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Received June 27, 1986 (2184)
Revised version May 4, 1987

STUDIA MATHEMATICA, T. XC. (1988)

Groups of isometries on operator algebras

by

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Abstract. Let ϱ be a C_0-group of isometries on a unital C^*-algebra A. If $u(t) = \varrho(t)1$ and $\alpha(t)a = u(t)^*\varrho(t)a$, then $\varrho(t)a = u(t)\alpha(t)a$, α is a C_0-group of $*$-automorphisms on A and u is a unitary 1-cocycle. We study this decomposition of ϱ; as a consequence we obtain a classification of the generators of C_0-groups of isometries on A.

Introduction. In [18] Kadison proved that an isometry of a unital C^*-algebra A onto itself can be decomposed into a C^*-homomorphism followed by multiplication by a unitary. We study the consequences of applying this decomposition to a strongly continuous isometric representation ϱ of a topological group on A. We prove that the C^*-homomorphic part of ϱ is a strongly continuous group of $*$-automorphisms and that ϱ is norm-continuous if A is a von Neumann algebra. We establish conditions, global as well as local, which are satisfied by ϱ if and only if it is a group of $*$-automorphisms.

Using perturbation theory for $*$-automorphism groups we prove that if ϱ is a one-parameter group of isometries on A with generator δ, then there exist (γ, v, h), where γ is the generator of a one-parameter group of $*$-automorphisms on A, v is a unitary in A and h is a selfadjoint element of A, such that $\mathscr{D}(\delta) = v^*\mathscr{D}(\gamma)$ and

$$\delta(a) = v^*\gamma(va) + iv^*hva$$

for a in $\mathscr{D}(\delta)$. Using this we give local and global conditions equivalent to the fact that the unitary part of ϱ is a group.

In the next part of the paper we specialize to the case where ϱ is a one-parameter group. We observe that in some representations of A, $\varrho(t)a = u(t)av(t)$, where u and v are strongly continuous unitary groups. We study the generators of (semi-) groups of this form.

This study was motivated by applications to quantum mechanics (e.g. [15], [22], [25]) and partially inspired by the corresponding problems for a one-parameter semigroup on a Hilbert space if each element of the semigroup is polar decomposed [11], [12].

§0. **Notation.** If X is a Banach space, then $B(X)$ denotes the algebra of all bounded linear maps from X into X. If G is a topological space and ϱ is a map from G into $B(X)$, then ϱ is *strongly continuous* if $g \to \varrho(g)f$ is a continuous map from G into X for each f in X.

If G is a group, then a map ϱ from G into $B(X)$ is called a *representation* of G on X if $\varrho(e) = 1$ and $\varrho(gh) = \varrho(g)\varrho(h)$ for g and h in G, where e is the unit in G and 1 is the identity on X. A representation of the additive group of real numbers will sometimes be called a *one-parameter group*.

Let $\boldsymbol{R}_+$ denote the set of nonnegative real numbers. A *strongly* (resp. *weak**) *continuous semigroup* on X (resp. X^*) is a map ϱ from $\boldsymbol{R}_+$ into $B(X)$ (resp. $B(X^*)$) such that ϱ is strongly continuous, $\varrho(0) = 1$ and $\varrho(s+t) = \varrho(s)\varrho(t)$ for s and t in $\boldsymbol{R}_+$ (resp. there exists a strongly continuous semigroup ϱ_* on X such that $\varrho(t) = \varrho_*(t)^*$ for t in $\boldsymbol{R}_+$). If ϱ is a strongly continuous semigroup, then the *generator* δ of ϱ is defined by

$$\delta(f) = \lim_{t \downarrow 0} (\varrho(t)f - f)/t,$$

the domain $\mathscr{D}(\delta)$ of δ being the f in X where the limit exists. If ϱ is weak* continuous, then the generator of ϱ is the adjoint of the generator of ϱ_*. If ϱ is a strongly (weak*) continuous semigroup with generator δ, then we write $\varrho(t) = \exp(t\delta)$.

If ϱ is a representation (resp. semigroup), then ϱ is an *isometric* (*contraction* etc.) *representation* (resp. *semigroup*) if each $\varrho(g)$ is an isometry (contraction etc.).

Let A be a C^*-algebra with unit 1. Denote by $U(A)$ the unitary group in A. For b in A denote by $L(b)$ (resp. $R(b)$) the element in $B(A)$ defined by $L(b)a = ba$ (resp. $R(b)a = ab$).

Fix α in $B(A)$. α is called a *C^*-homomorphism* if $\alpha(1) = 1$, $\alpha(a^*) = \alpha(a)^*$ for a in A, and $\alpha(a^2) = \alpha(a^2)$ for selfadjoint a in A. α is a $(*\text{-})$ *homomorphism* if ($\alpha(a^*) = \alpha(a)^*$ and) $\alpha(ab) = \alpha(a)\alpha(b)$ for a and b in A. α is an *anti-homomorphism* if $\alpha(ab) = \alpha(b)\alpha(a)$ for a and b in A.

