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The universal right K-property for some interpolation spaces

by

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Abstract. Under some conditions on a Banach couple $\vec{A}$ and the parameter Φ of the K-method we show that the couples $(\vec{A}_{L^\infty}, \vec{A}_\Phi)$, $(\vec{A}_\Phi, \vec{A}_{L^\infty_{1/s}})$ have the universal right K-property if and only if $\Phi = L^\infty_{1/\varphi}$, where φ is the fundamental function of the space Φ. These results are used to obtain a characterization of some symmetric spaces E on $(0, \infty)$ such that the couple (E, L^∞) has the universal right K-property. Moreover, it is proved that the couple (L^1, E) does not have that property.

1. Introduction. We recall some notation from interpolation theory (cf. [4], [13]).

A pair $\vec{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are both continuously imbedded in some Hausdorff topological vector space V.

For a Banach couple $\vec{A} = (A_0, A_1)$ we can form the *intersection* $A_0 \cap A_1$ and the *sum* $A_0 + A_1$. They are both Banach spaces in the natural norms $J(1, a; \vec{A})$ and $K(1, a; \vec{A})$, respectively, where

$$J(t, a; \vec{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad a \in A_0 \cap A_1,$$
$$K(t, a; \vec{A}) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1,$$

for $t \in \boldsymbol{R}_+ = (0, \infty)$.

Let a Banach space A be continuously imbedded in $A_0 + A_1$. The space which consists of all limits in $A_0 + A_1$ of bounded sequences in A is called the *Gagliardo completion* of A with respect to $A_0 + A_1$ and denoted by $A^\sim$. The space $A^\sim$ is equipped with the norm $\|a\|_{A^\sim} = \inf \sup_{n \geq 1} \|a_n\|_A$, where the infimum is taken over all sequences $\{a_n\}_{n=1}^\infty$ bounded in A such that $a_n \to a$ in $A_0 + A_1$. The closure of $A_0 \cap A_1 \subset A$ in A is denoted by A^0.

Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be two Banach couples. A linear operator acting from $A_0 + A_1$ into $B_0 + B_1$ will be called a linear mapping from the couple $\vec{A}$ into the couple $\vec{B}$, written $T: \vec{A} \to \vec{B}$, if T maps continuously A_i into B_i, $i = 0, 1$.

$\mathscr{F}$ is an *interpolation functor* if for any Banach couple $\vec{A}$, $\mathscr{F}(\vec{A})$ is a Banach space intermediate with respect to $\vec{A}$, i.e., $A_0 \cap A_1 \subset \mathscr{F}(\vec{A}) \subset A_0 + A_1$, and every operator $T\colon \vec{A} \to \vec{B}$ maps $\mathscr{F}(\vec{A})$ into $\mathscr{F}(\vec{B})$ for any two Banach couples $\vec{A}$ and $\vec{B}$. In this case the spaces $\mathscr{F}(\vec{A})$ and $\mathscr{F}(\vec{B})$ are called *interpolation spaces* with respect to $\vec{A}$ and $\vec{B}$.

Important families of interpolation functors are generated by the *K-method*. The interpolation functor K_Φ is defined by the formula

$$K_\Phi(\vec{A}) = \vec{A}_\Phi = \{a \in A_0 + A_1\colon \|a\|_{\vec{A}_\Phi} = \|K(\cdot, a; \vec{A})\|_\Phi < \infty\},$$

where Φ is a Banach lattice of (equivalence classes of) measurable functions on $(\boldsymbol{R}_+, dt/t)$ such that $\min(1, t) \in \Phi$.

Let $\vec{A}$ and $\vec{B}$ be two Banach couples. We say that $\vec{A}$ has the *Calderón property relative to* $\vec{B}$ if the condition

$$K(t, b; \vec{B}) \leqslant K(t, a; \vec{A}) \quad \text{for all } t > 0$$

implies the existence of an operator $T\colon \vec{A} \to \vec{B}$ such that $Ta = b$. If $\vec{A} = \vec{B}$ we say that $\vec{A}$ is a *Calderón couple*.

We say that $\vec{B}$ has the *universal right K-property* if every Banach couple $\vec{A}$ has the Calderón property relative to $\vec{B}$ (see [10]).

Conventions. We say that a positive function f on $\boldsymbol{R}_+$ is dominated by another function g (notation: $f \prec g$ or $f(t) \prec g(t)$) if there is a positive constant c such that $f(t) \leqslant cg(t)$ for all $t \in \boldsymbol{R}_+$. The functions f and g are equivalent ($f \sim g$) if $f \prec g$ and $g \prec f$. We denote by f^{-1} the inverse function of f (if it exists). Two Banach spaces A and B are considered equal (identically equal) if the linear spaces A, B are identical and their norms are equivalent (equal); we then write $A = B$ ($A \equiv B$). The characteristic function of a set e is denoted by $\mathbf{1}_e$.

