i \, d\mu$$

$$= \frac{1}{2L+1} \int \sum_{i=-L}^{L} \Big[\sum_{k=-N}^{N} \Big(\frac{S^i |T^k f|}{|k|+1}\Big)^r\Big]^{p/r} J_i \, d\mu$$

$$= \frac{1}{2L+1} \int \sum_{i=-L}^{L} \Big[\sum_{k=-N}^{N} \Big(\frac{g_{i+k} S^{i+k} |f|}{|k|+1}\Big)^r\Big]^{p/r} g_i^{-p} J_i \, d\mu.$$

On the other hand, since $\|T_{k,k} f\|_p \leqslant C\|f\|_p$ for all $k \geqslant 0$ and $f \in L_p$ by hypothesis, Theorem A shows that for almost all x in X the function $u_x(i) = g_i^{-p}(x) J_i(x)$ on the integers satisfies (A_p) with a constant C independent of x. Hence by Lemma A, for almost all x in X,

$$\sum_{i=-L}^{L} \left[\sum_{k=-N}^{N} \left(\frac{g_{i+k}(x) S^{i+k} |f|(x)}{|k|+1} \right)^r \right]^{p/r} g_i^{-p}(x) J_i(x)$$

$$\leqslant C_r \sum_{i=-L-N}^{L+N} (g_i(x) S^i |f|(x))^p g_i^{-p}(x) J_i(x)$$

$$= C_r \sum_{i=-L-N}^{L+N} S^i |f|^p(x) J_i(x),$$

and thus

$$\int |Q_N f|^p d\mu \leqslant \frac{1}{2L+1} C_r \sum_{i=-L-N}^{L+N} \int S^i |f|^p(x) J_i(x) d\mu$$

$$= \frac{2(L+N)+1}{2L+1} C_r \int |f|^p d\mu.$$

Letting $L \uparrow \infty$ and then $N \uparrow \infty$, we have $\int |Qf|^p d\mu \leqslant C_r \int |f|^p d\mu$, and the proof is complete.

Remark. It seems interesting to note that the converse of Theorem 1 is not true even if T is induced by a point transformation; an example can be found in [4].

2. An application. In this section let T be an invertible Lamperti operator on L_p with $1 < p < \infty$. Thus, as is easily seen (cf. [3]), T^{-1} is also a Lamperti operator on L_p, and there exists an invertible positive linear operator S acting on measurable functions such that S is multiplicative and $S1 = 1$, and a sequence $\{h_i\}_{i=-\infty}^{\infty}$ of measurable functions on X such that for each i, T^i has the form $T^i f(x) = h_i(x) S^i f(x)$. Clearly, $h_{i+j}(x) = h_i(x) S^i h_j(x)$ a.e. on X for any i and j. Let

$$\tau f = |h_1| Sf.$$

It then follows that τ is an invertible positive linear operator on L_p and, for each integer i, τ^i has the form $\tau^i f(x) = |h_i(x)| S^i f(x)$. We call τ the *linear modulus of* T. In this section we apply the result obtained in the preceding section to prove the following theorem due to Martín-Reyes and de la Torre [5] and the author [7].

THEOREM 2. *If T is an invertible Lamperti operator on L_p, $1 < p < \infty$, and if the linear modulus τ of T satisfies $\sup_{k \geqslant 0} \|\tau_{k,k}\|_p < \infty$, then for any $f \in L_p$ the limits*

$$\lim_{n \to \infty} n^{-1} \sum_{i=0}^{n-1} T^i f \quad \textit{and} \quad \lim_{n \to \infty} \sum_{k=1}^{n} k^{-1}(T^k f - T^{-k} f)$$

exist almost everywhere and in the norm topology of L_p.

Proof. We first notice that for any $f \in L_p$

(5) $$\lim_{n \to \infty} n^{-1} T^n f = 0 \quad \text{a.e. on } X \text{ and in the norm of } L_p,$$

(6) $$\sum_{k=1}^{\infty} (1/k)^2 |T^k f| < \infty \quad \text{a.e. on } X.$$

In fact, letting $r = p$ in the proof of Theorem 1, we see that

$$\sum_{k=-\infty}^{\infty} \int (|k|+1)^{-p} |\tau^k f|^p d\mu < \infty,$$

from which (5) follows. Further, since $\sum_{k=1}^{\infty} k^{-p} |T^k f|^p < \infty$ a.e. on X, Hölder's inequality implies that

$$\sum_{k=1}^{\infty} k^{-2} |T^k f| \leqslant [\sum_{k=1}^{\infty} k^{-p} |T^k f|^p]^{1/p} [\sum_{k=1}^{\infty} k^{-q}]^{1/q} < \infty$$

a.e. on X, where $1/p + 1/q = 1$. Thus (6) follows.

Since L_p is a reflexive Banach space and

$$\sup_{n \geqslant 1} \|n^{-1} \sum_{i=0}^{n-1} T^i\|_p < \infty,$$

a mean ergodic theorem (see e.g. Theorem VIII.5.1 in [1]) together with (5) proves the norm convergence of the ergodic averages $n^{-1} \sum_{i=0}^{n-1} T^i f$. It follows that the set $\{g + (f - Tf);\ Tg = g \text{ and } f \in L_p\}$ is a dense subset of L_p. Hence, by Theorem A and (5), we may apply Banach's convergence theorem (see e.g. Theorem IV.11.2 in [1]) to infer that the ergodic averages converge a.e. on X for any $f \in L_p$.