Let δ be a linear (unbounded) map on A, with domain $\mathscr{D}(\delta)$. Then δ is *symmetric* if $a \in \mathscr{D}(\delta)$ implies $a^* \in \mathscr{D}(\delta)$ and $\delta(a^*) = \delta(a)^*$, δ is a *derivation* if $a, b \in \mathscr{D}(\delta)$ imply $ab \in \mathscr{D}(\delta)$ and $\delta(ab) = \delta(a)b + a\delta(b)$. A symmetric derivation is called a *$*$-derivation*.

For background material on C^*-algebras, representations and semigroups, we refer the reader to [3], [8], [24] and [27].

§1. **Groups on C^*-algebras.** We introduce the polar decomposition of an isometric representation ϱ of a group G on a unital C^*-algebra A, and we investigate some of its basic properties.

The following result is an immediate consequence of [18, Theorem 7] and [26, Corollary 2].

THEOREM 1.1. *Let A be a unital C^*-algebra and let ϱ be a bounded linear map on A. If ϱ is surjective, then ϱ is an isometry if and only if $\varrho(U(A)) \subseteq U(A)$. If $\varrho(U(A)) \subseteq U(A)$ then $\varrho = L(u)\alpha$, where u is in $U(A)$ and α is a C^*-homomorphism; the decomposition is unique.*

DEFINITION 1.2. The decomposition $\varrho = L(u)\alpha$ in Theorem 1.1 will be called the *polar decomposition* of ϱ, because it is analogous to the polar decomposition of Hilbert space operators.

If ϱ is a map from a set G into $B(A)$ and $\varrho(g)U(A) \subseteq U(A)$ for each g in G, then we define u: $G \to U(A)$ and α: $G \to B(A)$ by the requirement that $\varrho(g) = L(u(g))\alpha(g)$ is the polar decomposition of $\varrho(g)$ for each g in G. We call u (resp. α) the *unitary* (resp. *positive* or *C^*-homomorphic*) *part* of ϱ, and the pair (u, α) is called the *polar decomposition* of ϱ. Note that (u, α) is determined by

$$u(g) = \varrho(g)1 \in U(A), \quad \alpha(g) = L(u(g)^*)\varrho(g) \in B(A)$$

for g in G. In the following two results we study the continuity properties of u and α.

PROPOSITION 1.3. *Let G be a topological space and let ϱ: $G \to B(A)$ be strongly continuous. Assume that $\varrho(g)U(A) \subseteq U(A)$, and let (u, α) be the polar decomposition of ϱ. Then $g \to u(g)$ is a continuous function from G into A, and α is strongly continuous.*

Proof. $u(g) = \varrho(g)1$ is continuous by assumption. Since

$$\|\alpha(g)a - \alpha(h)a\| \leqslant \|u(g) - u(h)\|\,\|a\| + \|\varrho(g)a - \varrho(h)a\|$$

for g, h in G and a in A we conclude that α is strongly continuous.

THEOREM 1.4. *Let G be a connected topological space, and let α be a strongly continuous map from G into the surjective C^*-homomorphisms on a unital C^*-algebra A. If there exists e in G such that $\alpha(e)$ is a $*$-homomorphism, then $\alpha(g)$ is a $*$-homomorphism for each g in G.*

Proof. We only need to prove that $\alpha(g)$ is a homomorphism for each g in G. Let π be an irreducible representation of A, and let $G_{\mathrm{h}}(\pi)$ (resp. $G_{\mathrm{ah}}(\pi)$) be the set of g in G for which $\pi\alpha(g)$ is a homomorphism (resp. anti-homomorphism). By [19, Theorem 2.6], G is the union of $G_{\mathrm{h}}(\pi)$ and $G_{\mathrm{ah}}(\pi)$. We will prove that $G = G_{\mathrm{h}}(\pi)$. It is easy to see that both $G_{\mathrm{h}}(\pi)$ and $G_{\mathrm{ah}}(\pi)$ are closed. If the intersection of $G_{\mathrm{h}}(\pi)$ and $G_{\mathrm{ah}}(\pi)$ is empty, then the proof is complete. Choose g in the intersection of $G_{\mathrm{h}}(\pi)$ and $G_{\mathrm{ah}}(\pi)$. Since $\pi\alpha(g)$ maps A onto $\pi(A)$ the existence of such a g implies that $\pi(A)$ is abelian, hence $G = G_{\mathrm{h}}(\pi) = G_{\mathrm{ah}}(\pi)$. Since the direct sum of all irreducible representations of A is faithful, each $\alpha(g)$ is a homomorphism, and this completes the proof of the theorem.

The following formula is the major tool used in the case where ϱ is an isometric representation of a group.

LEMMA 1.5. *Let ϱ be an isometric representation of a group G on a unital C^*-algebra A. If (u, α) is the polar decomposition of ϱ, then u is a 1-cocycle w.r.t. α, i.e.*

$$u(gh) = u(g)\alpha(g)\big(u(h)\big)$$

for g and h in G.