2. The universal right K-property for the couple $(\vec{A}_{L^\infty}, \vec{A}_\Phi)$. Let (Ω, Σ, μ) be a complete σ-finite measure space and $L^0 = L^0(\Omega, \Sigma, \mu)$ the space of all equivalence classes of μ-measurable real-valued functions, equipped with the topology of convergence in measure on μ-finite sets. We will say that a Banach space X is a *Banach lattice* (on (Ω, Σ, μ)) if X is a Banach subspace of L^0 with the property that if $x \in X$, $y \in L^0$ and $|y| \leqslant |x|$ μ-a.e. on Ω, then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$.

We say that a Banach lattice X on (Ω, Σ, μ) has the *Fatou property* if for every μ-a.e. pointwise increasing sequence $\{x_n\}_{n=1}^{\infty}$ of nonnegative functions in X with $\sup_{n \geqslant 1} \|x_n\|_X < \infty$, the function $x = \lim_{n\to\infty} x_n$ is in X with $\|x\|_X = \lim_{n\to\infty} \|x_n\|_X$.

Let X be a Banach lattice on (Ω, Σ, μ) and w a *weight function* (μ-a.e. positive measurable function on Ω). By X_w we shall denote the space of all functions x such that $xw \in X$ with the norm $\|x\|_{X_w} = \|xw\|_X$.

In the sequel let Φ be a Banach lattice on $(\boldsymbol{R}_+, dt/t)$ intermediate with respect to $\vec{L}^\infty = (L^\infty, L^\infty_{1/s})$, and let $\hat{\Phi} = \vec{L}^\infty_\Phi$. Moreover, for every function $f \in L^\infty + L^\infty_{1/s}$ let

$$P_u f = \overline{f \mathbf{1}_{(0,u)}}, \qquad Q_u f = \overline{f \mathbf{1}_{[u,\infty)}} \qquad (u > 0),$$

where

$$\bar{f} = \inf\{g\colon g \geqslant |f| \text{ a.e., } g\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+ \text{ concave}\}.$$

Note (see [5]) that for each $f \in L^\infty + L^\infty_{1/s}$ and all $t > 0$

$$\bar{f}(t) = K(t, f; \vec{L}^\infty). \tag{2.1}$$

Let $\vec{A} = (A_0, A_1)$ be a Banach couple and let A be a Banach space intermediate with respect to $\vec{A}$. We say that A is of the *class* $C_K(\psi, \vec{A})$ (cf. [11]) if

$$K(t, a; \vec{A}) \leqslant c\psi(t)\|a\|_A, \qquad a \in A,$$

and of the *class* $C_J(\psi, \vec{A})$ if

$$\psi(t)\|a\|_A \leqslant cJ(t, a; \vec{A}), \qquad a \in A_0 \cap A_1,$$

where the function ψ belongs to the set $\mathscr{P}$ of all *quasi-concave* functions on $\boldsymbol{R}_+$, i.e., $0 < \psi(s) \leqslant \max(1, s/t)\psi(t)$ for all $s, t \in \boldsymbol{R}_+$.

Remark 2.1. By the concavity of the K-functional and the inequality $K(s, a; \vec{A}) \leqslant \min(1, s/t) J(t, a; \vec{A})$ for $a \in A_0 \cap A_1$ we infer that for the interpolation space $\vec{A}_\Phi$ we have

$$K(t, a; \vec{A}) \leqslant \varphi(t)\|a\|_{\vec{A}_\Phi}, \qquad a \in \vec{A}_\Phi, \tag{2.2}$$

$$\varphi(t)\|a\|_{\vec{A}_\Phi} \leqslant J(t, a; \vec{A}), \qquad a \in A_0 \cap A_1, \tag{2.3}$$

where

$$\varphi(t) = \varphi_\Phi(t) = 1/\|\min(1, s/t)\|_\Phi.$$

Hence $\vec{A}_\Phi$ is of the class $C_K(\varphi, \vec{A}) \cap C_J(\varphi, \vec{A})$. In the sequel we often write φ instead of φ_Φ for a given Φ.

A function ψ in $\mathscr{P}$ is said to belong to $\mathscr{P}^{+-}$ if $\min(1, 1/t) s_\psi(t) \to 0$ as $t \to 0, \infty$, where $s_\psi(t) = \sup_{u>0}(\psi(ut)/\psi(u))$. If we introduce the numbers

$$\alpha_\psi = \sup_{0<t<1} \frac{\ln s_\psi(t)}{\ln t}, \qquad \beta_\psi = \inf_{1<t<\infty} \frac{\ln s_\psi(t)}{\ln t},$$

then $\psi \in \mathscr{P}^{+-}$ if and only if $0 < \alpha_\psi \leqslant \beta_\psi < 1$.

Let $\psi \in \mathscr{P}$ and $p = \infty$, or $\psi \in \mathscr{P}^{+-}$ and $1 \leqslant p < \infty$. Then for $\Phi = L^p_{1/\psi}(\boldsymbol{R}_+, dt/t)$ the interpolation space $\vec{A}_\Phi$ is denoted by $\vec{A}_{\psi,p}$. If $f(t) = t^\theta$ $(0 < \theta < 1)$, then we write $\vec{A}_{\theta,p}$.

By the equivalence theorem (see [11], Theorem 2.2) the proof of the following proposition is standard (cf. [4], p. 66).