To prove the rest of the theorem, we need the following result due to the author [7] (cf. Lemma in [7]): *For all $f \in L_p$*

(7) $$\|H^* f\|_p \leqslant C\|f\|_p, \quad \textit{where } H^* f = \sup_{n \geqslant 1} \Big| \sum_{k=1}^{n} k^{-1}(T^k f - T^{-k} f) \Big|.$$

By (7) and Lebesgue's convergence theorem, it suffices to prove the a.e. convergence of the ergodic series $\sum_{k=1}^{\infty} k^{-1}(T^k f - T^{-k} f)$. Since $\{g + (f - Tf):\ Tg = g \text{ and } f \in L_p\}$ is a dense subset of L_p, we again apply Banach's convergence theorem together with (7) and see that it suffices to prove the a.e.

convergence of

$$\sum_{k=1}^{n} k^{-1}[T^k(f-Tf)-T^{-k}(f-Tf)]$$
$$= f+Tf-\frac{1}{n}(T^{n+1}f+T^{-n}f)-\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)(T^{k+1}f+T^{-k}f)$$

as $n\uparrow\infty$; by (5) and (6) we see that the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n} k^{-1}[T^k(f-Tf)-T^{-k}(f-Tf)]$$

exists a.e. on X. This completes the proof.

References

[1] N. Dunford and J. T. Schwartz, *Linear operators*, Part I: *General theory*, Interscience, New York 1958.

[2] R. L. Jones, *Inequalities for the ergodic maximal function*, Studia Math. 60 (1977), 111–129.

[3] C.-H. Kan, *Ergodic properties of Lamperti operators*, Canad. J. Math. 30 (1978), 1206–1214.

[4] F. J. Martin-Reyes, *Weights for ergodic square functions*, Ann. Inst. H. Poincaré Probab. Statist. 22 (1986), 333–345.

[5] F. J. Martin-Reyes and A. de la Torre, *The dominated ergodic theorem for invertible, positive operators*, Semesterbericht Funktionalanalysis Tübingen 8 (1985), 143–150.

[6] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[7] R. Sato, *A remark on the ergodic Hilbert transform*, Math. J. Okayama Univ. 28 (1986), 159–163.

DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE
OKAYAMA UNIVERSITY
Okayama, 700 Japan

Received December 24, 1986 (2263)
Revised version April 21, 1987

STUDIA MATHEMATICA, T. XC. (1988)

On the geometry of spaces of $C_0 K$-valued operators

by

EHRHARD BEHRENDS (Berlin)

Abstract. Let K be a locally compact Hausdorff space and X a Banach space. We consider operator spaces W in $L(X, C_0 K)$ which contain the compact operators and have the property that $T\in W$ implies $M_h\circ T\in W$ for every bounded continuous scalar-valued function h on K (M_h denotes the multiplication operator $f\mapsto hf$ on $C_0 K$).

Our main results center around the M-structure properties of such spaces W. We characterize the centralizer of W if the centralizer of X' is small, and for many classes of Banach spaces X (including e.g. the L^1-preduals) we are able to describe all M-ideals of W, at least in the case of compact K.

These characterizations generalize results of Flinn and Smith who discussed the case $W = L(CK, CK)$ if the scalars are complex.

With our methods we also can treat questions as "Is K determined by W?" or "When can W be a dual space?". We are able to derive answers which generalize recent results of Cambern and Greim.

1. Introduction. Let X be a real or complex Banach space (the scalar field, $\boldsymbol{R}$ or $\boldsymbol{C}$, will be denoted by $\boldsymbol{K}$ in the sequel). The following basic definitions from M-structure theory will be of importance:

1.1. Definition. (i) Let $J\subset X$ be a closed linear subspace. J is called an *M-summand* (resp. *L-summand*) if there is a closed subspace $J^\perp\subset X$ such that $X = J\oplus J^\perp$ algebraically and $\|x+x^\perp\| = \max\{\|x\|, \|x^\perp\|\}$ (resp. $=\|x\|+\|x^\perp\|$) whenever $x\in J$, $x^\perp\in J^\perp$. J is called an *M-ideal* if J^π, the annihilator of J in X', is an L-summand.

(ii) Let $T\colon X\to X$ be an operator. T is called a *multiplier* if for every extreme functional p (i.e. for every extreme point p of the dual unit ball) there is a scalar $a_T(p)$ such that $p\circ T = a_T(p)p$. Mult(X) will denote the collection of all multipliers.

A multiplier S is called the *adjoint* of a multiplier T (and we write $S = T^*$ in this case) if $a_S(p)$ is the complex conjugate of $a_T(p)$ for every p.

$Z(X)$, the *centralizer* of X, is the set of all multipliers which admit an adjoint.

These definitions have been introduced by Cunningham ([8, 9]) and Alfsen–Effros ([1]); for a systematic introduction the reader is referred to Behrends ([2]).

Here we only note that $Z(X)$ and Mult(X) are always commutative