Proof. Since ϱ is a representation,

$$u(gh) = \varrho(gh)1 = \varrho(g)\big(\varrho(h)1\big) = u(g)\alpha(g)\big(u(h)\big)$$

which proves the lemma.

The following theorem is the major consequence of the results obtained above.

THEOREM 1.6. *Let G be a connected topological group, A a unital C^*-algebra and ϱ a strongly continuous isometric representation of G on A. Let $u(g) = \varrho(g)1$ and $\alpha(g) = L(u(g)^*)\varrho(g)$. Then α is a strongly continuous representation of G on A by $*$-automorphisms.*

Proof. By Proposition 1.3 and Theorem 1.4 it is enough to prove that $\alpha(gh) = \alpha(g)\alpha(h)$ for g and h in G. By Lemma 1.5

$$u(gh)\alpha(gh)a = \varrho(gh)a = \varrho(g)\big(\varrho(h)a\big) = u(g)\alpha(g)\big(u(h)\alpha(h)a\big)$$
$$= u(g)\alpha(g)\big(u(h)\big)\alpha(g)\big(\alpha(h)a\big) = u(gh)\alpha(g)\big(\alpha(h)a\big)$$

for a in A. This proves the theorem.

COROLLARY 1.7. *Let ϱ be a strongly continuous isometric representation of a connected topological group G on a von Neumann algebra. If the topology on G is metrizable, then ϱ is norm-continuous, i.e. $\|\varrho(g)-1\| \to 0$ as $g \to e$.*

Proof. α is norm-continuous by [10] or [20] hence $\varrho(g) = L\big(u(g)\big)\alpha(g)$ is norm-continuous by Proposition 1.3.

The case where the von Neumann algebra is all of $B(\mathscr{H})$ was considered in [2]. Note that if a topological group G satisfies the first axiom of countability, then the topology on G is metrizable [23].

Remark 1.8. We have the following converse of Theorem 1.6. Let G be a group, u a map from G into $U(A)$ and α a representation of G as $*$-automorphisms of A. If u is a 1-cocycle w.r.t. α, then $\varrho(g) = L\big(u(g)\big)\alpha(g)$ defines an isometric representation of G on A with polar decomposition (u, α).

§2. The unitary part. Let G be a group and ϱ an isometric representation of G on a unital C^*-algebra A. In this section we give global conditions which make the unitary part of the polar decomposition (u, α) of ϱ a representation of G.

The following result follows immediately from the cocycle property of u (Lemma 1.5).

PROPOSITION 2.1. *Fix g and h in G. Then $u(gh) = u(g)u(h)$ if and only if $\alpha(g)\big(u(h)\big) = u(h)$.*

PROPOSITION 2.2. *If G is a connected topological group and ϱ is strongly continuous, then the following three conditions are equivalent:*

(1) *u is a representation.*

(2) *The range of u is a subset of the fixed point algebra for the action of G on A by α.*

(3) *$L\big(u(g)\big)$ and $\alpha(h)$ commute for all g and h in G.*

Proof. (1)$\Leftrightarrow$(2) is a trivial consequence of Proposition 2.1. Since $\alpha(h)$ is a homomorphism (Theorem 1.4) we get

$$\alpha(h)L\big(u(g)\big)a = \alpha(h)\big(u(g)\big)\alpha(h)a$$

for a in A. Using this it is easy to see that (2) and (3) are equivalent.

DEFINITION 2.3. Let $\varrho \in B(A)$ with $\varrho U(A) \subseteq U(A)$ and let $\varrho = L(u)\alpha$ be the polar decomposition of ϱ. ϱ is said to be *quasi-normal* if $L(u)$ and α commute.

Remark 2.4. The term "quasi-normal" is chosen because a linear map H on a Hilbert space is quasi-normal if and only if $PU = UP$, where $H = UP$ is the polar decomposition of H.

The following theorem is similar to [12, Theorem 2] (cf. also [11, Theorem 6]).

THEOREM 2.5. *Let G be either the group of real numbers or the group of complex numbers of modulus one. If ϱ is a strongly continuous isometric representation of G on a unital C^*-algebra, then the following two conditions are equivalent:*

(1) *$\varrho(g)$ is quasi-normal for each g in G.*

(2) *$u(g) = \varrho(g)1$ is a representation of G.*

Proof. (2)$\Rightarrow$(1) follows from Proposition 2.2.