PROPOSITION 2.2. *If a Banach space A is of the class $C_J(\psi, \vec{A})$ with $\psi \in \mathscr{P}^{+-}$, then $\vec{A}_{\psi,1} \subset A$ with continuous imbedding.*

PROPOSITION 2.3 (see [2], [15]). *Let a Banach lattice Φ be such that φ is increasing and $\varphi(\mathbf{R}_+) = \mathbf{R}_+$. Then*

$$K(t, a; A_0, \vec{A}_\Phi) \sim t\,\|Q_u K(\cdot, a; \vec{A})\|_\Phi$$

for each Banach couple $\vec{A}$ and all $a \in A_0 + \vec{A}_\Phi$, where $u = \varphi^{-1}(t)$.

COROLLARY 2.4. *Let a Banach lattice Φ be such that $\varphi \in \mathscr{P}^{+-}$. Then*

$$K(t, f; L^\infty, \hat{\Phi}) \sim \sqrt{t}$$

for $f(s) = \sqrt{\bar{\varphi}(s)}$.

Proof. By Proposition 2.2 and Remark 2.1 we get $\vec{L}^\infty_{\varphi,1} \subset \hat{\Phi}$. Since $\hat{\Phi} \subset \Phi$, by Proposition 2.3 and equality (2.1) we obtain for $u = \varphi^{-1}(t)$

$$\begin{aligned}\|Q_u K(\cdot, f; \vec{L}^\infty)\|_\Phi &\prec \int_0^\infty \frac{K(s, Q_u f; \vec{L}^\infty)}{\varphi(s)}\frac{ds}{s} = \int_0^\infty \frac{Q_u f(s)}{\varphi(s)}\frac{ds}{s} \\ &= \frac{f(u)}{u}\int_0^u \frac{s}{\varphi(s)}\frac{ds}{s} + \int_u^\infty \frac{f(s)}{\varphi(s)}\frac{ds}{s} \\ &\prec \frac{f(u)}{u}\cdot\frac{u}{\varphi(u)} + \frac{1}{\sqrt{\varphi(u)}} \prec t^{-1/2}.\end{aligned}$$

Consequently,

$$K(t, f; L^\infty, \hat{\Phi}) \prec \sqrt{t}.$$

On the other hand,

$$\begin{aligned}K(t, f; L^\infty, \hat{\Phi}) &\succ t\,\|Q_u K(\cdot, f; \vec{L}^\infty)\|_\Phi = t\,\|Q_u \bar{f}\|_\Phi \\ &\geqslant t f(u)\,\|\min(1, s/u)\|_\Phi = t f(u)/\varphi(u) \geqslant \sqrt{t}\end{aligned}$$

and the proof is complete.

Now we give a simple example of a couple $\vec{B}$ such that no couple $\vec{A}$ has the Calderón property relative to $\vec{B}$. It is enough to note that if $(B_0 \cap B_1)^\sim \neq B_0 \cap B_1$, then $\vec{A}$ does not have the Calderón property relative to $\vec{B}$ (cf. [8]). Indeed, let $b \in (B_0 \cap B_1)^\sim \setminus (B_0 \cap B_1)$. Then from the equality $\vec{B}_{L^\infty \cap L^\infty_{1/s}} \equiv (B_0 \cap B_1)^\sim$ (see [3], [1]) we have

$$K(t, b; \vec{B}) \sim \min(1, t).$$

Let $0 \neq a \in A_0 \cap A_1$. Then

$$K(t, b; \vec{B}) \prec K(t, a; \vec{A}).$$

However, $b = Ta$ for no bounded operator $T: \vec{A} \to \vec{B}$ since $Ta \in B_0 \cap B_1$.

COROLLARY 2.5. *If $\Phi = (L^\infty_{s^{-\theta}})^0$, $0 < \theta < 1$, then no Banach couple $\vec{A}$ has the Calderón property relative to (L^∞, Φ).*

Proof. It is easy to see that $\hat{\Phi} \equiv \Phi$ and $\varphi(t) = t^\theta$. We have

$$\begin{aligned}\Phi &= \{f \in L^\infty + L^\infty_{1/s}:\ \lim_{t \to 0, \infty} t^{-\theta} K(t, f; \vec{L}^\infty) = 0\} \\ &= \{f \in L^\infty + L^\infty_{1/s}:\ \lim_{t \to 0, \infty} t^{-\theta} \bar{f}(t) = 0\},\end{aligned}$$

by Theorem 3.4.2(c) in [4] and equality (2.1).

For $f(s) = s^\theta \mathbf{1}_{(0,1)}$, we have $f \in L^\infty + \Phi$, $f \notin \Phi$ and

$$K(t, f; L^\infty, \Phi) = K(t, f; L^\infty, \hat{\Phi}) \prec t\,\|Q_u \bar{f}\|_{L^\infty_{s^{-\theta}}} \prec t\,\|s^\theta\|_{L^\infty_{s^{-\theta}}},$$

by Proposition 2.3. Therefore

$$\|f\|_{(L^\infty \cap \Phi)^\sim} = \sup_{t > 0} \frac{K(t, f; L^\infty, \Phi)}{\min(1, t)} < \infty.$$

This implies that $f \in (L^\infty \cap \Phi)^\sim$. Consequently, $(L^\infty \cap \Phi)^\sim \neq L^\infty \cap \Phi$ and the proof is finished.