(1)$\Rightarrow$(2). First we consider the case where G is the additive group of reals. Since ϱ and α are representations we get

$$L\big(u(nt)\big)\alpha(nt) = L\big(u(t)\big)^n\alpha(nt)$$

for t in G and $n = 1, 2, 3, \ldots$; therefore

$$u(nt) = u(t)^n$$

for t in G and $n = 1, 2, 3, \ldots$; in particular,

$$u(p/q+r/s) = u(1/(qs))^{ps+qr} = u(p/q)\,u(r/s)$$

for $p, q, r, s = 1, 2, 3, \ldots$; by continuity

$$u(s+t) = u(s)\,u(t)$$

for $s, t \geqslant 0$. Similarly one proves that $u(s+t) = u(s)u(t)$ for $s, t \leqslant 0$. By these equalities and the norm continuity of u

$$\lim_{t\downarrow 0}(\varrho(t)1-1)/t \quad \text{and} \quad \lim_{t\uparrow 0}(\varrho(t)1-1)/t$$

exist, hence $1 \in \mathscr{D}(\delta)$, where δ is the generator of ϱ [3], [8]. We deduce that

$$\lim_{t\downarrow 0}(u(t)-1)/t = \delta(1) = \lim_{t\uparrow 0}(u(t)-1)/t.$$

Hence $u(t) = \exp(t\delta(1))$ for t in G, in particular

$$u(s+t) = u(s)\,u(t)$$

for s and t in G.

If G is the multiplicative group of complex numbers of modulus one, apply the result above to $t \to u(e^{it})$. This completes the proof.

Remark 2.6. If $T(t)$ is a strongly continuous one-parameter (semi-) group on a Hilbert space and $T(t) = U(t)P(t)$ is the polar decomposition of each $T(t)$, then [12] $U(t)$ is a (semi-) group if $P(t)$ is a (semi-) group.

Next we indicate how one may construct a strongly continuous one-parameter group ϱ of isometries on a unital C^*-algebra A such that the unitary part of ϱ is not a representation. The construction is carried out in terms of the polar decomposition (u, α) of ϱ (cf. Remark 1.8).

Fix a strongly continuous one-parameter group α of $*$-automorphisms on A. For each selfadjoint h in A, the solution to

$$\frac{d}{ds}u(s) = iu(s)\alpha(s)h, \qquad u(0) = 1$$

is a norm-differentiable unitary 1-cocycle w.r.t. α [1, Theorem 2]. It is easy to see that $\alpha(t)u(s) = u(s)$ (all real s and t) implies that $\alpha(t)H = h$ (all t); hence by Proposition 2.1, u is a representation if and only if $\alpha(t)h = h$ for all t.

§3. Generator results. In this section, we obtain a characterization of the generators of isometric one-parameter groups on a unital C^*-algebra. Special

attention is paid to the situation where the unitary part is differentiable. Further, quasi-normal groups are classified in terms of their generators.

THEOREM 3.1. *Let A be a unital C^*-algebra and let δ be a linear map on A with domain $\mathscr{D}(\delta)$. δ is the generator of a strongly continuous one-parameter group of isometries on A if and only if there exists a selfadjoint h in A and a unitary v in A such that*

$$(*) \qquad \gamma(a) = v\delta(v^*a) - iha,$$

for a in $\mathscr{D}(\gamma) = v\mathscr{D}(\delta)$, is the generator of a strongly continuous one-parameter group of $*$-automorphisms on A.

Proof. Let $\varrho(t) = \exp(t\delta)$ be a group of isometries, and let (u, α) be the polar decomposition of ϱ. Since u is a 1-cocycle w.r.t. α, it follows from [6, Corollary 4.4] that there exists v in $U(A)$ and a (norm-) differentiable 1-cocycle w such that $u(t) = v^*w(t)\alpha(t)(v)$. By [1] and [6, Proposition 4.6], w is the unique solution to the differential equation $(d/dt)w(t) = iw(t)\alpha(t)h$, $w(0) = 1$, where $-i(d/dt)w(t)|_{t=0} = h = h^* \in A$. Hence, we get $(*)$ and γ is the generator of α.

Conversely, assume that γ is the generator of a group α of $*$-automorphisms on A. Let w be the solution to $(d/dt)w(t) = iw(t)\alpha(t)(h)$, $w(0) = 1$; then $\varrho(t)a = v^*w(t)\alpha(t)(va)$ is a strongly continuous one-parameter group of isometries on A. It is easy to see that the generator δ of ϱ is determined by $(*)$. This completes the proof.

Remark 3.2. (a) The decomposition of δ is unique in the sense that γ is the generator of the positive part of $\exp(t\delta)$ and for each unitary v^* in $\mathscr{D}(\delta)$, there exists exactly one selfadjoint h in A such that (δ, γ, v, h) satisfy $(*)$. In fact, $h = -iv\delta(v^*)$.

If (v^*, w^*) is a pair of unitaries in $\mathscr{D}(\delta)$ and (h, k) are the corresponding selfadjoint elements of A, i.e. (v, h) and (w, k) satisfy $(*)$, then $i(k-h) = \gamma(vw^*)$.

(b) The above theorem is the infinitesimal version of the fact (Theorem 1.6) that all strongly continuous one-parameter groups ϱ of isometries on a unital C^*-algebra A are of the form $\varrho(t) = L(u(t))\alpha(t)$, where α is a strongly continuous group of $*$-automorphisms on A and u is a continuous unitary 1-cocycle w.r.t. α.

COROLLARY 3.3. *Let δ be the generator of a strongly continuous one-parameter group of isometries on a unital C^*-algebra. If 1 is in $\mathscr{D}(\delta)$, then $\mathscr{D}(\delta)$ is a $*$-subalgebra of A, $\delta(1)^* = -\delta(1)$ and $\gamma = \delta - L(\delta(1))$ is a $*$-derivation.*

Proof. Let h be the selfadjoint element of A corresponding to the unitary $1 \in \mathscr{D}(\delta)$. Then $ih = (d/dt)u(0) = \delta(1)$, hence $\delta(1)^* = -ih^* = -\delta(1)$. The other statements are part of Theorem 3.1.

Our next result gives an algebraic characterization of the linear maps δ for which $\delta - L(\delta(1))$ is a $*$-derivation.

PROPOSITION 3.4. *Let A be a C^*-algebra and let δ be a linear map on A whose domain $\mathscr{D}(\delta)$ is a $*$-algebra. Fix h in A and let $\gamma = \delta - L(h)$. The following two conditions are equivalent:*

(1) (a) $\delta(a^*) = \delta(a)^* + ha^* - a^*h^*$.
    (b) $\delta(ab) = \delta(a)b + a\delta(b) - ahb$.

(2) *γ is a $*$-derivation.*

Further, if δ satisfies (1), *if A has a unit and $1 \in \mathscr{D}(\delta)$, then $h = \delta(1)$.*

Proof. It is easy to see that δ satisfies 1(a) (resp. 1(b)) if and only if γ is symmetric (resp. a derivation).

If $1 \in \mathscr{D}(\delta)$ then $0 = \gamma(1) = \delta(1) - h$, i.e. $h = \delta(1)$. The proof is complete.

COROLLARY 3.5. *Let (δ, h) satisfy* (1) *of Proposition* 3.4, *let ω be a state on A and let $(\mathscr{H}, \pi, \Omega)$ be the cyclic representation associated with ω. If $\mathscr{D}(\delta)$ is dense in A and*

$$\omega(\delta(a)) = \omega(ha)$$

for a in $\mathscr{D}(\delta)$, then there exists a symmetric operator H on $\mathscr{H}$ such that

$$\mathscr{D}(H) = \pi(\mathscr{D}(\delta))\Omega, \qquad \pi(\delta(a))f = i((H - ih)\pi(a) - \pi(a)H)f$$

for a in $\mathscr{D}(\delta)$ and f in $\mathscr{D}(H)$.

Further, δ is closable if π is faithful.

Proof. By assumption $\gamma = \delta - L(h)$ is a $*$-derivation and $\omega(\gamma(a)) = 0$ for a in $\mathscr{D}(\gamma)$. By [3, Corollary 3.2.27], δ is closable if π is faithful. By [3, Proposition 3.2.28] there exists a symmetric operator H on $\mathscr{H}$ such that

$$\pi(\gamma(a))f = i(H\pi(a) - \pi(a)H)t$$

for a in $\mathscr{D}(\gamma)$ and f in $\mathscr{D}(H) = \pi(\mathscr{D}(\gamma))\Omega$. From this the proof is easily completed.

Remark 3.6. The study of linear maps δ satisfying the conclusions of Corollary 3.5 was proposed in [14], where a set of sufficient conditions are stated. These conditions are very strong, in fact, they imply that δ is a $*$-derivation (i.e. $h = 0$).

The next corollary should be compared with [13], where a similar result is obtained for a strongly continuous one-parameter group of 2-positive maps.

COROLLARY 3.7. *Let G be a simply connected Lie group and let $X_1, \ldots, X_d$ be a basis for the Lie algebra of G. Let ϱ be a strongly continuous isometric representation of G on a unital C^*-algebra and let δ_j be the generator of $t \to \varrho(\exp tX_j)$ for all j. If $1 \in \mathscr{D}(\delta_j)$ for all j, then the following four conditions are equivalent:*

(1) *Each δ_j is a $*$-derivation.*

(2) *$\delta_j(1) = 0$ for all j.*

(3) *$\varrho(g)$ is a $*$-automorphism for all g in G.*

(4) *$\varrho(g)1 = 1$ for all g in G.*

Proof. (1)$\Leftrightarrow$(2) by Corollary 3.3.

(3)$\Leftrightarrow$(4) by the uniqueness of the polar decomposition.

(4)$\Rightarrow$(2) is trivial.

(2)$\Rightarrow$(4). Choose coordinates of the second kind

$$g = \exp(t_1(g)X_1)\ldots\exp(t_d(g)X_d)$$

in a neighbourhood of the unit e in G. Then

$$\varrho(g) = \exp(t_1(g)\delta_1)\ldots\exp(t_d(g)\delta_d).$$

Hence ϱ has the wanted property in a neighbourhood of e and therefore everywhere [7]. The proof is complete.