In the sequel we shall need some further notation (cf. [19]).

A Banach couple $\vec{A}$ is called *K_0-surjective* if for every function ψ such that $\min(1, 1/t)\psi(t) \to 0$ as $t \to 0, \infty$ one can find $a \in (A_0 + A_1)^0$ satisfying

$$K(t, a; \vec{A}) \sim \psi(t).$$

For example, the couples $(L^{p_0}(\mathbf{R}_+), L^{p_1}(\mathbf{R}_+))$, $1 \leqslant p_0 < p_1 \leqslant \infty$, $(L^\infty, L^\infty_{1/s})$ are K_0-surjective.

Remark 2.6. If for a Banach couple $\vec{A}$ there exists an element $a \in A_0 + A_1$ such that $K(t, a; \vec{A}) \sim \sqrt{t}$, then $\vec{A}$ is K_0-surjective (Yu. A. Brudnyi and N. Ya. Kruglyak–personal communication).

Let $\vec{X} = (X_0, X_1)$ be a couple of Banach lattices (on (Ω, Σ, μ)). The *Calderón space* (see [6]) of all $x \in L^0$ such that $|x| \leqslant \lambda |x_0|^{1-\theta} |x_1|^\theta$ μ-a.e. for some $x_i \in X_i$, $\|x_i\|_{X_i} \leqslant 1$, $i = 0, 1$, and some $\lambda > 0$ is denoted by $X_0^{1-\theta} X_1^\theta$. We put

$$\|x\|_{X_0^{1-\theta} X_1^\theta} = \inf \lambda.$$

Let X be a Banach lattice on (Ω, Σ, μ) and let $1 < p < \infty$. The *p-convexification* of X, denoted by $X^{(p)}$, is defined as follows: $x \in X^{(p)}$ if $|x|^p \in X$.

We put

$$\|x\|_{X(p)} = \| |x|^p \|_X^{1/p}.$$

It is easy to show the following

PROPOSITION 2.7. *Let X be a Banach lattice on (Ω, Σ, μ) and let w be a weight function. Then*

$$X^{1-\theta}(L_w^\infty)^\theta \equiv X_{w^\theta}^{(p)}, \qquad p = 1/(1-\theta),$$

for each $0 < \theta < 1$.

THEOREM 2.8. *Let a Banach lattice Φ with the Fatou property be such that $\varphi \in \mathscr{P}^{+-}$. Assume that a Banach couple $\vec{A}$ is K_0-surjective. If the couple $(\vec{A}_{L^\infty}, \vec{A}_\Phi)$ has the universal right K-property, then $\hat{\Phi} = L_{1/\varphi}^\infty$.*

Proof. Choose $a \in A_0 + A_1$ such that $K(s, a; \vec{A}) \sim f(s) = \sqrt{\varphi(s)}$. Then from the reiteration theorem (see [5], [18]) we infer that

$$K(t, a; \vec{A}_{L^\infty}, \vec{A}_\Phi) \sim K(t, K(s, a; \vec{A}); L^\infty, \hat{\Phi}) \sim K(t, f; L^\infty, \hat{\Phi}).$$

By Corollary 2.4 we obtain the inequality

$$K(t, a; \vec{A}_{L^\infty}, \vec{A}_\Phi) \leqslant K(t, x; \vec{L}^\infty),$$

where $x(s) = c\sqrt{s}$, with some positive constant c.

Now, by Proposition 2.7 and Theorem 2.1 in [20], $x \in L_{1/\sqrt{s}}^\infty = \langle \vec{L}^\infty; 1/2 \rangle$ (here $\langle \vec{X}; \theta \rangle$ is the $\pm$ method due to Gustavsson–Peetre; see [12], [20]). So if the couple $(\vec{A}_{L^\infty}, \vec{A}_\Phi)$ has the universal right K-property then

$$a \in \langle (\vec{A}_{L^\infty}, \vec{A}_\Phi); 1/2 \rangle.$$

By the maximal property of K_Ψ (see [5]) we have

$$\langle (\vec{A}_{L^\infty}, \vec{A}_\Phi); 1/2 \rangle \subset \vec{A}_{\langle (L^\infty, \Phi); 1/2 \rangle}$$

with continuous imbedding. Since

$$\langle (L^\infty, \Phi); 1/2 \rangle \subset \big((L^\infty)^{1/2} \Phi^{1/2} \big)^\sim,$$

by Theorem 2.1 in [20], it follows that

$$\langle (\vec{A}_{L^\infty}, \vec{A}_\Phi); 1/2 \rangle \subset \vec{A}_{\Phi(2)}$$

with continuous imbedding. Hence

$$\|a\|_{\vec{A}_{\Phi(2)}} = \|K(\cdot, a; \vec{A})\|_{\Phi(2)} \sim \|\sqrt{\varphi(\cdot)}\|_{\Phi(2)} = \|\varphi\|_\Phi^{1/2} < \infty.$$

Consequently, $\varphi \in \Phi$. Therefore if $x \in L_{1/\varphi}^\infty$, then $\bar{x}(s) \leqslant \bar{\varphi}(s) \|x\|_{L_{1/\varphi}^\infty}$ and $\|x\|_{\hat{\Phi}} \leqslant c \|x\|_{L_{1/\varphi}^\infty}$. Hence $L_{1/\varphi}^\infty \subset \hat{\Phi}$.