THEOREM 3.8. *Let ϱ be a strongly continuous isometric one-parameter group on a unital C^*-algebra, let (u, α) be the polar decomposition of ϱ and let δ be the generator of ϱ. The following eight conditions are equivalent:*

(1) *$u(s)u(t) = u(s+t)$ for all s and t.*

(2) *$u(s) = \alpha(t)u(s)$ for all s and t.*

(3) *$L(u(t))\alpha(t) = \alpha(t)L(u(t))$ for all t.*

(4) *$1 \in \mathscr{D}(\delta)$ and $u(t) = \exp(t\delta(1))$.*

(5) *$1 \in \mathscr{D}(\delta)$ and $\delta(1)u(s) = \alpha(t)(\delta(1)u(s))$ for all s and t.*

(6) *$1 \in \mathscr{D}(\delta)$ and $\delta(1) = \alpha(t)\delta(1)$ for all t.*

(7) *$u(s) \in \mathscr{D}(\delta)$ and $\delta(u(s)) = \delta(1)u(s)$ for all s.*

(8) *$1 \in \mathscr{D}(\delta)$, $\delta(1) \in \mathscr{D}(\delta)$ and $\delta^2(1) = \delta(1)^2$.*

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4) by Proposition 2.2 and (the proof of) Theorem 2.5. (5)$\Rightarrow$(6) is trivial.

(2)$\Leftrightarrow$(7) and (6)$\Leftrightarrow$(8) since $\gamma = \delta - L(\delta(1))$ is the generator of α by Corollary 3.3.

(2)$\Rightarrow$(5). Since (2) implies (1) and (4) we see that

$$\begin{aligned} 0 = \frac{d}{dt}u(s) = \frac{d}{dt}\alpha(t)u(s) &= \frac{d}{dt}u(-t)\varrho(t)u(s) \\ &= u(-t)\varrho(t)\delta(u(s)) - \delta(u(-t))\varrho(t)u(s) \\ &= \alpha(t)(\delta(1)u(s)) - \delta(1)u(s). \end{aligned}$$

(6) ⇒ (4). Since

$$\frac{d}{dt}u(t) = \varrho(t)\delta(1) = R(\delta(1))u(t)$$

the uniqueness theorem from semigroup theory [8, Theorem 1.7] implies

$$u(t) = \exp(tR(\delta(1)))u(0) = \sum_{n=0}^{\infty} t^n \delta(1)^n/n! = \exp(t\delta(1)).$$

This completes the proof.

The equivalence (1) ⇔ (8) in Theorem 3.8 can be formulated as follows:

COROLLARY 3.9. *The one-parameter group* $\exp(t\delta)$ *is quasi-normal if and only if we may take* $v = 1$ *in Theorem* 3.1 *and the corresponding* $h = h^*$ *satisfies* $h \in \mathscr{D}(\gamma)$ *and* $\gamma(h) = 0$.

It is not likely that one may take $v \neq 1$ in Corollary 3.9.

Remark 3.10. Note that [5, Lemma 2.1] is a trivial consequence of Corollary 3.3, because if ϱ is constructed as in Remark 1.8, then the assumption in [5, Lemma 2.1] is $1 \in \mathscr{D}(\delta)$.

§ 4. **Implemented groups.** In this section we consider operators on (subalgebras of) $B(\mathscr{H})$ of the form $a \to i(Ha - aK)$, where H and K are operators on the Hilbert space $\mathscr{H}$. Our first result shows that the generator of a norm-continuous one-parameter group of isometries is of this form, with H and K bounded and selfadjoint.

THEOREM 4.1. *Let G be a connected locally compact abelian group or a simply connected Lie group. If* ϱ *is a norm-continuous isometric representation of G on a unital* C^**-algebra A, then there exist two norm-continuous unitary representations U and V of G in* A'' *(the enveloping von Neumann algebra of A) such that*

$$\varrho(g)a = U(g)aV(g)^*$$

for all g in G and a in A.

Proof. Let (u, α) be the polar decomposition of ϱ. By Theorem 1.6, α is a norm-continuous representation of G on A by $*$-automorphisms, hence [9], [24, Theorem 8.5.2] there exists a norm-continuous unitary representation V of G in A'' such that $\alpha(g)a = V(g)aV(g)^*$. Let $U(g) = u(g)V(g)^*$. Applying Lemma 1.5 one proves that U is a representation of G. This completes the proof.

Remark 4.2. If G is abelian then $g \to V(g)^*$ is a representation of G.

Let ϱ be a strongly continuous one-parameter group of isometries on a unital C^*-algebra A, and let (u, α) be the polar decomposition of ϱ. By [21, Theorem A1], A may be represented on a Hilbert space $\mathscr{H}$ so that α is convariant, i.e. there exists a strongly continuous unitary group V on $\mathscr{H}$ such that $\alpha(t)a = V(t)^* aV(t)$. Let $U(t) = u(t)V(t)^*$. Then $\varrho(t)a = U(t)aV(t)$, where U and V are strongly continuous unitary groups on $\mathscr{H}$. If iH (resp. iK) denotes the generator of U (resp. V), then there exists a unitary operator φ and a selfadjoint operator h on $\mathscr{H}$ such that $\mathscr{D}(K) = \varphi\mathscr{D}(H)$ and $Kf = \varphi H\varphi^* f + hf$ for f in $\mathscr{D}(K)$ (cf. e.g. [6, Theorem 4.3]). The following discussion is motivated by the observations above.

DEFINITION 4.3. Let S and T be two densely defined unbounded linear operators on $\mathscr{H}$. Denote by $\delta_{S,T}$ the linear map on $B(\mathscr{H})$ defined by $\mathscr{D}(\delta_{S,T}) = \{a \in B(\mathscr{H}) \mid a\mathscr{D}(T) \subseteq \mathscr{D}(S)$ and $f \in \mathscr{D}(T) \to (Sa - aT)f$ is bounded$\}$ and

$$\delta_{S,T}(a)f = i(Sa - aT)f$$

for f in $\mathscr{D}(T)$ and a in $\mathscr{D}(\delta_{S,T})$. Let $\mathscr{D}_{S,T}$ be the set of a in $B(\mathscr{H})$ for which $(f, g) \in \mathscr{D}(T) \times \mathscr{D}(S^*) \to (af, S^* g) - (aTf, g)$ is bounded.

LEMMA 4.4. *If S is closable with closure* $\bar{S}$, *then* $\mathscr{D}_{S,T}$ *equals* $\mathscr{D}(\delta_{\bar{S},T})$.

Proof. Fix a in $\mathscr{D}_{S,T}$ and f in $\mathscr{D}(T)$. By assumption, $g \in \mathscr{D}(S^*) \to (af, S^* g)$ is bounded, hence $af \in \mathscr{D}(\bar{S})$ and therefore $a \in \mathscr{D}(\delta_{\bar{S},T})$. The converse inclusion is obvious.

Similarly to the proof of [3, Proposition 3.2.55] one proves

THEOREM 4.5. *Let* $\exp(itH)$ *and* $\exp(-itK)$ *be two strongly continuous semigroups on the Hilbert space* $\mathscr{H}$, *and let* η *be the weak* continuous semigroup on* $B(\mathscr{H})$ *defined by*

$$\eta(t)a = \exp(itH)a\exp(-itK)$$

for a in $B(\mathscr{H})$ *and* $t \geqslant 0$. *The generator of* η *is* $\delta_{H,K}$.

COROLLARY 4.6. *Let* $A \subseteq B(\mathscr{H})$ *be a* C^**-algebra and let* ϱ *be a strongly continuous semigroup on A. If* $\varrho(t)a = \eta(t)a$ *for all a in A and* $t \geqslant 0$, *then the restriction* δ_A *of* $\delta_{H,K}$ *to the set of a in* $\mathscr{D}_{H,K} \cap A$ *for which* $\delta_{H,K}(a) \in A$ *is the generator of* ϱ.

Proof. Let δ be the generator of ϱ and let $\tilde{\delta} = \delta_{H,K}$. It is obvious that $\delta \subseteq \delta_A \subseteq \tilde{\delta}$. Conversely, if $a \in \mathscr{D}(\delta_A)$, then $(1-\tilde{\delta})a \in A$, hence if $b = (1-\delta)^{-1}(1-\tilde{\delta})a$, then

$$(1-\tilde{\delta})b = (1-\delta)b = (1-\tilde{\delta})a$$

and therefore $a = b \in \mathscr{D}(\delta)$. This completes the proof.

PROPOSITION 4.7. *If S is closed, then $\delta_{S,T}$ is both norm-norm and strong-strong closed, and further*

$$\{a \in B(\mathscr{H}) \mid \operatorname{ran} a \subseteq \mathscr{D}(S),\ \operatorname{ran} a^* \subseteq \mathscr{D}(T^*)\} \subseteq \mathscr{D}(\delta_{S,T}).$$

Proof. Only the last assertion needs a proof. Fix g in $\mathscr{D}(S^*)$ with $\|g\| \leqslant 1$, and let $T_g(f) = (af, S^* g)$ for f in $\mathscr{H}$. Since $|T_g(f)| \leqslant \|Saf\|$, the Banach–Steinhaus Theorem implies $|(af, S^* g)| \leqslant M\|f\|\,\|g\|$ for f in $\mathscr{H}$, g in $\mathscr{D}(S^*)$ and some fixed $M \geqslant 0$. Likewise $|(aTf, g)| \leqslant M\|f\|\,\|g\|$ for f in $\mathscr{D}(T)$ and g in $\mathscr{H}$. An application of Lemma 4.4 completes the proof.

THEOREM 4.8. *Let T be a densely defined linear map on a Hilbert space and let $\delta = \delta_{T,T^*}$.*

(a) *δ is a derivation if T is symmetric.*

(b) *δ is symmetric if T is closed.*

(c) *T is symmetric if δ is a $*$-derivation and T is closed.*

In particular, if T is closed, then T is symmetric if and only if δ_{T,T^*} is a $*$-derivation.