On the other hand, by inequality (2.2) we have $\bar{x}(t) = K(t, x; \vec{L}^\infty) \leqslant \varphi(t) \|x\|_{\hat{\Phi}}$ for all $x \in \hat{\Phi}$ and $t > 0$. Thus $\hat{\Phi} \subset L_{1/\varphi}^\infty$ and the proof is complete.

COROLLARY 2.9. *Suppose that a Banach couple $\vec{A}$ is K_0-surjective. Then if a couple $(\vec{A}_{0,\infty}, \vec{A}_{f,p})$ with $f \in \mathscr{P}^{+-}$ has the universal right K-property it follows that $p = \infty$.*

Proof. The Banach lattice $\Phi = L_{1/f}^p(\mathbf{R}_+, dt/t)$ has the Fatou property and it is easy to show that $\varphi \sim f$. Hence $\varphi \in \mathscr{P}^{+-}$. From Theorem 2.8 we obtain $\hat{\Phi} = L_{1/f}^\infty$, so $\|f\|_{\hat{\Phi}} \sim \|f\|_\Phi < \infty$ and this implies $p = \infty$.

From Theorem 2.8 and Theorem 4.2 in [10] we get the following result:

THEOREM 2.10. *Let a Banach lattice Φ with the Fatou property be such that $\varphi \in \mathscr{P}^{+-}$. Assume that $\vec{A}$ is a K_0-surjective Banach couple such that the couple $(L^\infty, L_{1/\varphi}^\infty)$ has the Calderón property relative to $(\vec{A}_{0,\infty}, \vec{A}_{\varphi,\infty})$. Then $(\vec{A}_{L^\infty}, \vec{A}_\Phi)$ has the universal right K-property if and only if $\hat{\Phi} = L_{1/\varphi}^\infty$.*

Remark 2.11. By Theorem 4.2 in [10] it is easy to see that if a Banach couple $\vec{A}$ has the universal right K-property, then so does the couple $(\vec{A}_{0,\infty}, \vec{A}_{f,\infty})$ for each quasi-concave function f. The couples with the universal right K-property include $(L^{p_0,\infty}, L^{p_1,\infty})$ (i.e., "weak L^p" spaces) and $(\mathrm{Lip}(\psi_0), \mathrm{Lip}(\psi_1))$ for $\psi_0, \psi_1 \in \mathscr{P}^{+-}$. In fact, all known examples of such couples can be subsumed under the general results in [5], [18] which imply that the couple $(\vec{B}_{\psi_0,\infty}, \vec{B}_{\psi_1,\infty})$ where $\psi_0, \psi_1 \in \mathscr{P}^{+-}$ has the universal right K-property for any Banach couple $\vec{B} = (B_0, B_1)$.

3. Further results and remarks. Now we give a theorem about the universal right K-property for a couple (L^1, E) of symmetric spaces. In order to obtain this result we introduce some further notation.

The *associated space* X^1 of a Banach lattice X on (Ω, Σ, μ) is the collection of all measurable functions $x \in L^0$ for which

$$\|x\|_{X^1} = \sup_{\|y\|_X \leqslant 1} \int_\Omega |x(t) y(t)| \, d\mu < \infty.$$

A Banach subspace E of $L^0(\mathbf{R}_+, m)$, where m is the Lebesgue measure, is said to be *symmetric* on $\mathbf{R}_+$ if for each $y \in E$ and all measurable functions x such that $x^*(t) \leqslant y^*(t)$ for $t > 0$, we have $x \in E$ and $\|x\|_E \leqslant \|y\|_E$ (cf. [13]). Here x^* denotes the nonincreasing rearrangement of $|x|$.

The *fundamental function* of a symmetric space E on $\mathbf{R}_+$ is defined for $t > 0$ as $\psi_E(t) = \|\mathbf{1}_{(0,t)}\|_E$.

For each symmetric space E on $\boldsymbol{R}_+$ with fundamental function ψ_E the following continuous inclusions are valid (see [13]):

$$\Lambda(E) \subset E \subset M(E), \tag{3.1}$$

where

$$\Lambda(E) = \{x \in L^0 \colon \|x\|_{\Lambda(E)} = \int_0^\infty x^*(s)\,d\psi_E(s) < \infty\},$$

$$M(E) = \left\{x \in L^0 \colon \|x\|_{M(E)} = \sup_{t>0}\left(\frac{\psi_E(t)}{t}\int_0^t x^*(s)\,ds\right) < \infty\right\}.$$

Let Φ' be the dual Banach space to Φ with respect to the bilinear form

$$(f, g) = \int_0^\infty f(t)\,g\left(\frac{1}{t}\right)\frac{dt}{t}.$$

By J_Ψ we denote the *J-method* (see [5], [10], [18] for more details).

Note that $K_\Phi(\vec{X})$ and $J_\Psi(\vec{X})$ are Banach lattices if $\vec{X} = (X_0, X_1)$ is a couple of Banach lattices.

THEOREM 3.1. *Assume that* (X_0, X_1) *is a couple of Banach lattices on* (Ω, Σ, μ). *Then*:

(a) $J_\Psi(X_0, X_1)^1 = K_{\Psi'}(X_0^1, X_1^1)$.

(b) *If* $\Phi \equiv \hat{\Phi}$ *and* $\Phi \cap L^\infty \neq L^\infty \cap L^\infty_{1/s}$, $\Phi \cap L^\infty_{1/s} \neq L^\infty \cap L^\infty_{1/s}$, *then* $K_\Phi(X_0, X_1)^1 = J_{\Phi'}(X_0^1, X_1^1)$.

Proof. This follows by applying the results of Lozanovskii [14] and a modification of the proof of Theorem 10.1 in [5].

THEOREM 3.2. *Let E be a symmetric space on* $\boldsymbol{R}_+$ *with the Fatou property such that the fundamental function* $\psi_E \in \mathscr{P}^{+-}$. *Then the couple* (L^1, E) *does not have the universal right K-property.*

Proof. Since E has the Fatou property, E is an interpolation space with respect to (L^1, L^∞) (see [13], Theorem 4.9, p. 142). But (L^1, L^∞) is a Calderón couple (see [7]) so $E = K_{\Phi_0}(L^1, L^\infty)$ with some $\Phi_0 \equiv \hat{\Phi}_0$, by the results of [5]. It is easy to see that $\varphi_{\Phi_0}(t) = t/\psi_E(t) \in \mathscr{P}^{+-}$.

Moreover, observe that $\Phi_0 \cap L^\infty \neq L^\infty \cap L^\infty_{1/s}$ and $\Phi_0 \cap L^\infty_{1/s} \neq L^\infty \cap L^\infty_{1/s}$. Indeed, if for example $\Phi_0 \cap L^\infty = L^\infty \cap L^\infty_{1/s}$, then $\Phi_0 \subset L^\infty_{1/s}$ and $\varphi_{\Phi_0}(t) \leqslant c\varphi_{L^\infty_{1/s}}(t) = ct$, so

$$\lim_{t\to 0} t^{-1}\varphi_{\Phi_0}(t) \leqslant c < \infty.$$

On the other hand,

$$\lim_{t\to 0} t^{-1}\varphi_{\Phi_0}(t) = \lim_{t\to 0} 1/\psi_E(t) = \infty$$

and we get a contradiction.

Since $E = E^{11}$, it follows from Theorem 3.1 that

$$E = K_\Phi(L^1, L^\infty),$$

where $\Phi \equiv \Phi_0''$. Obviously, Φ has the Fatou property. Since $\varphi_\Phi(t) \sim t/\psi_E(t)$ we have $\varphi \in \mathscr{P}^{+-}$. Therefore if the couple (L^1, E) has the universal right K-property, then so does the couple $(K_{L^\infty}(L^1, L^\infty), K_\Phi(L^1, L^\infty))$. Hence by Theorem 2.10 we deduce that $\hat{\Phi} = L^\infty_{1/\varphi}$ and consequently

$$E = K_\Phi(L^1, L^\infty) = K_{\hat{\Phi}}(L^1, L^\infty) = (L^1, L^\infty)_{\varphi,\infty}$$
$$= (L^1, L^\infty)_{t/\psi_E(t),\infty} = M(E).$$

But the couple $(L^1, M(E))$ does not have the universal right K-property (see [9]). Thus the proof is complete.

Remark 3.3. If a symmetric space E does not have the Fatou property, then in general (L^1, E) is not a Calderón couple. For example, this is the case if E is the closure of $L^1 \cap L^\infty$ in M_α, $0 < \alpha < 1$, where $M_\alpha = (L^1, L^\infty)_{1-\alpha,\infty}$ (see [17]).

4. The universal right K-property for the couple $(\vec{A}_\Phi, \vec{A}_{1,\infty})$. We give some results for the couple $(\vec{A}_\Phi, \vec{A}_{1,\infty})$.

PROPOSITION 4.1 (see [2], [15]). *Let a Banach lattice Φ be such that the function* $\varphi_*(t) = t/\varphi(t)$ *is increasing and* $\varphi_*(\boldsymbol{R}_+) = \boldsymbol{R}_+$. *Then*

$$K(t, a; \vec{A}_\Phi, A_1) \sim \|P_u K(\cdot, a; \vec{A})\|_\Phi$$

for each Banach couple $\vec{A}$ *and all* $a \in \vec{A}_\Phi + A_1$, *where* $u = \varphi_*^{-1}(t)$.