Proof. (a) Fix a and b in $\mathscr{D}(\delta)$ and f in $\mathscr{D}(T^*)$. We have $ab\,\mathscr{D}(T^*) \subseteq a\mathscr{D}(T) \subseteq \mathscr{D}(T)$ and

$$i(Tab - ab\,T^*)f = iTabf + a\big(\delta(b) - iTb\big)f = \delta(a)\,bf + a\delta(b)\,f.$$

(b) If f and g are in $\mathscr{D}(T^*)$ and a is in $\mathscr{D}(\delta)$, then the formula

$$(a^* f,\ T^* g) - (a^* T^* f,\ g) = (f,\ aT^* g) - (T^* f,\ ag)$$

shows $a^* \in \mathscr{D}(\delta)$ and $\delta(a^*) = \delta(a)^*$.

(c) Fix a and b in $\mathscr{D}(\delta)$ and let f and g be in $\mathscr{D}(T^*)$. Since δ is a $*$-derivation we get

$$(Tabf,\ g) - (abT^* f,\ g) = (aTbf,\ g) - (abT^* f,\ g) - (bf,\ Ta^* g) + (bf,\ a^* T^* g).$$

Hence $(aTbf, g) = (bf, Ta^* g)$, and therefore bf is in $\mathscr{D}(T^*)$ and $(Tbf, a^* g) = (T^* bf, a^* g)$. Choosing a and b suitably one gets $\mathscr{D}(T) \subseteq \mathscr{D}(T^*)$ and $Tf = T^* f$ for f in $\mathscr{D}(T)$. Assertions (a) through (c) combined prove the last assertion of the theorem.

Finally, we briefly discuss the extension problem for the $*$-derivations $\delta_S = \delta_{S,S^*}$. Trivially $S \subseteq T$ implies $\delta_S \subseteq \delta_T$. Hence, if S is symmetric and T is a maximal symmetric extension of S, then either δ_T or $-\delta_T = \delta_{-T}$ is the generator of a weak* continuous semigroup α of $*$-homomorphisms on $B(\mathscr{H})$. It is clear that α cannot be extended to a one-parameter group unless T is selfadjoint. Note that $\delta_S \subseteq \operatorname{ad} S \subseteq \delta_{S^*}$; this gives a connection with the extension problem as formulated in [4], [16] and [17].

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Received October 1, 1986 (2219)
Revised version April 22, 1987

Added in proof (January 1988). After having finished the work on this paper, the author learned that the norm-continuous case of Theorem 3.1 was handled in: A. M. Sinclair, *Jordan homomorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. 24 (1970), 209–214.

STUDIA MATHEMATICA, T. XC. (1988)

The universal right K-property for some interpolation spaces

by

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Abstract. Under some conditions on a Banach couple $\vec{A}$ and the parameter Φ of the K-method we show that the couples $(\vec{A}_{L^\infty}, \vec{A}_\Phi)$, $(\vec{A}_\Phi, \vec{A}_{L^\infty_{1/s}})$ have the universal right K-property if and only if $\Phi = L^\infty_{1/\varphi}$, where φ is the fundamental function of the space Φ. These results are used to obtain a characterization of some symmetric spaces E on $(0, \infty)$ such that the couple (E, L^∞) has the universal right K-property. Moreover, it is proved that the couple (L^1, E) does not have that property.

1. Introduction. We recall some notation from interpolation theory (cf. [4], [13]).

A pair $\vec{A} = (A_0, A_1)$ of Banach spaces is called a *Banach couple* if A_0 and A_1 are both continuously imbedded in some Hausdorff topological vector space V.

For a Banach couple $\vec{A} = (A_0, A_1)$ we can form the *intersection* $A_0 \cap A_1$ and the *sum* $A_0 + A_1$. They are both Banach spaces in the natural norms $J(1, a; \vec{A})$ and $K(1, a; \vec{A})$, respectively, where

$$J(t, a; \vec{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad a \in A_0 \cap A_1,$$

$$K(t, a; \vec{A}) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1,$$

for $t \in \mathbf{R}_+ = (0, \infty)$.

Let a Banach space A be continuously imbedded in $A_0 + A_1$. The space which consists of all limits in $A_0 + A_1$ of bounded sequences in A is called the *Gagliardo completion* of A with respect to $A_0 + A_1$ and denoted by $A^\sim$. The space $A^\sim$ is equipped with the norm $\|a\|_{A^\sim} = \inf \sup_{n \geq 1} \|a_n\|_A$, where the infimum is taken over all sequences $\{a_n\}_{n=1}^\infty$ bounded in A such that $a_n \to a$ in $A_0 + A_1$. The closure of $A_0 \cap A_1 \subset A$ in A is denoted by A^0.

Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be two Banach couples. A linear operator acting from $A_0 + A_1$ into $B_0 + B_1$ will be called a linear mapping from the couple $\vec{A}$ into the couple $\vec{B}$, written $T: \vec{A} \to \vec{B}$, if T maps continuously A_i into B_i, $i = 0, 1$.