COROLLARY 4.2. *Let a Banach lattice Φ be such that the function φ is in* $\mathscr{P}^{+-}$. *Then*

$$K(t, f; \hat{\Phi}, L^\infty_{1/s}) \sim \sqrt{t}$$

for $f(s) = \sqrt{s\varphi(s)}$.

The proof is similar to that of Corollary 2.4.

COROLLARY 4.3. *If* $\Phi = (L^\infty_{s^{-\theta}})^0$, $0 < \theta < 1$, *then no Banach couple $\vec{A}$ has the Calderón property relative to* $(\Phi, L^\infty_{1/s})$.

Proof. Similarly to the proof of Corollary 2.5 it can be seen that the function $f(s) = s^\theta \mathbf{1}_{[1,\infty)}$ is in $(\Phi \cap L^\infty_{1/s})^\sim$ but not in $\Phi \cap L^\infty_{1/s}$.

THEOREM 4.4. *Let a Banach lattice Φ with the Fatou property be such that $\varphi\in\mathscr{P}^{+-}$. Moreover, let $\vec{A}$ be a K_0-surjective Banach couple. If the couple $(\vec{A}_\Phi, \vec{A}_{1,\infty})$ has the universal right K-property, then $\hat{\Phi} = L^\infty_{1/\varphi}$.*

Proof. Choose $a\in A_0+A_1$ such that $K(s, a; \vec{A}) \sim f(s) = \sqrt{s\varphi(s)}$. Then from the reiteration theorem (see [5], [18]) we get

$$K(t, a; \vec{A}_\Phi, \vec{A}_{1,\infty}) \sim K(t, K(\cdot, a; \vec{A}); \hat{\Phi}, L^\infty_{1/s}) \sim K(t, f; \hat{\Phi}, L^\infty_{1/s}).$$

By Corollary 4.2,

$$K(t, a; \vec{A}_\Phi, \vec{A}_{1,\infty}) \leqslant K(t, x; \vec{L}^\infty),$$

where $x(s) = c\sqrt{s}$, $c > 0$.

We have $x\in L^\infty_{1/\sqrt{s}} = \langle \vec{L}^\infty; 1/2\rangle$ by Proposition 2.7 and Theorem 2.1 in [20]. So if $\vec{X} = (\vec{A}_\Phi, \vec{A}_{1,\infty})$ has the universal right K-property, then $a\in\langle\vec{X}; 1/2\rangle$ with $\|a\|_{\langle \vec{X};1/2\rangle}$ bounded by some positive constant. Since $\langle(\Phi, L^\infty_{1/s}); 1/2\rangle \subset (\Phi^{1/2}(L^\infty_{1/s})^{1/2})^\sim$, by Theorem 2.1 in [20], it follows that

$$\langle(\vec{A}_\Phi, \vec{A}_{1,\infty}); 1/2\rangle \subset \vec{A}_{\langle(\Phi,L^\infty_{1/s});1/2\rangle} \subset \vec{A}_{(\Phi^{1/2}(L^\infty_{1/s})^{1/2})^\sim} = \vec{A}_F$$

with continuous imbeddings by Proposition 2.7, where $F = \Phi^{(2)}_{s^{-1/2}}$. Hence we obtain

$$\|a\|_{\vec{A}_F} = \|K(s, a; \vec{A})\|_F \sim \|(s^{-1/2}f(s))^2\|^{1/2}_\Phi = \|\varphi\|^{1/2}_\Phi < \infty.$$

Consequently, $\varphi\in\Phi$. This implies that $\hat{\Phi} = L^\infty_{1/\varphi}$, which finishes the proof.

COROLLARY 4.5. *Let $\vec{A}$ be a K_0-surjective Banach couple. If a couple $(\vec{A}_{f,p}, \vec{A}_{1,\infty})$, where $f\in\mathscr{P}^{+-}$, has the universal right K-property, then $p=\infty$.*

THEOREM 4.6. *Suppose a Banach lattice Φ has the Fatou property and $\varphi\in\mathscr{P}^{+-}$. Let, for a K_0-surjective couple $\vec{A}$, the couple $(L^\infty_{1/\varphi}, L^\infty_{1/s})$ have the Calderón property relative to $(\vec{A}_{\varphi,\infty}, \vec{A}_{1,\infty})$. Then the couple $(\vec{A}_\Phi, \vec{A}_{1,\infty})$ has the universal right K-property if and only if $\hat{\Phi} = L^\infty_{1/\varphi}$.*

COROLLARY 4.7. *Assume that a Banach lattice Φ has the Fatou property and $\varphi\in\mathscr{P}^{+-}$. Then the couple $(\hat{\Phi}, L^\infty_{1/s})$ has the universal right K-property if and only if $\hat{\Phi} = L^\infty_{1/\varphi}$.*

THEOREM 4.8. *Let E be a symmetric space on $\mathbf{R}_+$ with the Fatou property and let the fundamental function $\psi_E\in\mathscr{P}^{+-}$. Then the couple (E, L^∞) has the universal right K-property if and only if $E = M(E)$.*

Proof. Assume that the couple (E, L^∞) has the universal right K-property. Then by Theorems 3.1 and 4.4 we obtain $E = M(E)$ (the proof is similar to the proof of Theorem 3.2). On the other hand, the couple $(M(E), L^\infty)$ has the universal right K-property if $\psi_E\in\mathscr{P}^{+-}$ (see [9], [22]).

It is possible to give a direct proof of Theorem 4.8 (see [16]). We present it for the sake of completeness.

Since $\psi_E \sim \bar{\psi}_E$, we get $\bar{\psi}_E\in\mathscr{P}^{+-}$, so $\bar{\psi}_E(\mathbf{R}_+) = \mathbf{R}_+$ and $\bar{\psi}_E$ is increasing. Then

$$K(t, y; E, L^\infty) \prec K(t, y; \Lambda(E), L^\infty)$$
$$\leqslant \int_0^\infty (y\mathbf{1}_{(0,\bar{\psi}_E^{-1}(t))})^*(s)\,\bar{\psi}_E(s)\frac{ds}{s} \leqslant \int_0^{\bar{\psi}_E^{-1}(t)} \bar{\psi}_E(s)^{1/2}\frac{ds}{s} \prec \sqrt{t}$$

by (3.1) for $y(s) = \bar{\psi}_E(s)^{-1/2}$ and by the formula for $K(t, x; \Lambda(E), L^\infty)$. Consequently, we have

$$K(t, y; E, L^\infty) \leqslant K(t, x; \vec{L}^\infty) \quad \text{for } x(s) \sim \sqrt{s}.$$

By a result of Ovchinnikov (see [21]) and Proposition 2.7, the spaces $E^{1/2}(L^\infty)^{1/2} = E^{(2)}$ and $(L^\infty)^{1/2}(L^\infty_{1/s})^{1/2} = L^\infty_{1/\sqrt{s}}$, intermediate with respect to (E, L^∞) and $(L^\infty, L^\infty_{1/s})$ respectively, are interpolation spaces with respect to (E, L^∞) and $(L^\infty, L^\infty_{1/s})$. Therefore if the couple (E, L^∞) has the universal right K-property, then $(L^\infty, L^\infty_{1/s})$ has the Calderón property relative to (E, L^∞). Hence there exists a constant $c > 0$ such that

$$\|y\|_{E^{(2)}} \leqslant c\|x\|_{L^\infty_{1/\sqrt{s}}} < \infty.$$

Hence $\psi_E(\cdot)^{-1} \sim \bar{\psi}_E(\cdot)^{-1}\in E$. Since $x^*(t) \leqslant \psi_E(t)^{-1}\|x\|_{M(E)}$ for each $x\in M(E)$ and $t > 0$, we have

$$\|x\|_E = \|x^*\|_E \leqslant \|\psi_E(\cdot)^{-1}\|_E\|x\|_{M(E)} < \infty,$$

and so $M(E)\subset E$. Consequently, $E = M(E)$ by (3.1).

On the other hand, the couple $(M(E), L^\infty)$ has the universal right K-property if $\beta_{\psi_E} < 1$ (see [9]). Thus the proof is complete.

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On the ergodic power function for invertible positive operators

by

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Abstract. Let T be an invertible positive linear operator on L_p, $1 < p < \infty$, of a σ-finite measure space, and suppose T^{-1} is also positive. For $1 < r < \infty$, the *ergodic r-th power function* $P_r f$ of $f \in L_p$ (with respect to T) is defined by

$$P_r f = \Big[\sum_{k=0}^{\infty} |T_{k+1,0} f - T_{k,0} f|^r + |T_{0,k+1} f - T_{0,k} f|^r\Big]^{1/r}$$

where $T_{n,k} f = (n+k+1)^{-1} \sum_{i=-n}^{k} T^i f$ with $n, k \geqslant 0$. In this paper it is proved that if $T_{k,k}$ are uniformly bounded operators on L_p then $\|P_r f\|_p \leqslant C \|f\|_p$ for all $f \in L_p$. This generalizes a recent result of F. J. Martín-Reyes. An application is also given.

1. Introduction and the theorem. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T an invertible linear operator on $L_p = L_p(X, \mathcal{F}, \mu)$, with $1 < p < \infty$. If both T and T^{-1} are positive, then, as is well known (see e.g. [3]), T and T^{-1} are Lamperti operators, and there exists an invertible positive linear operator S acting on measurable functions such that S is multiplicative and $S1 = 1$, and a sequence $\{g_i\}_{i=-\infty}^{\infty}$ of positive measurable functions on X such that for each integer i, T^i has the form

$$T^i f(x) = g_i(x) S^i f(x). \tag{1}$$

It is immediately seen that

$$g_{i+j}(x) = g_i(x) S^i g_j(x) \quad \text{a.e. on } X. \tag{2}$$

Further, by the Radon–Nikodym theorem there exists a sequence $\{J_i\}_{i=-\infty}^{\infty}$ of positive measurable functions on X such that

$$\int J_i(x) S^i f(x)\, d\mu = \int f(x)\, d\mu \quad \text{for each } i \text{ and } f \in L_1. \tag{3}$$

Clearly,

$$J_{i+j}(x) = J_i(x) S^i J_j(x) \quad \text{a.e. on } X. \tag{4}$$